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*Research Article*

# Explicit Solutions and Conservation Laws for a New Integrable Lattice Hierarchy

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An integrable lattice hierarchy is derived on the basis of a new matrix spectral problem. Then, some properties of this hierarchy are shown, such as the Liouville integrability, the bi-Hamiltonian structure, and infinitely many conservation laws. After that, the Darboux transformation of the first integrable lattice equation in this hierarchy is constructed. Eventually, the explicitly exact solutions of the integrable lattice equation are investigated via graphs.

## 1. Introduction

Investigation of nonlinear integrable lattice equations [1–6] is an active topic in the field of nonlinear science because its solutions can explain some physical phenomena [7–10]. Researchers in soliton field have proposed many valuable nonlinear integrable lattice equations [11–15] until now, but there are still many integrable lattice equations that have not been discovered. Generally speaking, nonlinear integrable lattice equations are related to their matrix spectral problems, respectively. Therefore, finding suitable matrix spectral problems is of great significance for us to derive integrable lattice equations and investigate their explicitly exact solutions.

It is well known that it is not easy to get solutions for integrable lattice systems. Currently, there are several useful approaches which can help us obtain explicit solutions, for instance, the binary nonlinearization [16–18], Lie symmetry analysis [19, 20], and the Darboux transformation [21–29]. The Darboux matix method is one of important and effective techniques to obtain explicitly exact solutions of integrable lattice equations [30–34].

In this paper, we will focus on this spectral problem:

$$E\varphi_n = U_n(u,\lambda)\varphi_n,$$
$$U_n = \begin{pmatrix} \dfrac{u_n}{\lambda} & \dfrac{1+v_n}{\lambda} \\ u_n & 1 \end{pmatrix}, \tag{1}$$

and its auxiliary problem,

$$\varphi_{n,t} = V_n(u,\lambda)\varphi_n, \tag{2}$$

where $u = (u_n, v_n)^T$, $u_n$ and $v_n$ are potentials, and $\lambda$ is a spectral parameter independent of t. According to the above spectral problem, an integrable lattice hierarchy in Liouville sense is presented firstly. Then, we investigate its bi-Hamiltonian structures using the discrete trace identity [35] proposed by G. Z. Tu. Its infinitely many conservation laws are also presented. Finally, the explicitly exact solutions of the integrable lattice equation derived from this matrix spectral problem are discussed by DT method.

The structure of the article is as follows. In Section 2, we will obtain a new integrable hierarchy in Liouville sense and present its bi-Hamiltonian structure in accordance with the matrix spectral problem. In Section 3, we will discuss conservation laws of the integrable hierarchy proposed in Section 2. In Section 4, its N-fold DT will be constructed and its exact solutions will be obtained. In Section 5, some conclusions are given.

## 2. Integrability and Bi-Hamiltonian Structures of a Lattice Hierarchy

Let us first introduce a shift operator $E$, which obeys the following operations:

$$\begin{aligned}(Ef)(n) &= f(n+1),\\ \left(E^{-1}f\right)(n) &= f(n-1),\\ & n\in Z,\end{aligned} \tag{3}$$

where $f$ is a lattice function.

In order to obtain a hierarchy of integrable lattice equations, we assume that

$$V_n = \begin{pmatrix} a_n & b_n \\ \lambda c_n & -a_n \end{pmatrix}. \tag{4}$$

By solving the stationary discrete zero curvature equation

$$E(V_n)U_n - U_nV_n = 0, \tag{5}$$

the following relationships will be derived:

$$\begin{aligned} a_{n+1}\frac{u_n}{\lambda} + b_{n+1}u_n - a_n\frac{u_n}{\lambda} - c_n(1+v_n) &= 0,\\ a_{n+1}\frac{1+v_n}{\lambda} + b_{n+1} - b_n\frac{u_n}{\lambda} + a_n\frac{1+v_n}{\lambda} &= 0,\\ -(a_{n+1}+a_n)u_n + c_{n+1}u_n - \lambda c_n &= 0,\\ (1+v_n)c_{n+1} - u_nb_n - (a_{n+1}-a_n) &= 0. \end{aligned} \tag{6}$$

Substituting

$$\begin{aligned} a_n &= \sum_{m>=0} a_n^{(m)}\lambda^{-m},\\ b_n &= \sum_{m>=0} b_n^{(m)}\lambda^{-m},\\ c_n &= \sum_{m>=0} c_n^{(m)}\lambda^{-m} \end{aligned} \tag{7}$$

into (6) follows the recursion relations

$$\begin{aligned} u_na_{n+1}^{(m)} - u_na_n^{(m)} - (1+v_n)c_n^{(m+1)} + u_nb_{n+1}^{(m+1)} &= 0,\\ (1+v_n)a_{n+1}^{(m)} - u_nb_n^{(m)} + (1+v_n)a_n^{(m)} + b_{n+1}^{(m+1)} &= 0,\\ u_nc_{n+1}^{(m)} - u_na_{n+1}^{(m)} - u_na_n^{(m)} - c_n^{(m+1)} &= 0,\\ (1+v_n)c_{n+1}^{(m)} - a_{n+1}^{(m)} - u_nb_n^{(m)} + a_n^{(m)} &= 0. \end{aligned} \tag{8}$$

By calculation, we find $b_n^{(0)} = 0, c_n^{(0)} = 0$ and $a_n^{(0)}$ is a arbitrary constant. For simplicity of calculation, we take $a_n^{(0)} = 0$. Then, the recursion relations Equation (8) could uniquely determine $a_n^{(m)}, b_n^{(m)}, c_n^{(m)}$, and $m >= 1$. The first few quantities are given by

$$\begin{aligned} a_n^{(1)} &= u_n(1+v_{n-1}),\\ b_n^{(1)} &= 1+v_n,\\ c_n^{(1)} &= u_n,\\ a_n^{(2)} &= -u_n^2(1+v_{(n-1)})^2 - u_nu_{n-1}v_{n_1}(1+v_{n-2})\\ &\quad - u_nu_{n+1}v_n(1+v_{n-1}),\\ b_{n+1}^{(2)} &= -u_{n+1}(1+v_n)^2 - u_nv_n(1+v_{n-1}),\\ c_n^{(2)} &= -u_nu_{n+1}v_n - u_n^2(1+v_{n-1}) \end{aligned} \tag{9}$$

and so on. Now, for any integer $m >= 0$, we define

$$V_n^{[m]} = \lambda^m V_n = \begin{pmatrix} \sum_{i=0}^{m} a_n^{(i)}\lambda^{(m-i)} & \sum_{i=0}^{m} b_n^{(i)}\lambda^{(m-i)} \\ \sum_{i=0}^{m} c_n^{(i)}\lambda^{(m-i)} & -\sum_{i=0}^{m} a_n^{(i)}\lambda^{(m-i)} \end{pmatrix}. \tag{10}$$

