Chaos Synchronization in Time-Dependent Duplex Networks

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1. Introduction

Collective behaviors are ubiquitous in natural systems and human society. Among these, synchronization, one of the most interesting collective behaviors, has been investigated since the early days in natural science and has drawn great attention in the last decades [1–4]. There are various forms of synchronization, such as complete synchronization (CS) [5], cluster synchronization [6], remote synchronization [7], partial synchronization [8], phase synchronization [9], generalized synchronization [10], and lag synchronization [11]. Among these various types of synchronization, CS is the simplest one and its stability has been well analyzed by using the master stability function method [12, 13]. When complex networks are involved, it has been found that both the coupling function and the network structure can affect the stability of CS.

Previous studies on network science focused on single-layer networks. However, there are many complex systems ranging from social systems to technological systems which have to be modeled as multilayer networks other than single-layer ones [14–17]. For example, in social networks people may interact with each other through different channels such as WeChat, Twitter, and Facebook [18]. Each channel consists of an independent layer on which people interact with the coupling function of their own layer and all layers supported by different channels are integrated into a multilayer network. Among multilayer networks, multiplex networks refer to the ones in which different layers share the same group of nodes and each layer of the networks owns its unique interaction pattern [19–21]. Recently, CS has been investigated in multilayer networks [14, 22–24]. Different from single-layer networks on which the onset of the instability of CS is induced by the most unstable transversal network modes, it is still a challenge to figure out the mechanism for CS on multiplex networks to lose its stability. Except for some peculiar multiplex networks, there are no rigorous results for the stability of CS though some approximation methods have been proposed [25].

Interaction patterns may be time-dependent in many situations such as power transmission system [26], consensus problem [27], and person to person communications [28]. In modeling, the time-varying interaction patterns may be realized in several ways, for example, by rewiring connections in networks in the course of time [29], by assuming oscillators to move in space [30], and by allowing the switch of network topologies among several given interaction patterns [31]. Synchronization on time-varying networks has been
investigated when oscillators may move in space [32, 33]. It has been shown that synchronization time may depend nonmonotonically on the mobility of oscillators and the mechanism driving synchronization is different for different dynamical regimes [34–37]. However, in these literatures concerning with synchronization on time-varying networks, the coupling function is always same. In this paper, we construct a time-varying duplex networks by switching the interactions pattern and coupling function among oscillators between two single-layer networks with different coupling functions and study CS among chaotic Lorenz oscillators on them.

The paper is organized as follows. In Section 2, we present the model. The time-varying duplex networks are characterized by the parameters, the switching frequency \( \omega \) and the offset strength \( A \). In Section 3, we first consider two single-layer networks and study CS on each of them. Then we consider the time-varying duplex networks constructed from these two single-layer ones and investigate the dependence of CS on the switch frequency and the offset strength. The numerical results and some discussions are presented. In Section 4, a summary is made.

2. Model

We consider a network composed of \( N \) chaotic oscillators whose motion equations are governed by

\[
\dot{x}_i = F(x_i) + \varepsilon_1 \sum_{j=1}^{N} L_{ij}^{(1)}(t) H^{(1)}(x_j - x_i) + \varepsilon_2 \sum_{j=1}^{N} L_{ij}^{(2)}(t) H^{(2)}(x_j - x_i)
\]

where \( x_i = (x_i, y_i, z_i) \) is the 3-dimensional state variable of oscillator \( i \). We consider chaotic Lorenz dynamics which is described as \( F(x) = [10(y - x), 28x - y - xz, xy - z] \). \( L^{(1)}(t) = L^{(1)}(\Theta(\sin \omega t - A)) \) and \( L^{(2)}(t) = L^{(2)}(\Theta(A - \sin \omega t)) \) with the step function \( \Theta(x) = 1 \) if \( x > 0 \) and \( \Theta(x) = 0 \) otherwise. \( L^{(1)} \) and \( L^{(2)} \) are the adjacency matrices in the two layers, respectively. \( L_{ij}^{(1)} = 1 \) \( (L_{ij}^{(2)} = 1) \) when there exists a connection between oscillators \( i \) and \( j \) in the layer \( L^{(1)} \) \( (L^{(2)}) \) and, otherwise, \( L_{ij}^{(1)} = 0 \) \( (L_{ij}^{(2)} = 0) \). The parameter \( A \) is the offset strength.

