Research Article

The State-Dependent Impulsive Model with Action Threshold Depending on the Pest Density and Its Changing Rate

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Whether the integrated control measures are applied or not depends not only on the current density of pest population, but also on its current growth rate, and this undoubtedly brings challenges and new ideas to the state control measures that only rely on the pest density. To address this, utilizing the tactics of IPM, we constructed a Lotka-Volterra predator-prey system with action threshold depending on the pest density and its changing rate and examined its dynamical behavior. We present new criteria guaranteeing the existence, uniqueness, and global stability of periodic solutions. With the help of Lambert W function, the Poincaré map is constructed for the phase set, which can help us to provide the satisfactory conditions for the existence and stability of the semitrivial periodic solution and interior order-1 periodic solutions. Furthermore, the existence of order-2 and nonexistence of order- k ($k \geq 3$) periodic solutions are discussed. The idea of action threshold depending on the pest density and its changing rate is more general and can generate new remarkable directions as well compared with those represented in earlier studies. The analytical techniques developed in this paper can play a significant role in analyzing the impulsive models with complex phase set or impulsive set.

1. Introduction

The predator-prey interaction systems attracted wide attention since the basic idea exhibited by Lotka and Volterra in the 1920s and widely studied for their rich dynamics [1–3]. Many scholars have considered and investigated this type of models, and the structure of the Lotka-Volterra system has been tremendously enhanced and extended [4–7]. Since the complex and rich dynamics for interactive species are very common in the real life, therefore, a lot of researchers have explored the procedures that affect the dynamics of prey-predator models. They need to perceive what models represent the best interaction between different species.

In many natural phenomena in the real world, prompt change happens which is subject to the instantaneous changes and interferences and they are much of the time thought to be as impulses in the process of modeling. Therefore, impulsive differential equations give a characteristic portrayal for such jumping. Recently, a greater part of the researchers has given attention to the impulsive differential equations which have

played a key role in the area of life sciences [8, 9]. This type of equations is found in almost every area of applied sciences. Various illustrations can be seen in [10, 11].

The concept of impulsive differential equations is an important and new area of differential equations. It plays an essential role in the development of qualitative theory of differential equations. This type of equations appears to present a natural structure for mathematical modeling of many real phenomena. For example, by modeling impulsive differential equations, external effects of different possible changes in the populations can be incorporated into the proposed model. There are two main kinds of the classical impulsive differential equations: fixed-time impulse and state-dependent impulse. We refer readers to [12–24] for more details about the aforementioned kinds of classical impulsive differential equations and their application in different fields.

It is important to note that the most significant idea in integrated pest management (IPM) process is the use of the economic threshold (ET). The ET is the existing pest quantity in the field when it is needed to take control action, with the

objective that the economic injury level (EIL) is not touched and surpassed. Based on the two key definitions of ET and EIL, many ecological systems concerning IPM have been developed and studied [22–26]. Note that in all the above proposed models, the IPM tactics should be implemented only when the density of the pest population reaches the ET, which means that the impulsive set of the proposed state-dependent impulsive differential equations is a straight line. From mathematical point of view, this significantly reduces the difficulties in analyzing the models. From a biological point of view, the IPM measures based on the pest density are often inconsistent with reality due to the fact that a small density of the pest population could induce a fast growth rate.

In practical IPM based pest control strategies, the challenge is how to implement control tactics and when to implement them. As mentioned before, a basic hypothesis in all existing models is that the IPM should be implemented once the density of the pest population reaches the given economic threshold; i.e., the state-dependent impulsive models have been used. Note that, in this way, the two crucial points related to analyzing the global dynamics of the proposed models, the impulsive set and phase set, are two straight lines. However, there are two practical situations: one is that the number of pest population is relatively large, but its change rate is quite small; the other is that the number of population is small, but its change rate is significantly high. The latter is more obvious at the initial stage of the outbreak of the pest. The question is how to formulate the mathematical models with both density and its change rate dependent feedback control (so-called ratio-dependent action threshold), which will undoubtedly result in complex curves for impulsive and phase sets. So far, the above problem has not been well modeled, depicted, and studied.

Therefore, in order to overcome those shortcomings, based on the classical Lotka-Volterra system, we propose the model with pest density and its change rate dependent feedback control; i.e., the action threshold depending on the pest density and its change rate determines whether the control strategy is implemented:

$$\left\{ \begin{array}{l} \frac{dx(t)}{dt} = x(t)(a - by(t)), \\ \frac{dy(t)}{dt} = y(t)(cx(t) - d), \\ x(t^+) = (1 - p)x(t), \\ y(t^+) = y(t) + \tau, \end{array} \right\} \quad \begin{array}{l} a_1x + b_1 \frac{dx}{dt} < A, \\ a_1x + b_1 \frac{dx}{dt} = A, \end{array} \quad (1)$$

where a_1, b_1 and A are all positive constants with $a_1 + b_1 = 1$. x and y address the populations of prey and predator, respectively. The quantities of ax and dy successively represent the pest growing rate and predator decline rate in the absence of each other. b represents predation rate of the predator on the prey; c indicates the contribution of prey to the predator's growth. Whenever the pest population density reaches the action threshold condition $a_1x + b_1(dx/dt) = A$, the controlling measures are instantly performed and they are adjusted to $(1 - p)x$ and $y + \tau$, respectively, where $p \in [0, 1]$ signifies the proportion of pest reduction whenever it reaches the action threshold and $\tau \geq 0$ represents the release quantity

of predator at that time. It is noted that we can fully consider the more general pest and natural enemy systems, but in order to highlight the effects of threshold conditions and compare with previous work, the simplest pest natural enemy system is chosen here.