In order to derive the associated integrable lattice hierarchy, we give a modification $\Delta_n^{(m)}$,

$$\Delta_n^{(m)} = \begin{pmatrix} -2a_n^{(m)} + c_n^{(m)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{11}$$

Letting

$$V_n^{[m]} = V_n^{[m]} + \Delta_n^{(m)}, \tag{12}$$

we can obtain that

$$\begin{aligned} & E\left(V_n^{[m]}\right)U_n - U_nV_n^{[m]}\\ &= \begin{pmatrix} \frac{u_n}{\lambda}\left(a_n^{(m)} - a_{n+1}^{(m)} + c_{n+1}^{(m)} - c_n^{(m)}\right) & \frac{v_n}{\lambda}\left(a_n^{(m)} - a_{n+1}^{(m)}\right) \\ u_n\left(a_n^{(m)} - a_{n+1}^{(m)} + c_{n+1}^{(m)} - c_n^{(m)}\right) & 0 \end{pmatrix}. \end{aligned} \tag{13}$$

Then, the discrete zero-curvature equation

$$U_{n,t} - E\left(V_n^{[m]}\right)U_n + U_nV_n = 0, \quad m \ge 0 \tag{14}$$

is able to bring about the following integrable lattice hierarchy:

$$\begin{aligned} u_{n,t_m} &= u_n\left[a_n^{(m)} - a_{n+1}^{(m)} + c_{n+1}^{(m-1)} - c_n^{(m-1)}\right],\\ v_{n,t_m} &= v_n\left[a_n^{(m)} - a_{n+1}^{(m)}\right]. \end{aligned} \tag{15}$$

$E\varphi_n = U_n(u,\lambda)\varphi_n$ and $\varphi_{n,t} = V_n(u,\lambda)\varphi_n$ compose the Lax representation of the integrable lattice hierarchy Equation (15). When m=1, (15) is reduced to the following:

$$\begin{aligned} u_{n,t} &= u_n(u_nv_{n-1} - u_{n+1}v_n),\\ v_{n,t} &= v_n(u_n - u_{n+1}) + v_n(u_nv_{n-1} - u_{n+1}v_n), \end{aligned} \tag{16}$$

in which $\varphi_{n,t} = V_n^{[1]}\varphi_n V_n^{[1]}$ is

$$V_n^{[1]} = \begin{pmatrix} -\frac{1}{2}\lambda - u_n v_{n-1} & 1 + v_{n-1} \\ \lambda u_n & \frac{1}{2}\lambda - u_n - u_n v_{n-1} \end{pmatrix}. \quad (17)$$

Next, we try to construct the bi-Hamiltonian structures of (15). In order to achieve this goal, we denote $\Gamma_n = V_n U_n^{-1}$ and $\langle M, N\rangle$ stands for the trace of the product of two square matrices of the same orders M and N. Then,

$$\begin{aligned} \left\langle \Gamma_n, \frac{\partial U_n}{\partial \lambda} \right\rangle &= -a_{n+1}, \\ \left\langle \Gamma_n, \frac{\partial U_n}{\partial u_n} \right\rangle &= \frac{a_n}{u_n}, \\ \left\langle \Gamma_n, \frac{\partial U_n}{\partial v_n} \right\rangle &= \frac{1}{v_n}\left(a_{n+1} - c_{n+1}\right). \end{aligned} \quad (18)$$

The discrete trace identity [35] plays an important role in constructing Hamiltonian structures of integrable lattice equations and studying their integrability. Here, we try to construct Hamiltonian structure of (15) by using

$$\frac{\delta}{\delta u_n}\sum_{n\in Z}\left\langle \Gamma_n, \frac{\partial U_n}{\partial \lambda} \right\rangle = \left(\lambda^{-\varepsilon}\frac{\partial}{\partial \lambda}\lambda^{\varepsilon}\right)\left\langle \Gamma_n, \frac{\partial U_n}{\partial u_n^i} \right\rangle, \quad i = 1, 2. \quad (19)$$

Substituting the expansions $a_n = \sum_{m>=0} a_n^{(i)}\lambda^{-i}$ and $c_n = \sum_{m>=0} c_n^{(i)}\lambda^{-i}$ into (19), we have

$$\frac{\delta}{\delta u_n}\sum_{n\in Z}\left(-a_n^{(m+1)}\right) = (c - m)\begin{pmatrix} \frac{a_n^{(m)}}{u_n} \\ \frac{1}{v_n}\left[a_{n+1}^{(m)} - c_{n+1}^{(m)}\right] \end{pmatrix}. \quad (20)$$

To get the value of the constant $c$, we simply set $m = 0$ in (20) and then find $c = 0$. Therefore we have that

$$\frac{\delta H_n^{(m)}}{\delta u_n} = \begin{pmatrix} \frac{a_n^{(m)}}{u_n} \\ \frac{1}{v_n}\left[a_{n+1}^{(m)} - c_{n+1}^{(m)}\right] \end{pmatrix}, \quad (21)$$

$$H_n^{(m)} = \frac{\sum_{n\in z} a_n^{(m+1)}}{m}. \quad (22)$$

So we obtain the desired Hamiltonian form of (15),

$$\begin{aligned} \begin{pmatrix} u_n \\ v_n \end{pmatrix}_{t_m} &= J\frac{\delta H_n^{(m)}}{\delta u_n} = JL\frac{\delta H_n^{(m-1)}}{\delta u_n} = K\frac{\delta H_n^{(m-1)}}{\delta u_n} \\ &= KL^{m-1}\begin{pmatrix} \frac{1}{2u_n} \\ \frac{1}{2v_n} \end{pmatrix}, \end{aligned} \quad (23)$$

where

$$J = \begin{pmatrix} 0 & u_n v_n\left(-1 + E^{-1}\right) \\ u_n v_n (1 - E) & 0 \end{pmatrix}, \quad (24)$$