The time-dependent duplex network reduces to the single layer \( L^{(1)} \) when \( A \leq -1 \) and to the single layer \( L^{(2)} \) for \( A \geq 1 \). At any given time, oscillators interact with each other either through layer \( L^{(1)} \) or through layer \( L^{(2)} \) for \( -1 < A < 1 \). Increasing the offset strength \( A \) prolongs the layer \( L^{(2)} \) to be the acting network. The parameter \( \omega \) is the switching frequency accounting for the rate of the network topology alternating between the layers \( L^{(1)} \) and \( L^{(2)} \). In the layers \( L^{(1)} \) and \( L^{(2)} \), oscillators interact with each other through the coupling function \( H^{(1)} \) and \( H^{(2)} \). Unless specified, we set \( H^{(1)}(x_j - x_i) = [x_j - x_i, 0, 0] \) and \( H^{(2)}(x_j - x_i) = [y_j - y_i, 0, 0] \). \( \varepsilon_1 \) and \( \varepsilon_2 \) account for the coupling strengths in the layers \( L^{(1)} \) and \( L^{(2)} \), respectively.

### 3. Numerical Simulations

Before investigating CS on time-dependent duplex networks, we take a look at CS on single-layer networks. For simplicity, we let \( N = 6 \). We generate two random networks \( L^{(1)} \) and \( L^{(2)} \) which are illustrated in the insets in Figures 1(a) and 1(b). Using the master stability function method, the stability of CS on \( L^{(1)} \) or \( L^{(2)} \) can be analyzed by \( N \) decoupled linear differential equations

\[
\dot{x}^{(\alpha)}_i = [DF(s) + \varepsilon\lambda^{(\alpha)}_i DH^{(\alpha)}(s)]x^{(\alpha)}_i
\]

with \( \alpha = 1, 2 \). \( \lambda^{(\alpha)}_i \) \((i = 0, 1, \ldots, N - 1)\) are the eigenvalues of the Laplacian matrix constructed from \( L^{(\alpha)} \) and we suppose \( \lambda^{(0)}_0 \geq \lambda^{(0)}_1 \geq \cdots \geq \lambda^{(0)}_{N-1} \). Except for the eigenvalue \( \lambda^{(0)}_0 = 0 \) accounting for the synchronous network mode, other \( N - 1 \) nonzero \( \lambda^{(\alpha)}_i \) represent the transversal network modes \( y^{(\alpha)}_i \) to CS. \( DF(s) \) and \( DH(s) \) are the Jacobian matrices of the corresponding functions evaluated at CS. The stability of CS requires that the largest transversal Lyapunov exponent, \( \lambda^{(\alpha)}_1 \), for each nonzero \( \lambda^{(\alpha)}_i \) to be negative. Figures 1(a) and 1(b) show \( \lambda^{(\alpha)}_1 \) for \( N - 1 \) transversal network modes for the single-layer networks \( L^{(1)} \) and \( L^{(2)} \), respectively. Clearly, the network \( L^{(1)} \) supports stable CS for \( \varepsilon_1 > 3.3 \) while \( \varepsilon_2 \) is always unstable on \( L^{(2)} \). For an arbitrary perturbation to CS, it can always be expanded as a linear combination of all transversal network modes. With time elapsing, it is the network mode with the largest transversal Lyapunov exponent that takes responsibility for determining the stability of CS. According to Figures 1(a) and 1(b), we may find that, for the layer \( L^{(1)} \), the network mode \( y^{(1)}_1 \) is the dominant one which always has the largest \( \Lambda \). On the other hand, the dominant network mode depends on the coupling strength for layer \( L^{(2)} \). For example, the dominant mode is \( y^{(2)}_1 \) for \( \varepsilon_2 < 3.4 \) while it changes to \( y^{(2)}_3 \) for \( \varepsilon_2 > 3.4 \). Furthermore, we calculate the Lyapunov exponent spectrum for the model dynamics equation (1) on the networks \( L^{(1)} \) and \( L^{(2)} \). The first four largest Lyapunov exponents against the coupling strength \( \varepsilon_1 \) and \( \varepsilon_2 \) are presented in Figures 1(c) and 1(d), respectively. Beyond the critical coupling strength \( \varepsilon_1 = 3.3 \), CS is established for the network \( L^{(1)} \) since all Lyapunov exponents are negative except for the largest Lyapunov exponent taking the value close to that at \( \varepsilon_1 = 0 \) and the second largest Lyapunov exponent to be zero. On the other hand, CS is impossible for the network \( L^{(2)} \) and the model dynamics may be either chaotic or time-independent for sufficiently strong \( \varepsilon_2 \).