The contents of the paper are arranged as follows. In Section 2, according to the representation of model (1), the Poincaré map in the exact phase sets is constructed. The existence and stability conditions for the semitrivial periodic solution are acquired in Section 3. Moreover, in Section 4.1, the existence and stability of an order-1 periodic solution will be addressed. In Section 4.2, the nonexistence of order- k ($k \geq 3$) limit cycle for system (1) is discussed in detail. Furthermore, the specific conditions are given which assure the existence of order-2 periodic solution. To conclude the entire work, a brief discussion is given in the last section.

2. Construction of Poincaré Map

The quantities a_1, b_1 are dependent upon each other. So the increase or decrease in one dependent quantity will affect the other. If the estimation of b_1 reduces, a_1 will obviously approach 1. Consequently, the ratio dependent action threshold will be transformed to the density dependent ET which is already discussed in literatures [27–31]. It follows from $a_1x + b_1(dx/dt) = A$ and the first equation of model (1) that $y = ((a_1 + ab_1)x - A)/bb_1x$ with

$$\lim_{x \rightarrow +\infty} \frac{(a_1 + ab_1)x - A}{bb_1x} = \frac{a_1 + ab_1}{bb_1}. \quad (2)$$

The essential assumption in this paper is that the initial amount x_0^+ of the pest population must satisfy $a_1x_0^+ + b_1(dx_0^+/dt) < A$. In model (1) without impulsive effect, there always exists a saddle point $(0, 0)$, which is unstable and a stable centre $E_0(x_*, y_*) = (d/c, a/b)$.

Poincaré map plays a significant role in the investigation of the existence of order- k periodic solutions (see appendix for definition). So it is important to construct the Poincaré map first and to provide the exact domain and range for it. It is moreover needed to understand the conditions under which the solution initiating from (x_0^+, y_0^+) is free from impulsive effect; i.e., the solution originating from $y^+ = ((a_1 + ab_1)x^+ - A(1 - p))/bb_1x^+ + \tau$ cannot reach the curve $y = ((a_1 + ab_1)x - A)/bb_1x$ for maximum impulsive set.

Note that in the rest paper, unless otherwise specified, we always take an initial point (x_0^+, y_0^+) from the curve $y^+ = ((a_1 + ab_1)x^+ - A(1 - p))/bb_1x^+ + \tau$ to reveal the properties of the Poincaré map, and for the dynamics of system, we may only focus on solutions initiating from the phase set which will analytically be formulated later.

For convenience, we indicate the two curves $y = ((a_1 + ab_1)x - A)/bb_1x$ and $y^+ = ((a_1 + ab_1)x^+ - A(1 - p))/bb_1x^+ + \tau$ by Γ_{Im} and Γ_{Ph} , respectively, as shown in Figure 1. Let A/a_1 and $A/(a_1 + ab_1)$ be the horizontal coordinates of the curve Γ_{Im} at $y = 0$ and $y = a/b$, respectively. Then, according to the

FIGURE 1: Illustration diagrams of the phase sets and impulsive sets for Cases (i) and (ii). (a) $A/a_1 \leq x_*$; (b) $x_* \leq A/(a_1 + ab_1)$ and $x_{E_1} < A(1-p)/(a_1 + bb_1\tau)$. Γ_1 is a trajectory touches the curve Γ_{ph} at point $T = (x_T, y_T)$, and Γ_2 is a trajectory tangent to the curve Γ_{Im} at point $S = (x_S, y_S)$.

different locations of Γ_{Im} and x_* , we take the following two cases:

$$\begin{aligned} \text{Case (i)} \quad & \frac{A}{a_1} \leq x_*; \\ \text{Case (ii)} \quad & x_* \leq \frac{A}{a_1 + ab_1}. \end{aligned} \quad (3)$$

Now in view of the above cases, the exact domains for the impulsive and phase sets will be discussed in the following in more detail.

Case (i). In this case, the trajectory Γ_1 (as shown in Figure 1(a)) touches the curve Γ_{ph} at point $(A(1-p)/(a_1 + \tau bb_1), a/b)$, denoted by $T = (x_T, y_T)$. Thus, in order to define the maximum impulsive and phase sets, we denote

$$\begin{aligned} \mathcal{M}_1 &= \left\{ (x, y) \in R_+^2 \mid \frac{A}{(a_1 + ab_1)} \leq x \leq x_{Q_2}, 0 \leq y \leq y_{Q_2} \right\}, \\ \mathcal{N}_1 &= \left\{ (x^+, y^+) \in R_+^2 \mid \frac{A(1-p)}{(a_1 + ab_1)} \leq x^+ < \frac{A}{(a_1 + ab_1)}, \right. \\ &\quad \left. \tau \leq y^+ < \frac{p(a_1 + ab_1)}{bb_1} + \tau \right\}. \end{aligned} \quad (4)$$

In the following lemma, the analytical formula for y_{Q_2} is also given, which is useful for the coming analyses. Note that it is actually the maximal vertical coordinate of the solution for Case (i).