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \quad (25)$$

$$\begin{aligned} L_{11} &= -\frac{1}{u_n}(E-1)^{-1}(1+v_n)Ep_n^2 + \frac{1}{u_n}(E-1)^{-1}u_nE^{-1}u_nv_n + \frac{1}{u_n}(1-E)^{-1}u_n^2, \\ L_{12} &= -\frac{1}{u_n}(E-1)^{-1}(1+v_n)Eu_nv_n + \frac{1}{u_n}(E-1)^{-1}u_nE^{-1}\left(v_n+v_n^2\right), \\ L_{21} &= \frac{1}{v_n}\left(1-E^{-1}\right)^{-1}u_nE^{-1}u_nv_n - \frac{1}{v_n}\left(1-E^{-1}\right)^{-1}v_nEu_n^2, \\ L_{22} &= \frac{1}{v_n}\left(1-E^{-1}\right)^{-1}u_nE^{-1}\left(v_n+v_n^2\right) - \frac{1}{v_n}\left(1-E^{-1}\right)^{-1}v_nEu_nv_n - \frac{1}{v_n}\left(1-E^{-1}\right)^{-1}u_nv_n. \end{aligned} \quad (26)$$

In (23), the operator $J$ is a Hamiltonian operator for the hierarchy Equation (15), and the operator $L$ is a recursion operator for the hierarchy Equation (15). These two operators $J$ and $L$ satisfy the following relationships:

$$\begin{aligned} J^* &= -J, \\ JL &= L^*J, \end{aligned} \quad (27)$$

where $J^*$ stands for the formal conjugation of $J$ and

$$K = J = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \quad (28)$$

$$\begin{aligned} K_{11} &= u_nv_nEu_n^2 - u_n^2E^{-1}u_nv_n, \\ K_{12} &= u_nv_nEu_nv_n - u_n^2E^{-1}\left(v_n+v_n^2\right) + u_n^2v_n, \\ K_{21} &= \left(v_n+v_n^2\right)Eu_n^2 - u_nv_nE^{-1}u_nv_n - u_n^2v_n, \\ K_{22} &= \left(v_n+v_n^2\right)Eu_nv_n - u_nv_nE^{-1}\left(v_n+v_n^2\right). \end{aligned} \quad (29)$$

*Remark.* (1) The hierarchy of lattice equations (15) is an integrable Hamiltonian system in Liouville sense.

(2) The bi-Hamiltonian structure Equation (23) of the integrable lattice equation exists.

## 3. Conservation Laws of (16)

Based on Lax representation of the hierarchy of integrable lattice equations (15), we can obtain its conversation laws. Substituting $\Phi_n = (\phi_{1,n}, \phi_{2,n})^T$ into the spectral problem (1), we have

$$\begin{aligned}\phi_{1,n+1} &= \frac{u_n}{\lambda}\phi_{1,n} + \frac{1+v_n}{\lambda}\phi_{2,n},\\ \phi_{2,n+1} &= u_n\phi_{1,n} + \phi_{2,n},\end{aligned} \tag{30}$$

which demonstrates that

$$\lambda u_n\theta_n\theta_{n+1} + \lambda\theta_{n+1} - u_n\theta_n - (1+v_n) = 0, \tag{31}$$

where we have $\theta_n = \phi_{1,n}/\phi_{2,n}$.

Then, if we suppose that $\theta_n = \sum_{j=0}^{\infty}\theta_n^{(j)}\lambda^{-j}$, the following two recursion relationships will be found. One is

$$\begin{aligned}\theta_n^{(0)} &= \frac{1}{u_n},\\ \theta_n^{(1)} &= \frac{-v_n u_{n+1}}{u_n},\\ \theta_n^{(2)} &= v_n v_{n+1} u_{n+2} - v_n u_{n+1},\\ \theta_n^{(j)} &= -u_n\theta_n^{(j-1)} + u_n\sum_{m=1}^{j-1}\theta_n^{(m)}\theta_{n+1}^{(j-m)},\end{aligned} \tag{32}$$

and the other is

$$\begin{aligned}\theta_n^{(0)} &= 0,\\ \theta_n^{(1)} &= 1+v_n,\\ \theta_n^{(2)} &= -u_{n-1}v_{n-1}(1+v_{n-2}),\\ \theta_{n+1}^{(j)} &= u_n\theta_n^{(j-1)} - u_n\sum_{m=1}^{j-1}\theta_n^{(m)}\theta_n^{(j-m)}.\end{aligned} \tag{33}$$

Based on (2) and (17), a direct calculation gives

$$\begin{aligned}&[\ln(u_n\theta_n+1)]_t\\ &= (E-1)\left(-\lambda u_n\theta_n - \frac{1}{2}\lambda + u_n + u_n v_{n-1}\right).\end{aligned} \tag{34}$$

Then, we discuss conversation laws of the hierarchy of integrable lattice equations (15) according to the above two different recursion relationships, respectively. Substituting $\theta_n = \sum_{j=0}^{\infty}\theta_n^{(j)}\lambda^{-j}$ into (34), equating the coefficients of $\lambda$ on both sides of the equation, we can achieve an infinite number of conversation laws of (16) based on the (32). The first two of them are

$$\begin{aligned}\left(\frac{1}{2}u_n\right)_t &= (E-1)(u_n + u_n v_{n-1} + v_n u_{n+1}),\\ (-2v_n u_{n+1})_t &= (E-1)(-u_n v_n v_{n+1} u_{n+2} + u_n v_n u_{n+1}).\end{aligned} \tag{35}$$