Then, we consider the time-dependent duplex networks constructed from the networks \( L^{(1)} \) and \( L^{(2)} \) in Figure 1 and focus on the effects of the parameters \( A \) and \( \omega \) on the model dynamics in (1). To measure the synchronization in the model, we consider the synchronization error \( \Delta \), which is given by

\[
\Delta = \frac{2}{N(N-1)} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \left\| x_j - x_i \right\|_2^2
\]
The transversal Lyapunov exponents for different transversal network modes are plotted against the coupling strength in (a) for the layer $L^{(1)}$ and in (b) for the layer $L^{(2)}$. The insets in these two plots show the network structures of $L^{(1)}$ in (a) and of $L^{(2)}$ in (b). The first four largest Lyapunov exponents are plotted against the coupling strength in (c) for the layer $L^{(1)}$ and in (d) for the layer $L^{(2)}$. N = 6.

\[
\|x_j - x_i\|_2 = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}
\]

refers to the Euclidean norm between oscillators $i$ and $j$ and $\langle \cdot \rangle_t$ stands for the time average. For each parameter combination, the synchronization error is averaged over $10^3$ time units after a transient time around $10^4$ time units.

We first set $\varepsilon_1 = 5$ where CS is an attractor on the layer $L^{(1)}$ and $\varepsilon_2 = 4$ where chaotic desynchronous state is realized on the layer $L^{(2)}$. Increasing $A$ means that oscillators spend more time on the layer $L^{(2)}$, which tends to disfavor CS on the time-dependent duplex network. Compatible with the intuition, Figure 2(a) shows that $\Delta$ always increases with $A$. There seems to exist a critical $A_c$ for each $\omega$ above which $\Delta$ becomes nonzero and, to be stressed, $A_c$ increases rapidly at low $\omega$ and seems to saturate to a value at high $\omega$. Figure 2(b) presents $\Delta$ against $\omega$ for different $A$. For $A$ close to 1, $\Delta$ keeps nonzero and CS is impossible. On the other hand, $\Delta = 0$ is always possible for high $\omega$ when $A$ is not close to 1. Next, we set $\varepsilon_2 = 5$ where equilibrium is the attractor on the layer $L^{(2)}$. The dependence of $\Delta$ against $A$ and $\omega$ are presented in Figures 2(c) and 2(d) and we observe the similar behaviors to those at $\varepsilon_2 = 4$. Thus, CS on the time-dependent duplex network is insensitive to chaotic or regular dynamics on the isolated single-layer networks. To be mentioned, Figure 1(d) suggests a quenched state at $\varepsilon_2 = 5$ on the network $L^{(2)}$ where the maximum Lyapunov exponent is negative. Figure 2(b) further suggests that the quenched state is actually an inhomogeneous one according to the nonzero $\Delta$ at $A = 1$.

To get an overview on CS in the model (1), we present $\Delta$ in the plane of $A$ and $\omega$. Figures 3(a) and 3(b) display the results for the parameter combinations $(\varepsilon_1 = 5, \varepsilon_2 = 4)$ and $(\varepsilon_1 = 5, \varepsilon_2 = 5)$, respectively. It is clear that CS may be realized in a large range of domain in the parameter plane, for example, high $\omega$ and $A$ not close to 1. Actually, there exist two dynamical regimes depending on $\omega$. For high $\omega$, the critical $A_c$ separating CS from desynchronous states is roughly independent of $\omega$, which is around 0.8 for $(\varepsilon_1 = 5, \varepsilon_2 = 4)$ and 0.6 for $(\varepsilon_1 = 5, \varepsilon_2 = 5)$. In contrast, for low $\omega$, the parameter regimes supporting CS are narrow. The enlargement of the parameter plane in Figures 3(c) and 3(d) shows irregular boundary between CS and desynchronous states at low $\omega$.