Lemma 1. *The impulsive set for Case (i) is defined as $\mathcal{M}_1$. The maximum vertical coordinate for this is y_{Q_2} , where $y_{Q_2} = -(a/b)W(-e^{-1+A_{Q_2}/a})$ provided that $A_{Q_2} \leq 0$.*

Proof. Let a solution be initiating from the point $(A(1-p)/(a_1 + \tau bb_1), a/b)$, and it touches the curve Γ_{Im} at point (x_{Q_2}, y_{Q_2}) . Since it passes through the point $T = (A(1-p)/(a_1 + \tau bb_1), a/b)$, we can write

$$a \ln y - by + d \ln x - cx$$

$$= a \ln \frac{a}{b} - a + d \ln \frac{A(1-p)}{a_1 + \tau bb_1} - c \frac{A(1-p)}{a_1 + \tau bb_1}, \quad (5)$$

the curve Γ_1 intersects the curve Γ_{Im} at point (x_{Q_2}, y_{Q_2}) , and then (5) becomes

$$\begin{aligned} & a \ln y_{Q_2} - by_{Q_2} + d \ln x_{Q_2} - cx_{Q_2} \\ &= a \ln \frac{a}{b} - a + d \ln \frac{A(1-p)}{a_1 + \tau bb_1} - c \frac{A(1-p)}{a_1 + \tau bb_1}. \end{aligned} \quad (6)$$

Rearranging this equation for y_{Q_2} , we get

$$\left(-\frac{b}{a} y_{Q_2} \right) e^{-(b/a)y_{Q_2}} = -e^{-1+A_{Q_2}/a}, \quad (7)$$

where $A_{Q_2} = d(\ln(A(1-p)/(a_1 + \tau bb_1)) - \ln x_{Q_2}) + c(x_{Q_2} - A(1-p)/(a_1 + \tau bb_1))$. With the help of Lambert W function, we get

$$y_{Q_2} = -\frac{a}{b} W(-e^{-1+A_{Q_2}/a}), \quad (8)$$

and it is well defined due to $A_{Q_2} \leq 0$. This completes the proof. $\square$

Case (ii). Let us denote the intersection point of the closed trajectory Γ_2 with line $y = a/b$ (denoted by L_1) by $E_1 = (x_{E_1}, y_{E_1})$, as shown in Figure 1(b). The closed trajectory is tangent to the curve Γ_{Im} at the point $S = (x_S, y_S)$. Then, based on the positions of the curves Γ_{Im} and Γ_{ph} , we discuss the maximum impulsive and phase sets for this case as follows:

$$\begin{aligned} \mathcal{M}_2 &= \left\{ (x, y) \in R^2 \mid \frac{A}{(a_1 + ab_1)} \leq x \leq x_S, 0 \leq y \leq y_S \right\}, \\ \mathcal{N}_2 &= \left\{ (x^+, y^+) \in R_+^2 \mid x^+ \in \left[\frac{A(1-p)}{(a_1 + ab_1)}, x_{P_2} \right] \right. \\ &\quad \cup \left[x_{P_1}, \frac{A}{(a_1 + ab_1)} \right), y^+ \in [\tau, y_{P_2}] \\ &\quad \left. \cup \left[y_{P_1}, \frac{p(a_1 + ab_1)}{bb_1} + \tau \right) \right\}. \end{aligned} \quad (9)$$

From the phase set, it is clear that the closed trajectory is tangent to the curve Γ_{Im} at the point $S = (x_S, y_S)$. So, obviously any solution initiating from the interior of segment $\overline{P_1 P_2}$ will not reach the curve Γ_{Im} and hence will be free from impulsive effect.

In the following, we give some quantities which are not only helpful for finding the exact values of y_{P_1} and y_{P_2} , but also assume a significant role in finding the fixed point of Poincaré map $p(y_i^+)$. These quantities are listed as follows:

$$\begin{aligned} A_{P_1} &= d(\ln x_{P_1} - \ln x_S) + c(x_S - x_{P_1}), \\ A_{P_2} &= d(\ln x_{P_2} - \ln x_S) + c(x_S - x_{P_2}). \end{aligned} \quad (10)$$

Lemma 2. *The impulsive set for Case (ii) is defined as $\mathcal{M}_2$. In this case, any solution initiating from (y_{P_2}, y_{P_1}) will be free from impulse effect, where*

$$\begin{aligned} y_{P_1} &= -\frac{a}{b}W\left(-1, -\frac{b}{a}y_S e^{-(b/a)y_S - A_{P_1}/a}\right), \\ y_{P_2} &= -\frac{a}{b}W\left(-\frac{b}{a}y_S e^{-(b/a)y_S - A_{P_2}/a}\right) \end{aligned} \quad (11)$$

provided that $A_{P_1}, A_{P_2} \geq 0$

Proof. For this case, the closed trajectory is tangent to the curve Γ_{Im} at the point $S = (x_S, y_S)$, so we can write

$$\begin{aligned} a \ln y - by + d \ln x - cx \\ = a \ln y_S - by_S + d \ln x_S - cx_S. \end{aligned} \quad (12)$$