Similarly, we also have another infinite conservation laws of (16) based on (33). The first two of them are

$$\begin{aligned}&(u_n + u_n v_n)_t = (E-1)(-u_n u_{n-1} v_{n-1}(1+v_{n-1})),\\ &\left[-u_n(1+v_n)\left(v_{n+1} + \frac{1}{2}u_n + \frac{1}{2}v_n\right)\right]_t = (E-1)\\ &\quad\cdot\left[-u_n^2 v_n\left(-u_n(1+v_{n-1})^2 + u_{n-1}v_{n-1}(1+v_{n-2})\right)\right].\end{aligned} \tag{36}$$

## 4. The N-Fold DT and Exact Solutions of Eq. (16)

If a integrable lattice equation with Lax pair $E\Phi_n = U_n\Phi_n$ and $\Phi_{n,t} = V_n\Phi_n$ exists a gauge transformation

$$\widetilde{\Phi}_n = T_n\Phi_n, \tag{37}$$

where $T_n$ is a reversible matrix and

$$\begin{aligned}E\widetilde{\Phi}_n &= \widetilde{U}_n(u,\lambda)\widetilde{\Phi}_n,\\ \widetilde{\Phi}_{n,t} &= \widetilde{V}_n(u,\lambda)\widetilde{\Phi}_n.\end{aligned} \tag{38}$$

The new potentials $\widetilde{U}_n(u,\lambda)$ and $\widetilde{V}_n(u,\lambda)$ possess the same form with the old potentials $U_n(u,\lambda)$ and $V_n(u,\lambda)$, respectively. The gauge transformation in (37) composes Darboux transformation of an integrable lattice equation with the transformation between old potential functions $U_n(u,\lambda)$, $V_n(u,\lambda)$ and new potential functions $\widetilde{U}_n(u,\lambda)$ $\widetilde{V}_n(u,\lambda)$.

Based on (37) and (38), we are also able to obtain the relations, i.e.,

$$\begin{aligned}\widetilde{U}_n &= T_{n+1}\widetilde{U}_n T_n^{-1},\\ \widetilde{V}_n &= \left(T_{n,t} + T_n\widetilde{V}_n\right)T_n^{-1}.\end{aligned} \tag{39}$$

*4.1. The N-Fold DT of Eq. (16).* The selection of $T_n$ is very important in constructing the Darboux transformation. The solution of the soliton equation can be obtained much more easily by an appropriate Darboux matrix. For this integrable lattice equation (16), we take

$$\begin{aligned}T_n &= \begin{pmatrix} a_n & b_n\\ c_n & d_n\end{pmatrix}\\ &= \begin{pmatrix} \left(1-b_n^{(N-1)}\right)\lambda^N + \sum_{i=0}^{N-1}d_n^{(i)}\lambda^i & \sum_{i=0}^{N-1}b_n^{(i)}\lambda^i\\ 0 & \lambda^N + \sum_{i=0}^{N-1}d_n^{(i)}\lambda^i\end{pmatrix}\end{aligned} \tag{40}$$

with $b_n^{(i)}$ and $d_n^{(i)}$ being all functions with respect to $n$ and $t$ and $N$ being a positive integer number. It is easy to find that $\det(T_n)$ is a (2N)-th order polynomial with respect to $\lambda$. If we assume that the 2N parameters $\lambda_j(\lambda_j \neq \lambda_i, j \neq i; \lambda_j \neq 0, j = 1, 2, \ldots, 2N)$ are the roots of the $\det(T_n)$, $\det(T_n)$ can be expressed as

$$\det(T_n) = \left(1 - b_n^{(N-1)}\right) \prod_{i=1}^{2N} \left(\lambda - \lambda_j\right). \quad (41)$$

Setting $\varphi_n = (\varphi_{n,1}, \varphi_{n,2})^T$ and $\psi_n = (\psi_{n,1}, \psi_{n,2})^T$ being the column vectors of $\Phi_n$, we find that the column vectors of $\widetilde{\Phi}_n = T_n \Phi_n$ are linear dependent because of $\det T_n(\lambda_i) = 0$. Then for $\lambda = \lambda_j$, $j = 1, 2, \ldots, 2N$, according to (37), we have

$$\begin{aligned} \sum_{i=0}^{N-1} \left(d_n^{(i)} + \delta_{j,n} b_n^{(i)}\right) \lambda_j^i &= -\left(1 - b_n^{(N-1)}\right) \lambda_j^N, \\ \sum_{i=0}^{N-1} \delta_{j,n} d_n^{(i)} \lambda_j^i &= -\delta_{j,n} \lambda_j^N, \end{aligned} \quad (42)$$

in which

$$\begin{aligned} \delta_{j,n} &= \frac{\varphi_{2,n} + \gamma_j \psi_{2,n}}{\varphi_{1,n} + \gamma_j \psi_{1,n}}, \\ \delta_{j,n+1} &= \lambda_j \frac{u_n + \delta_{j,n}}{u_n + (1 + v_n) \delta_{j,n}}, \\ & \qquad i = 1, \ldots, 2N. \end{aligned} \quad (43)$$

and the parameters $\lambda_j$ and $\gamma_i$ are so chosen that the denominators in (43) are nonzero.

**Theorem 1.** *The matrix $\widetilde{U}_n$ defined by (38) has the same form as $U_n$, i.e.,*

$$\widetilde{U}_n = \begin{pmatrix} \dfrac{\widetilde{u}_n}{\lambda} & \dfrac{1 + \widetilde{v}_n}{\lambda} \\ \widetilde{u}_n & 1 \end{pmatrix}. \quad (44)$$

*where the transformation from the old potentials $u_n$ and $v_n$ into the new potentials $\widetilde{u}_n$ and $\widetilde{v}_n$ is given by*

$$\begin{aligned} \widetilde{u}_n &= \frac{u_n}{1 - b_n^{(N-1)}}, \\ \widetilde{v}_n &= \left(1 - b_{n+1}^{(N-1)}\right) v_n. \end{aligned} \quad (45)$$

*Proof.* Let $T_n^{-1} = T_n^* / \det T_n$ and

$$T_{n+1} U_n T_n^* = \begin{pmatrix} f_{11}(\lambda, n) & f_{12}(\lambda, n) \\ f_{21}(\lambda, n) & f_{22}(\lambda, n) \end{pmatrix}. \quad (46)$$