The results at high $\omega$ can be understood as follows. For high $\omega$, the interaction pattern among oscillators switches between the layers $L^{(1)}$ and $L^{(2)}$ rapidly. In this case, the fast-switching approximation [34, 38], which averages out the
Figure 2: The synchronization error $\Delta$ against $A$ for different $\omega$ in (a) and (b) and against $\omega$ for different $A$ in (c) and (d). $\varepsilon_1 = 5$ and $\varepsilon_2 = 4$ in (a) and (c); $\varepsilon_1 = 5$ and $\varepsilon_2 = 5$ in (b) and (d).

The effect of alternation between $L^{(1)}$ and $L^{(2)}$, may be applied. Consequently, the interaction pattern among oscillators may be approximated by a time-independent duplex network with $L^{(1)}$ and $L^{(2)}$ being its two layers and with the effective coupling strengths $\varepsilon_{1,e} = \varepsilon_1 \tau/T$ and $\varepsilon_{2,e} = \varepsilon_2(T - \tau)/T$ where $T = 2\pi/\omega$ and $\tau$ is the time duration that oscillators interact with each other in the time-dependent network through $L^{(1)}$. Just like social networks where each person may simultaneously communicate with others via Facebook network and Twitter network, in this static duplex network as an approximation, each oscillator may simultaneously interact with others through two different networks assisted by the coupling functions $H^{(1)}$ and $H^{(2)}$, respectively. Since $\tau$ is determined by $\sin \omega t - A > 0$, it is unchanged with $\omega$ for a given $A$, which means that $\varepsilon_{1,e}$ and $\varepsilon_{2,e}$ are determined by $A$ uniquely. We present the phase diagram in the plane of $\varepsilon_1$ and $\varepsilon_2$ for the time-independent duplex network with $L^{(1)}(t) = L^{(1)}$ and $L^{(2)}(t) = L^{(2)}$ in Figure 3(e) to show the CS region. Then we present the effective coupling strengths $\varepsilon_{1,e}$ and $\varepsilon_{2,e}$ acquired from the coupling strengths $\varepsilon_1$ and $\varepsilon_2$ on the time-dependent duplex network by changing $A$ from $-1$ to $1$ with the increment $0.1$. The green dots in Figure 3(e) are for the set ($\varepsilon_1 = 5$, $\varepsilon_2 = 4$) while the red dots for the set ($\varepsilon_1 = 5$, $\varepsilon_2 = 5$). According to the plot, we find the critical $A$ is around 0.8 for $\varepsilon_2 = 4$ and 0.6 for $\varepsilon_2 = 5$, which are in agreement with Figures 3(a) and 3(b).