Since the trajectory Γ_2 passes to the point (x_{P_2}, y_{P_2}) , (12) can be rewritten as follows:

$$\begin{aligned} a \ln y_{P_2} - by_{P_2} + d \ln x_{P_2} - cx_{P_2} \\ = a \ln y_S - by_S + d \ln x_S - cx_S. \end{aligned} \quad (13)$$

With the help of Lambert W function, the above equation can be solved as follows:

$$y_{P_2} = -\frac{a}{b}W\left(-\frac{b}{a}y_S e^{-(b/a)y_S - A_{P_2}/a}\right). \quad (14)$$

The value of y_{P_1} can be found similarly; i.e., we have

$$y_{P_1} = -\frac{a}{b}W\left(-1, -\frac{b}{a}y_S e^{-(b/a)y_S - A_{P_1}/a}\right). \quad (15)$$

Since $A_{P_1}, A_{P_2} \geq 0$, y_{P_1} and y_{P_2} are well defined. This completes the proof. $\square$

Note that for this case, $\mathcal{M}_2$ is the maximum impulsive set. As the weighted parameter b_1 increases, the tangent point of the closed trajectory with the curve Γ_{Im} also moves to another point having vertical coordinate less than a/b ; i.e., $y_S < a/b$. For Case (ii) $x_{E_1} \leq A(1-p)/(a_1 + bb_1\tau)$, where x_{E_1} is the horizontal coordinate of the point E_1 . If $x_{E_1} = A(1-p)/(a_1 + bb_1\tau)$, then the impulsive set $\mathcal{M}_2$ becomes $\mathcal{M}_1$; i.e., $\mathcal{M}_2 = \mathcal{M}_1$. From Figure 1(b), it is clear that $x_{E_1} < x_* < A/(a_1 + ab_1)$.

The impulsive and phase sets, which are needed for developing the Poincaré map, have already been formed. These can be now used to determine the Poincaré map.

2.1. Poincaré Map. Let a trajectory initiate with initial value (x_0^+, y_0^+) . We assume that it repeats the impulse processes k times which may be finite or infinite. Let us denote the coordinates with the impulsive set $\mathcal{M}_1$ (or $\mathcal{M}_2$) by $p_i = (x_i, y_i)$ and phase set $\mathcal{N}_1$ (or $\mathcal{N}_2$) by $p_i^+ = (x_i^+, y_i^+)$. If these points lie in the same trajectory, then the following relation must be satisfied:

$$\begin{aligned} a \ln \frac{y_{i+1}}{y_i^+} - b(y_{i+1} - y_i^+) \\ = d \ln x_i^+ - d \ln x_{i+1} + cx_{i+1} - cx_i^+, \end{aligned} \quad (16)$$

and denote $A_l = d(\ln x_i^+ - \ln x_{i+1}) + c(x_{i+1} - x_i^+)$. With the help of Lambert W function, the above equation can be solved for y_{i+1} as

$$y_{i+1} = -\frac{a}{b}W\left[-\frac{b}{a}y_i^+ \left(\exp\left(-\frac{b}{a}y_i^+ + \frac{A_l}{a}\right)\right)\right], \quad (17)$$

from this, we can write

$$\begin{aligned} y_{i+1}^+ &= -\frac{a}{b}W\left[-\frac{b}{a}y_i^+ \left(\exp\left(-\frac{b}{a}y_i^+ + \frac{A_l}{a}\right)\right)\right] + \tau \\ &= p(y_i^+). \end{aligned} \quad (18)$$

Note that if $A_l \leq 0$, then for any $y_i^+ \geq 0$, (17) and (18) are well defined. Actually, if we denote $g(y) = -(b/a)y \exp(-(b/a)y)$, then it can easily be shown that

$$g'(y) = \frac{b^2}{a^2} \exp\left(-\frac{b}{a}y\right) \left(y - \frac{a}{b}\right) \quad (19)$$

and $g(y)$ attains its minimum value $-e^{-1}$ at the point $y = a/b$. Therefore, $-(b/a)y \exp(-(b/a)y) \exp(A_l/a) \in [-e^{-1}, 0)$ for all $A_l \leq 0$ and $y > 0$.