It can be see that $f_{11}$ and $f_{12}$ are (2N-1)-th order polynomials with respect to $\lambda$, and $f_{21}$ and $f_{22}$ are (2N)-th order polynomials with respect to $\lambda$. From (42), we have

$$a_n(\lambda_j) = -\delta_{j,n} b_n(\lambda_j). \quad (47)$$

Moreover, (46) can be described as

$$T_{n+1} U_n T_n^* = (\det T_n) P_n, \quad (48)$$

together with

$$P_n = \begin{pmatrix} \lambda^{-1} P_{11}(\lambda, n) & \lambda^{-1} P_{12}(\lambda, n) \\ P_{21}(\lambda, n) & P_{22}(\lambda, n) \end{pmatrix}, \quad (49)$$

where $\lambda_i (i = 1, 2, \ldots, 2N)$ are roots of $f_{st}(\lambda, n)(s, t = 1, 2)$, $P_n$ is a 2-order square matrix, and $P_{jk}(j, k = 1, 2)$ are the functions with respect to n and t. Thus, (48) can be written as

$$T_{n+1} U_n = P_n T_n. \quad (50)$$

Equating the coefficients of $\lambda^N$ and $\lambda^{N-1}$ on both sides of (50), we find

$$\begin{aligned} P_{11} &= \frac{u_n}{1 - b_n^{(N-1)}} = \widetilde{u}_n, \\ P_{12} &= 1 + v_n \left(1 - b_{n+1}^{(N-1)}\right) = 1 + \widetilde{v}_n, \\ P_{21} &= \frac{u_n}{1 - b_n^{(N-1)}} = \widetilde{u}_n, \\ P_{22} &= 1. \end{aligned} \quad (51)$$

From (51), we can see that $P_n = \widetilde{U}_n$. The proof is completed. □

**Theorem 2.** *Under the transformation (37) and (45), the matrix $\widetilde{V}_n$ defined by (38) has the same form as $V_n$, i.e.,*

$$\widetilde{V}_n = \begin{pmatrix} -\dfrac{1}{2} - \widetilde{u}_n \widetilde{v}_{n-1} & 1 + \widetilde{v}_{n-1} \\ \lambda \widetilde{u}_n & \dfrac{1}{2} - \widetilde{u}_n (1 + \widetilde{v}_{n-1}) \end{pmatrix}. \quad (52)$$

*Proof.* Let $T_n^{-1} = T_n^* / \det T_n$ and

$$(T_{n,t} + T_n V_n) T_n^* = \begin{pmatrix} g_{11}(\lambda, n) & g_{12}(\lambda, n) \\ g_{21}(\lambda, n) & g_{22}(\lambda, n) \end{pmatrix}, \quad (53)$$

in which all $g_{jk}(j, k = 1, 2)$ are the functions of n and t. Through a series of calculations, we can figure out that $g_{11}(\lambda, n)$, $g_{21}(\lambda, n)$, and $g_{22}(\lambda, n)$ are (2N+1)-th order polynomials with respect to $\lambda$, and $g_{12}(\lambda, n)$ is (2N)-th order polynomial with respect to $\lambda$. We can verify from (47) that $\lambda_i (i = 1, 2, \ldots, 2N)$ are the roots of $g_{jk}(j, k = 1, 2)$. So, we know

$$(T_{n,t} + T_n V_n) T_n^* = (\det T_n) R_n, \quad (54)$$

together with

$$R_n = \begin{pmatrix} \lambda R_{11}^{(1)}(\lambda, n) + R_{11}^{(0)}(\lambda, n) & R_{12}^{(0)}(\lambda, n) \\ \lambda R_{21}^{(1)}(\lambda, n) + R_{21}^{(0)}(\lambda, n) & \lambda R_{22}^{(1)}(\lambda, n) + R_{22}^{(0)}(\lambda, n) \end{pmatrix}. \quad (55)$$

where all the $R_{jk}(j,k=1,2)$ are the functions with respect to $n$ and $t$. Thus, (54) can be rewritten as

$$T_{n,t}+T_nV_n=R_nT_n, \tag{56}$$

Equating the coefficients of $\lambda^{N+1}$ and $\lambda^N$ on both sides of (56), we find that

$$\begin{aligned}
R_{11}^{(1)}&=-\frac{1}{2},\\
R_{11}^{(0)}&=-\frac{u_n}{1-b_n^{(N-1)}}\left(1-b_{n-1}^{(N-1)}\right)v_{n-1}=-\widetilde{u}_n\widetilde{v}_{n-1},\\
R_{12}^{(0)}&=1+\left(1-b_n^{N-1}\right)=1+\widetilde{v}_{n-1},\\
R_{21}^{(1)}&=\frac{u_n}{1-b_n^{(N-1)}}=\widetilde{u}_n,\\
R_{21}^{(0)}&=0,\\
R_{22}^{(1)}&=\frac{1}{2},\\
R_{22}^{(0)}&=-\widetilde{u}_n\left(1+\widetilde{v}_{n-1}\right).
\end{aligned} \tag{57}$$

From (38) and (57), we are able to see that $R_n=\widetilde{V}_n$. The proof is completed. □

*4.2. Exact Solutions of (16).* In this section, the explicit solutions of the integrable lattice equation (16) will be discussed under its Darboux transformation. We can find $u_n=1, v_n=1$ is a trivial solution of (16) easily. Starting from this trivial solution, the solutions of the Lax Pair Equations (1) and (17) can be obtained,

$$\begin{aligned}
\varphi_n&=\begin{pmatrix}\varphi_{n,1}\\ \varphi_{n,2}\end{pmatrix}=\begin{pmatrix}\tau_1^n e^{\rho_1 t}\\ \tau_1^{n+1}e^{\rho_1 t}\end{pmatrix},\\
\psi_n&=\begin{pmatrix}\psi_{n,1}\\ \psi_{n,2}\end{pmatrix}=\begin{pmatrix}\tau_2^n e^{\rho_2 t}\\ \tau_2^{n+1}e^{\rho_2 t}\end{pmatrix},
\end{aligned} \tag{58}$$

in which

$$\begin{aligned}
\tau_1&=-\sqrt{\lambda_i^2+6\lambda_i+1}+\lambda_i-1,\\
\tau_2&=\sqrt{\lambda_i^2+6\lambda_i+1}+\lambda_i-1
\end{aligned} \tag{59}$$

$$\begin{aligned}
\rho_1&=\frac{1}{2}\left(3+\sqrt{\lambda_i^2+6\lambda_i+1}\right),\\
\rho_2&=-\frac{1}{2}\left(-3+\sqrt{\lambda_i^2+6\lambda_i+1}\right).
\end{aligned} \tag{60}$$