On the other hand, the dynamics is quite complicated at low $\omega$. However, for the extremely low $\omega$, we could present some plausible explanations. For the extremely low $\omega$, the system has to spend sufficiently long time on one layer before switching to the other one. Consequently, the transversal network modes $\psi_1^{(1)}$ on the layer $L^{(1)}$ and $\psi_2^{(2)}$ on the layer $L^{(2)}$ will have different effects on model dynamics. To sort out these effects from different network modes, we firstly take the transversal network modes with the largest transversal Lyapunov exponent on each layer into considerations, for example, $\psi_1^{(1)}$ on the layer $L^{(1)}$ and, on the layer $L^{(2)}$, $\psi_2^{(2)}$ for $\varepsilon_2 < 3.4$ and $\psi_5^{(2)}$ for $\varepsilon_2 > 3.4$. Supposing the transversal Lyapunov exponent of the network mode in the layer $L^{(1)}$ to be $\Lambda^{(1)}$ and in the layer $L^{(2)}$ to be $\Lambda^{(2)}$, any perturbation away from CS would evolve in the form $e^{\Lambda^{(1)} \tau + \Lambda^{(2)} (T - \tau)}$. For
the time-independent duplex network with two layers represented by Figure 3(f). There is strong discrepancy between the onset of desynchronization acquired in this way is also with the largest network (c) and (d) The enlargement view of (a) and (b) in the range of \( \omega \in (0,0.08) \), respectively. (e) The phase diagram in the plane of \( \epsilon_1 \) and \( \epsilon_2 \) for the time-independent duplex network with two layers represented by \( L^{(1)} \) and \( L^{(2)} \). The shaded area corresponds to the CS region. The dot lines denote the effective \( \epsilon_{1\lambda} \) and \( \epsilon_{3\lambda} \) acquired by the fast-switching approximation. (f) The critical \( A_\star \) are plotted against \( \epsilon_2 \) with \( A_\star^1 \) in black, \( A_\star^2 \) in blue, and \( A_\star \) acquired from the synchronization error in red. In addition, \( A_\star \) against \( \epsilon_2 \) in the time-dependent duplex network with \( L^{(2)} \) replaced by \( L^{(3)} \) is plotted in green. \( \epsilon_1 = 5 \) for the solid symbols and \( \epsilon_1 = 6 \) for the open symbols. The inset in (f) shows the single-layer network \( L^{(3)} \).
the inset of Figure 3(f), which keeps the overall interaction pattern unchanged but with rearranged oscillators on the layer. The Laplacian of $L^{(3)}$ has the same eigenvalues with $L^{(2)}$ and, therefore, the stability of CS on $L^{(3)}$ is exactly the same as that on $L^{(2)}$. Different from $L^{(2)}$, the product between $v^{(3)}_1$ and $v^{(3)}_2$ is around 0.66, which assures that the network mode $v^{(3)}_1$ is the dominant one in determining $A_c$. Consequently, $A_c$, acquired according to $\Delta$, in the range of $\varepsilon_2 \in (1.5, 3.5)$ is greatly lowered and the discrepancy between $A_c$ and $A_c^*$ is greatly reduced, as shown in Figure 3(f).

We have found that CS is possibly established on a time-dependent duplex network when one of the layers does not support CS on its own. Actually, CS can be realized on time-dependent duplex networks even when two layers do not allow for the stable CS on their own. To illustrate it, we consider the coupling functions $H_1(x_j - x_i) = [y_j - y_i, 0, 0]$ and $H_2(x_j - x_i) = [z_j - z_i, 0, 0]$ where CS is unstable on either $L^{(1)}$ or $L^{(2)}$. We present the synchronization error $\Delta$ in the plane of $A$ and $\omega$ in Figure 4(a) with $\varepsilon_1 = \varepsilon_2 = 5$ and Figure 4(b) with $\varepsilon_1 = 8$ and $\varepsilon_2 = 4$. Except for the parameter regimes close to $A = -1$ and $A = 1$, CS can always be produced at high $\omega$ where the critical $A$ for CS is insensitive to $\omega$. Figure 4(c) shows the stability diagram in the plane of $\varepsilon_1$ and $\varepsilon_2$ for the time-independent duplex network. The critical $A$ in Figures 4(a) and 4(b) can be acquired by plotting the effective coupling strengths $\varepsilon_{1,e}$ and $\varepsilon_{2,e}$ at different $A$ in Figure 4(c).

4. Conclusion

In this work, we constructed time-dependent duplex networks in which the network topology alternates periodically between two single-layer networks. The temporal properties of the networks are characterized by the offset strength $A$ and the switching frequency $\omega$. We studied the complete chaos synchronization on this type of time-dependent duplex networks. By monitoring the synchronization error, we find that there are two dynamical regimes depending on the switching frequency $\omega$. For high $\omega$ where the fast-switching approximation is valid, the time-dependent duplex networks can be approximated by time-independent duplex network with effective coupling strength. The critical $A$ for the onset of stable CS is independent of $\omega$. For low $\omega$, the critical $A$ strongly depends on $\omega$ and the coupling strengths on the two layers. At extremely low $\omega$, we find that the dependence of the critical $A$ on the coupling strengths can be roughly explained based on the picture in which only one dominant network mode in each layer is taken into considerations. We found that which network mode is the dominant one is determined
by its transversal Lyapunov exponent and its inner product with the dominant one on the other layer.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References