If $A_l > 0$, then we have $-(b/a)y \exp(-(b/a)y) \exp(A_l/a) \geq -e^{-1}$. From this, we obtain the following inequality:

$$\frac{b}{a}y \exp\left(-\frac{b}{a}y\right) \leq \exp\left(-1 - \frac{A_l}{a}\right). \quad (20)$$

The solution gives $y \in (0, y_{\min}] \cup [y_{\max}, p(a_1 + ab_1)/bb_1 + \tau)$, where from Lemma 2 we know that

$$\begin{aligned} y_{\min} &= -\frac{a}{b}W\left(-\frac{b}{a}y_S e^{-(b/a)y_S - A_{P_2}/a}\right), \\ y_{\max} &= -\frac{a}{b}W\left(-1, -\frac{b}{a}y_S e^{-(b/a)y_S - A_{P_1}/a}\right) \end{aligned} \quad (21)$$

and from the properties of the Lambert W function, it is clear that

$$y_{\min} < \frac{a}{b} < y_{\max}. \quad (22)$$

Hence, the Poincaré map on the impulsive points can be determined as

$$y_{i+1}^+ = \begin{cases} p(y_i^+), & \tau \leq y_i^+ < \frac{p(a_1 + ab_1)}{bb_1} + \tau & \text{if } A_{Q_2} \leq 0, \\ p(y_i^+), & y_i^+ \in [\tau, y_{\min}] \cup \left[y_{\max}, \frac{p(a_1 + ab_1)}{bb_1} + \tau \right) & \text{if } A_{P_1}, A_{P_2} \geq 0. \end{cases} \quad (23)$$

Note that $A_{Q_2} \leq 0$ is equivalent to $A_I \leq 0$, and $A_I \geq 0 \iff A_{P_1}, A_{P_2} \geq 0$. Thus, in the following discussions for convenience, we use A_I rather than A_{Q_2} or A_{P_1} or A_{P_2} . Difference equation (23) which defines the Poincaré map reveals the relations between the impulsive points y_i^+ and y_{i+1}^+ , so the existence and stability of fixed point of (23) indicates the existence and stability of order-1 periodic solution of system (1). Therefore, we conclude that the properties of the Poincaré map play the crucial role in analyzing the impulsive semidynamical system.

3. Characterization of Periodic

Solution for $\tau = 0$

To study the existence and stability of the order-1 periodic solution, we generally examine the fixed point of the Poincaré map $p(y_i^+)$. The impulsive point series determine this fixed point, which can be expressed by Lambert W function. In this section, we address the fixed point for the special case, i.e., $\tau = 0$. The analytical formula for the Poincaré map is already obtained in (23), which allows us to utilize this analytical formula to discuss the fixed point. Let y^* be the fixed point; then we have $y^* = p(y^*)$; i.e.,

$$y^* = -\frac{a}{b}W \left[-\frac{b}{a}y^* \left(\exp \left(-\frac{b}{a}y^* + \frac{A_I}{a} \right) \right) \right]. \quad (24)$$

To demonstrate the fixed point, we consider the following two cases for A_I . If $A_I = 0$, then the following relation is satisfied for the fixed point of the Poincaré map:

$$y^* = -\frac{a}{b}W \left[-\frac{b}{a}y^* \left(\exp \left(-\frac{b}{a}y^* \right) \right) \right], \quad (25)$$

and according to the definition of Lambert W function, from (25), we get $-(b/a)y^*e^{-(b/a)y^*} = -(b/a)y^*e^{-(b/a)y^*}$, which shows that any $y^* \geq 0$ is the fixed point of the Poincaré map.

If $A_I \neq 0$, then the following relation is satisfied for the fixed point of the Poincaré map:

$$y^* = -\frac{a}{b}W \left[-\frac{b}{a}y^* \left(\exp \left(-\frac{b}{a}y^* + \frac{A_I}{a} \right) \right) \right], \quad (26)$$

which indicates $-(b/a)y^*e^{-(b/a)y^*} = -(b/a)y^*e^{-(b/a)y^* + A_I/a}$, i.e., $e^{A_I/a} = 1$, by simple calculation. It follows from $A_I \neq 0$ that $y^* \neq 0$ cannot be a fixed point of Poincaré map and only $y^* = 0$ is a fixed point in this case.

In the following, we will show that a semitrivial periodic solution exists for system (1) when $y(t) = 0$ if and only if

$\tau = 0$. If $y(t) = 0$ and $\tau = 0$, then system (1) can be reduced to the following subsystem:

$$\begin{aligned} \frac{dx(t)}{dt} &= ax(t), \quad x < \frac{A}{a_1 + ab_1}, \\ x(t^+) &= (1-p)x(t), \quad x = \frac{A}{a_1 + ab_1}, \end{aligned} \quad (27)$$

solving the first equation with initial condition $x(0^+) = (1-p)(A/(a_1 + ab_1))$, one has

$$x^T(t) = (1-p) \frac{A}{a_1 + ab_1} \exp(at). \quad (28)$$

Further, letting $A/(a_1 + ab_1) = (1-p)(A/(a_1 + ab_1))\exp(aT)$ and solving it for T , we have $T = -(1/a)\ln(1-p)$. Hence, system (1) holds a semitrivial periodic solution with period T as follows:

$$(x^T(t), 0) = \left((1-p) \frac{A}{a_1 + ab_1} \exp(at), 0 \right). \quad (29)$$

We can examine the stability of the semitrivial periodic solution by using the idea of Poincaré map presented in Section 2 and the stability criteria shown in the appendix.