According to (43), we have

$$\delta_{i,n}=\frac{\tau_1^{n+1}e^{\rho_1 t}+\gamma_i\tau_2^{n+1}e^{\rho_2 t}}{\tau_1^n e^{\rho_1 t}+\gamma_i\tau_2^n e^{\rho_2 t}} \tag{61}$$

By using of the Cramer rules, solving (42) follows

$$b_n^{(N-1)}=\frac{\Delta b_n^{(N-1)}}{\Delta} \tag{62}$$

together with

$$\Delta=\begin{pmatrix}
1 & \lambda_1 & \dots & \lambda_1^{N-1} & \delta_1 & \delta_1\lambda_1 & \dots & \delta_1\lambda_1^{N-2} & \left(\delta_1-\lambda_1\right)\lambda_1^{N-1}\\
1 & \lambda_2 & \dots & \lambda_2^{N-1} & \delta_2 & \delta_2\lambda_2 & \dots & \delta_2\lambda_2^{N-2} & \left(\delta_2-\lambda_2\right)\lambda_2^{N-1}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & \lambda_{2N} & \dots & \lambda_{2N}^{N-1} & \delta_{2N} & \delta_{2N}\lambda_{2N} & \dots & \delta_{2N}\lambda_{2N}^{N-2} & \left(\delta_{2N}-\lambda_{2N}\right)\lambda_{2N}^{N-1}
\end{pmatrix}, \tag{63}$$

and $\Delta b_n^{(N-1)}$ is obtained from $\Delta$ by replacing its (2N)-th column with $(-\lambda_1^N,-\lambda_2^N,\dots,-\lambda_{2N}^N)^T$, i.e.,

$$\Delta b_n^{(N-1)}=\begin{pmatrix}
1 & \lambda_1 & \dots & \lambda_1^{N-1} & \delta_1 & \delta_1\lambda_1 & \dots & \delta_1\lambda_1^{N-2} & -\lambda_1^N\\
1 & \lambda_2 & \dots & \lambda_2^{N-1} & \delta_2 & \delta_2\lambda_2 & \dots & \delta_2\lambda_2^{N-2} & -\lambda_2^N\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & \lambda_{2N} & \dots & \lambda_{2N}^{N-1} & \delta_{2N} & \delta_{2N}\lambda_{2N} & \dots & \delta_{2N}\lambda_{2N}^{N-2} & -\lambda_{2N}^N
\end{pmatrix}. \tag{64}$$

By using (41), (45), and (62), the solution of (16) can be derived, and it is expressed as

$$\begin{aligned}
\widetilde{u}_n&=\frac{1}{1-b_n^{(N-1)}},\\
\widetilde{v}_n&=1-b_{n+1}^{(N-1)}.
\end{aligned} \tag{65}$$

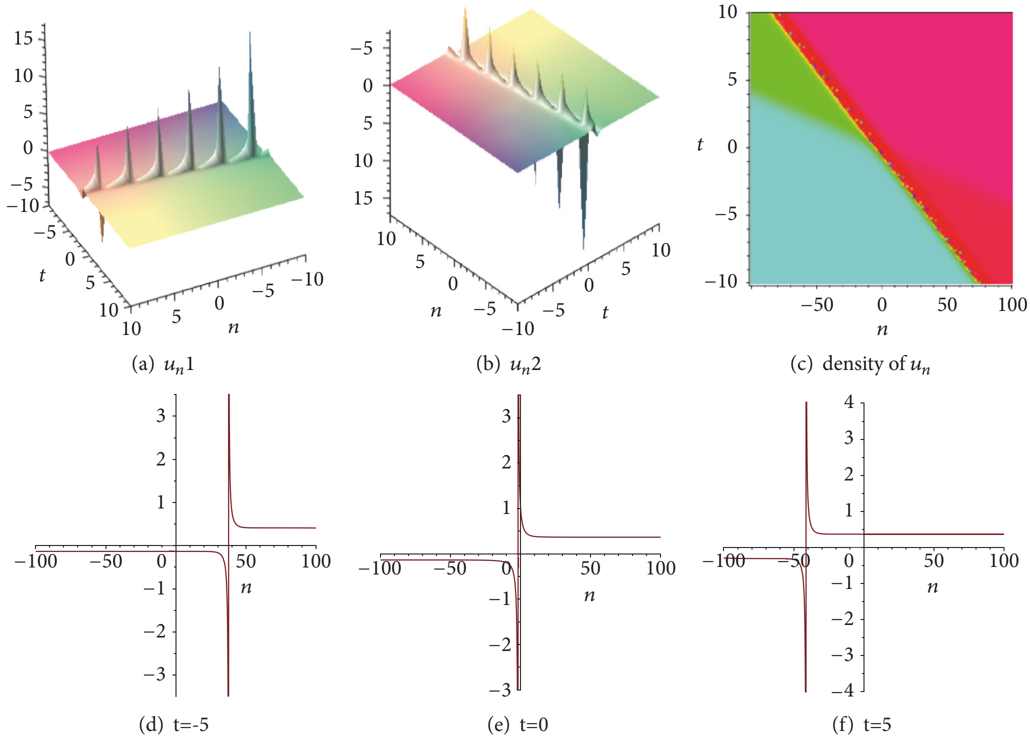
In order to investigate the structure of the solutions (65) fully, we plot their structure figures as shown in Figures 1–4 when $N=1$ and $N=2$.