Theorem 3. *If $\tau = 0$ and $A_I = d \ln(1-p) + c(A/(a_1 + ab_1))p < 0$, then the semitrivial periodic solution $(x^T(t), 0)$ of system (1) is globally asymptotically stable.*

Proof. To discuss the stability of the semitrivial periodic solution, we employ Lemma A.3 shown in the appendix. Let us denote

$$\begin{aligned} P(x, y) &= x(a - by), \\ Q(x, y) &= y(cx - d), \\ \alpha(x, y) &= -px, \\ \beta(x, y) &= \tau, \\ \phi(x, y) &= (a_1 + ab_1)x - bb_1xy - A, \end{aligned} \quad (30)$$

$$(x^T(T), y^T(T)) = \left(\frac{A}{a_1 + ab_1}, 0 \right),$$

$$(x^T(T^+), y^T(T^+)) = \left((1-p) \frac{A}{a_1 + ab_1}, 0 \right).$$

Then, by simple calculations, we have

$$\frac{\partial P}{\partial x} = a - by,$$

$$\frac{\partial Q}{\partial y} = cx - d,$$

$$\frac{\partial \alpha}{\partial x} = -p,$$

$$\frac{\partial \phi}{\partial x} = a_1 + ab_1 - bb_1 y,$$

$$\frac{\partial \phi}{\partial y} = -bb_1 x,$$

$$\frac{\partial \alpha}{\partial y} = \frac{\partial \beta}{\partial x} = \frac{\partial \beta}{\partial y} = 0,$$

(31)

and

$$\begin{aligned} \Delta_1 &= \frac{P_+ ((\partial \beta / \partial y) (\partial \phi / \partial x) - (\partial \beta / \partial x) (\partial \phi / \partial y) + \partial \phi / \partial x) + Q_+ ((\partial \alpha / \partial x) (\partial \phi / \partial y) - (\partial \alpha / \partial y) (\partial \phi / \partial x) + \partial \phi / \partial y)}{P (\partial \phi / \partial x) + Q (\partial \phi / \partial y)} \\ &= \frac{P^+ (x^T(T^+), y^T(T^+)) (a_1 + ab_1 - bb_1 y) + Q^+ (x^T(T^+), y^T(T^+)) (pbb_1 x - bb_1 x)}{P (x^T(T), y^T(T)) (a_1 + ab_1 - bb_1 y) - Q (x^T(T), y^T(T)) (bb_1 x)} = (1 - p). \end{aligned} \quad (32)$$

In addition, we also have

$$\begin{aligned} \exp \left(\int_0^T \left[\frac{\partial P}{\partial x} (x^T(t), y^T(t)) \right. \right. \\ \left. \left. + \frac{\partial Q}{\partial y} (x^T(t), y^T(t)) \right] dt \right) &= \exp \left(\int_0^T \left[a \right. \right. \\ &\left. \left. + c(1-p) \frac{A}{a_1 + ab_1} \exp(at) - d \right] dt \right) \\ &= \exp \left((a-d)T + c(1-p) \frac{A}{a_1 + ab_1} \frac{\exp(at)}{a} \Big|_0^T \right) \quad (33) \\ &= \exp \left((a-d)T + c(1-p) \frac{A}{a_1 + ab_1} \right. \\ &\quad \cdot \left. \frac{\exp(aT) - 1}{a} \right) = \exp \left(\ln \frac{1}{(1-p)} + \frac{d}{a} \ln(1-p) \right. \\ &\quad \left. + \frac{c}{a} \frac{A}{a_1 + ab_1} p \right) = \exp \left(\ln \frac{1}{(1-p)} + \frac{A_l}{a} \right). \end{aligned}$$

Therefore, the Floquet multiplier μ_2 can be directly attained as

$$\begin{aligned} \mu_2 &= \Delta_1 \exp \left(\int_0^T \left[\frac{\partial P}{\partial x} (x^T(t), y^T(t)) \right. \right. \\ &\quad \left. \left. + \frac{\partial Q}{\partial y} (x^T(t), y^T(t)) \right] dt \right) = (1-p) \\ &\quad \cdot \exp \left(\ln \frac{1}{1-p} + \frac{A_l}{a} \right) = \exp \left(\frac{A_l}{a} \right). \end{aligned} \quad (34)$$

Furthermore, it follows from the last equation that $|\mu_2| < 1$ provided that $A_l < 0$, which shows that the semitrivial periodic solution $(x^T(t), 0)$ of system (1) is orbitally asymptotically stable.

Method 2. We can also check the stability directly from the Poincaré map. If $\tau = 0$, then from (18), we get

$$p(y_i^+) = -\frac{a}{b} W \left[-\frac{b}{a} y_i^+ \left(\exp \left(-\frac{b}{a} y_i^+ + \frac{A_l}{a} \right) \right) \right]. \quad (35)$$

Taking the derivative of (35), we get

$$\begin{aligned} \frac{dp(y_i^+)}{dy_i^+} \Big|_{y_i^+ = y^*} &= \frac{d}{dy_i^+} \Big|_{y_i^+ = y^*} \left(-\frac{a}{b} \right. \\ &\quad \cdot \left. W \left[-\frac{b}{a} y_i^+ \left(\exp \left(-\frac{b}{a} y_i^+ + \frac{A_l}{a} \right) \right) \right] \right) \\ &= \frac{(-a/b) W \left[-(b/a) y^* (\exp(-(b/a) y^* + A_l/a)) \right]}{1 + W \left[-(b/a) y^* (\exp(-(b/a) y^* + A_l/a)) \right]} \left(\frac{1}{y^*} \right. \\ &\quad \left. - \frac{b}{a} \right) = h(y^*). \end{aligned} \quad (36)$$

The semitrivial periodic solution is stable if and only if $|h(y^*)| < 1$. By taking limit of $h(y^*)$, we get

$$\lim_{y^* \rightarrow 0} h(y^*) = e^{A_l/a}, \quad (37)$$

from this limitation, it is clear that if $A_l < 0$, then $|h(y^*)| < 1$, and hence the semitrivial periodic solution is locally asymptotically stable.

Next, we show that the semitrivial periodic solution $(x^T(t), 0)$ is globally attractive. Let L_2 and L_3 be the vertical lines $A/(a_1 + ab_1)$ and $A(1-p)/(a_1 + ab_1)$, respectively, and let $p_0^+ = (A(1-p)/(a_1 + ab_1), y_i^+) \in L_3$ and $p_1 = (A/(a_1 +$

FIGURE 2: Stability of semitrivial order-1 periodic solution $(x^T(t), 0)$. (a)-(c) Stable semitrivial order-1 limit cycle with $L_2 = A/(a_1 + ab_1) = 1.3$ and $A_l = -0.0266$, $d = 0.5$; (d)-(f) Unstable semitrivial order-1 limit cycle with $L_2 = 1.3$ and $A_l = 0.0049$, $d = 0.2$. All other parameter values are fixed as $a = 1$, $b = 0.1$, $c = 0.2$, $a_1 = 0.5$, $b_1 = 0.5$, $p = 0.1$, and $\tau = 0$.

$ab_1), y_{i+1}) \in L_2$; then for the points p_0^+ and p_1 , the following relation can easily be obtained:

$$\begin{aligned} A_l &= d \ln(1-p) + c \frac{A}{a_1 + ab_1} p \\ &= a \ln \frac{y_{i+1}}{y_i^+} - b(y_{i+1} - y_i^+). \end{aligned} \quad (38)$$

It follows from $A_l \neq 0$ that $y_{i+1} \neq y_i^+$. Here, we define a function $f(y) = a \ln y - by$; then, it can easily be shown that $f'(y) = a/y - b$. This shows that $f(y)$ is monotonically increasing for all those values y of the domain, such that $y < a/b$.

If $A_l > 0$, then from (38), we get $a \ln(y_{i+1}/y_i^+) - b(y_{i+1} - y_i^+) > 0$, which becomes $a \ln(y_{i+1}/y_i) - b(y_{i+1} - y_i) > 0$ due to $\tau = 0$. This shows that $y_{i+1} > y_i$, and hence the impulsive point sequence $\{y_k^+\}_{k=0}^\infty$ is strictly increasing for $A_l > 0$. Similarly, if $A_l < 0$, then $y_{i+1} < y_i$ indicating that impulsive point sequence $\{y_k^+\}_{k=0}^\infty$ is strictly decreasing. From this, we conclude that if $A_l < 0$, then any trajectory y_i^+ originating from $[0, y_{Q_2})$ repeats its pulse action infinite number of times and satisfies $\lim_{k \rightarrow \infty} y_k = y^*$. Hence, all the above results affirm that the semitrivial periodic solution $(x^T(t), 0)$ is globally asymptotically stable. This completes the proof. $\square$

In order to evaluate the main results shown in Theorem 3, we fix all parameter values as those shown in Figure 2 and reveal the effects of A_l on the stability of the semitrivial periodic solution. It follows from Figure 2 that if $A_l < 0$, then the semitrivial periodic solution is orbitally asymptotically stable, while if $A_l \geq 0$, then it becomes unstable, and

the solution is eventually determined by the original Lotka-Volterra system without pulse effects after a finite impulsive perturbations, as shown in Figures 2(d)–2(f).

4. Characterization of Periodic Solution for $\tau > 0$

To investigate and check the periodicity of Poincaré map, we first prove an important lemma, which will be used in upcoming results.

Lemma 4. *If $A_l \geq 0$, then the following inequality is satisfied for the Poincaré map:*

$$p(y_i^+) > y_i^+, \quad \text{for all } y_i^+ \in (0, y_{P_2}). \quad (39)$$

Proof. Let a trajectory pass through the point $p_i^+ = (x_i^+, y_i^+)$ and touch the curve Γ_{Im} at point $p_{i+1} = (x_{i+1}, y_{i+1})$ below the line L_1 . Here, we also assume that y_i^+ is less than a/b . Then, for the points p_i^+ and p_{i+1} , the following relation can easily be obtained:

$$\begin{aligned} a \ln y_{i+1} - by_{i+1} + d \ln x_{i+1} - cx_{i+1} \\ = a \ln y_i^+ - by_i^+ + d \ln x_i^+ - cx_i^+, \end{aligned} \quad (40)$$

from (40), we get

$$-\frac{b}{a} y_{i+1} e^{(-(b/a)y_{i+1})} = -\frac{b}{a} y_i^+ e^{(-(b/a)y_i^+ + A_l/a)}. \quad (41)$$