(I) In particular, when $N=1$, solving (42) leads to

$$\begin{aligned}
\widetilde{u}_n&=\frac{1}{1-b_n^{(0)}},\\
\widetilde{v}_n&=1-b_{n+1}^{(0)}.
\end{aligned} \tag{66}$$

where

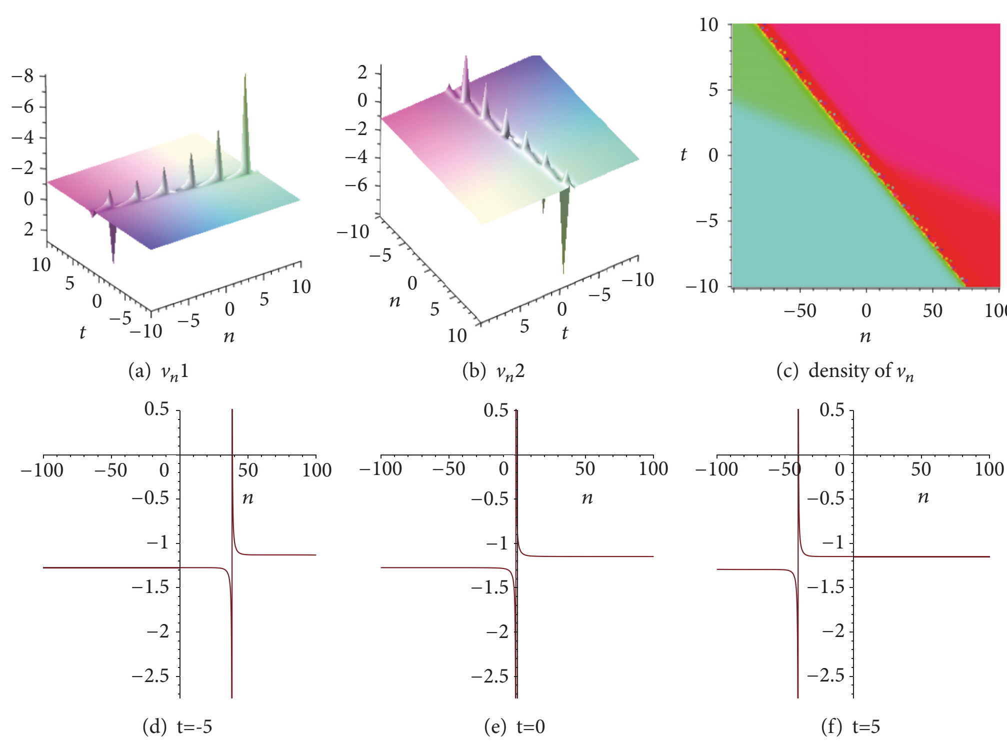
$$\begin{aligned}
\Delta&=\begin{pmatrix}1 & \left(\delta_{1,n}-\lambda_1\right)\\ 1 & \left(\delta_{2,n}-\lambda_2\right)\end{pmatrix},\\
\Delta b_n^{(0)}&=\begin{pmatrix}1 & -\lambda_1\\ 1 & -\lambda_2\end{pmatrix}.
\end{aligned} \tag{67}$$



(a) $u_n 1$ (b) $u_n 2$ (c) density of $u_n$

(d) t=-5 (e) t=0 (f) t=5

FIGURE 1: A structure of soliton solution for $u_n$ in (66) with $\lambda_1 = 0.5, \lambda_2 = 2, \gamma_1 = 1, \gamma_2 = -1$. Figures (a) and (b) are three-dimensional graphs of $u_n$. Figure (c) is the density plot of $u_n$ and Figures (d), (e), and (f) are structures of the solution $u_n$ at different time.



(a) $v_n 1$ (b) $v_n 2$ (c) density of $v_n$

(d) t=-5 (e) t=0 (f) t=5

FIGURE 2: A structure of soliton solution for $v_n$ in (66) with $\lambda_1 = 0.5, \lambda_2 = 2, \gamma_1 = 1, \gamma_2 = -1$. Figures (a) and (b) are three-dimensional graphs of $v_n$. Figure (c) is the density plot of $v_n$ and Figures (d), (e), and (f) are structures of the solution $v_n$ at different time.

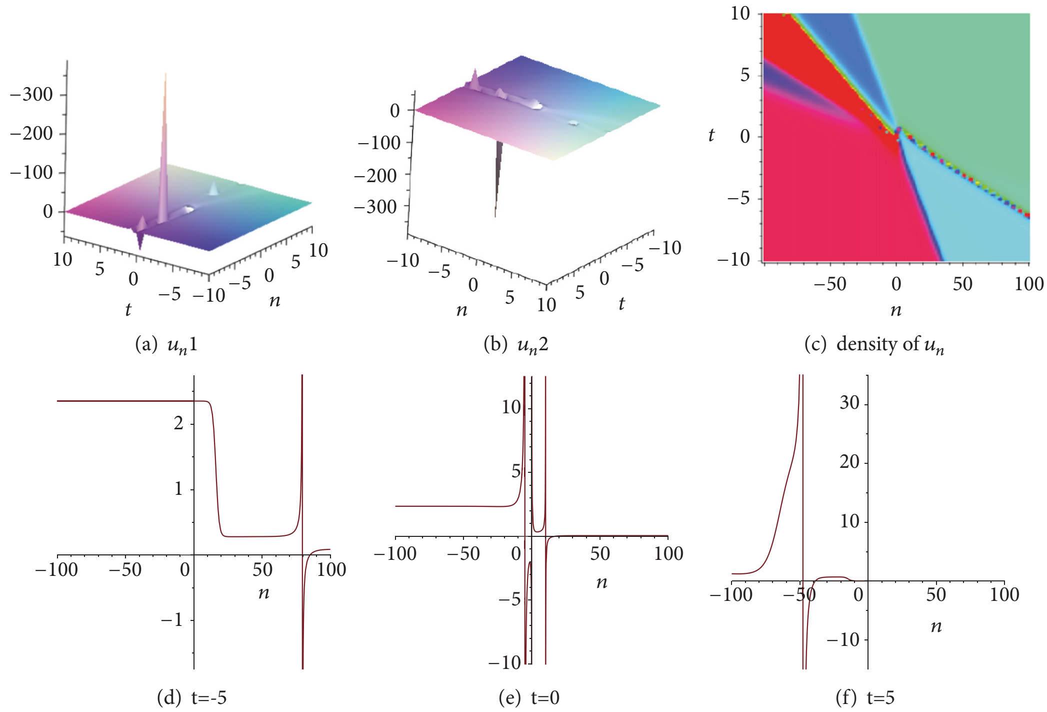



FIGURE 3: A structure of soliton solution for $u_n$ in (68) with $\lambda_1 = 0.5, \lambda_2 = 2, \lambda_3 = 1, \lambda_4 = 4, \gamma_1 = 1, \gamma_2 = -1, \gamma_3 = 2, \gamma_4 = -2$. Figures (a) and (b) are three-dimensional graphs of $u_n$, Figure (c) is the density plot of $u_n$, and Figures (d), (e), and (f) are structures of the solution $u_n$ at different time.

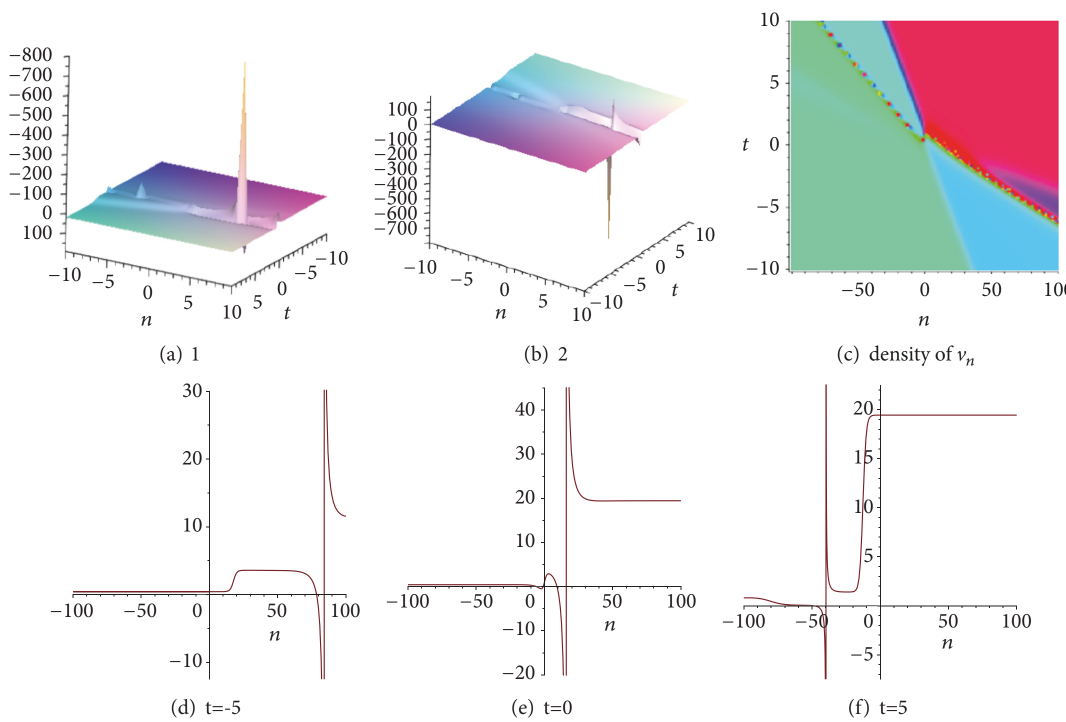



FIGURE 4: A structure of soliton solution for $v_n$ in (68) with $\lambda_1 = 0.5, \lambda_2 = 2, \lambda_3 = 1, \lambda_4 = 4, \gamma_1 = 1, \gamma_2 = -1, \gamma_3 = 2, \gamma_4 = -2$. Figures (a) and (b) are three-dimensional graphs of $v_n$, Figure (c) is the density plot of $v_n$, and Figures (d), (e), and (f) are structures of the solution $v_n$ at different time.

Figures 1 and 2 show the structures of the soliton solutions $\widetilde{u}_n$ and $\widetilde{v}_n$ in (65).

(II) When N=2, solving (42) leads to

$$\begin{aligned} \widetilde{u}_n &= \frac{1}{1-b_n^{(1)}}, \\ \widetilde{v}_n &= 1-b_{n+1}^{(1)}. \end{aligned} \tag{68}$$

where

$$\begin{aligned} \Delta &= \begin{pmatrix} 1 & \lambda_1 & \delta_1 & (\delta_{1,n}-\lambda_1)\lambda_1 \\ 1 & \lambda_2 & \delta_2 & (\delta_{2,n}-\lambda_2)\lambda_2 \\ 1 & \lambda_3 & \delta_3 & (\delta_{3,n}-\lambda_3)\lambda_3 \\ 1 & \lambda_4 & \delta_4 & (\delta_{4,n}-\lambda_4)\lambda_4 \end{pmatrix}, \\ \Delta b_n^{(1)} &= \begin{pmatrix} 1 & \lambda_1 & \delta_1 & \lambda_1^2 \\ 1 & \lambda_2 & \delta_2 & \lambda_2^2 \\ 1 & \lambda_3 & \delta_3 & \lambda_3^2 \\ 1 & \lambda_4 & \delta_4 & \lambda_4^2 \end{pmatrix}. \end{aligned} \tag{69}$$

Figures 3 and 4 show the structures of the soliton solutions $\widetilde{u}_n$ and $\widetilde{v}_n$ in (68).

## 5. Conclusions

In this article, we have derived an integrable lattice hierachy on the basis of a new matrix spectral problem. Then, some properties of this hierarchy have been shown, such as the Liouville integrability, the bi-Hamiltonian structures, and infinitely many conservation laws. Eventually, the Darboux transformation of the first integrable equation in this hierarchy has been constructed and the exact solutions of the integrable equation have been investigated via graphs. The binary nonlinearization [36–38] is also a useful method to investigate explicit solutions. We will investigate the binary nonlinearization with regard to (16) associated with spectral problems (1) and (2) in subsequent papers.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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