If $A_l \geq 0$, then we get the following inequality:

$$-\frac{b}{a} y_{i+1} e^{(-(b/a)y_{i+1})} \leq -\frac{b}{a} y_i^+ e^{(-(b/a)y_i^+)}. \quad (42)$$

Let $f(y) = -y \exp(-y)$; then from calculation, it is obvious that $f'(y) < 0$ if $y \in (0, 1)$ and $f'(y) > 0$ if $y > 1$. It follows from $(b/a)y_{i+1}$ and $(b/a)y_i^+ \in (0, 1)$ that we have $y_{i+1} \geq y_i^+$. Also from $y_{i+1}^+ = y_{i+1} + \tau$ and $p(y_i^+) = y_{i+1}^+$, we get $p(y_i^+) > y_i^+$ for all $y_i^+ \in (0, y_{P_2})$. The proof is completed. $\square$

4.1. Existence and Stability of Order-1 Periodic Solution. This section focuses on the investigation of the existence and stability of the order-1 periodic solutions of system (1), i.e., interior periodic solutions. In order to achieve this, we first check the behaviour of Poincaré map in different intervals.

Theorem 5. *The monotonicities of Poincaré map $p(y_i^+)$ for Cases (i) and (ii), respectively, are as follows:*

(i) *It is increasing on $[0, y_T]$ and decreasing on $[y_T, p(a_1 + ab_1)/bb_1 + \tau]$.*

(ii) *It is increasing on $[0, y_{P_2}]$ and decreasing on $[y_{P_2}, p(a_1 + ab_1)/bb_1 + \tau]$.*

Proof.

Case (i). From the vector field of system (1) without impulsive effect, we know that the domain of Poincaré map for Case (i) is $[0, p(a_1 + ab_1)/bb_1 + \tau]$. Let $y_{k_1}^+, y_{k_2}^+ \in [0, y_T]$, such that $y_{k_1}^+ < y_{k_2}^+$. Then, from this, it is easy to get $y_{k_1+1} < y_{k_2+1}$, and from the uniqueness of the trajectories, we get $p(y_{k_1}^+) < p(y_{k_2}^+)$. Similarly, if $y_{k_1}^+, y_{k_2}^+ \in [y_T, p(a_1 + ab_1)/bb_1 + \tau]$, such that $y_{k_1}^+ < y_{k_2}^+$. The trajectories initiating from the points $y_{k_1}^+, y_{k_2}^+$ will cross the curve Γ_{ph} and will touch the curve Γ_{Im} . Let us denote the vertical coordinate of the trajectories starting from $y_{k_1}^+$ and $y_{k_2}^+$ intersecting the curve Γ_{ph} by $y_{l_1}^+$ and $y_{l_2}^+$, respectively. It is interesting to note that the order of new positions is reversed (i.e., $y_{l_1}^+ > y_{l_2}^+$), and from the uniqueness of trajectories consequently, we get $p(y_{k_1}^+) = p(y_{l_1}^+) > p(y_{l_2}^+) = p(y_{k_2}^+)$. Hence, according to the definition of $p(y_i^+)$, it is increasing on $[0, y_T]$ and decreasing on $[y_T, p(a_1 + ab_1)/bb_1 + \tau]$.

Equation (A.10) also shows that the function Lambert $W(z)$ is strictly increasing for all $z \in [-e^{-1}, 0]$. From (19), it follows that the function $g(z) = -(b/a)z \exp(-(a/b)z)$ is strictly increasing on $(0, a/b]$ and strictly decreasing on $[a/b, p(a_1 + ab_1)/bb_1 + \tau]$.

Case (ii). In this case, the impulsive function is increasing on $[0, y_S]$. Hence, according to the definition of $p(y_i^+)$, it is increasing on $[0, y_{P_2}]$ and decreasing on $[y_{P_2}, p(a_1 + ab_1)/bb_1 + \tau]$. $\square$

Theorem 6. *For Case (i), the fixed point of the Poincaré map $p(y_i^+)$ exists and hence a periodic solution of order-1 exists for system (1).*

Proof. Let us assume that the curve Γ_1 starts from the point $T = (A(1 - p)/(a_1 + \tau bb_1), a/b)$. It then reaches the curve Γ_{Im} at the point (x_{Q_2}, y_{Q_2}) , and then due to the one-time pulse action at the point Q_2 , it is adjusted to $y_{Q_2}^+$ at the curve Γ_{ph} . Now we determine the location of the point $y_{Q_2}^+$. If

$p(y_T) = y_{Q_2}^+ = y_T = a/b$, then obviously the curve $\widehat{TQ_2}$ forms a periodic solution of order-1 for system (1). If $y_{Q_2}^+ \neq y_T$, then we have two possible cases either (c_1) $y_{Q_2}^+ > y_T$ or (c_2) $y_{Q_2}^+ < y_T$. For case (c_1) , the trajectory initiating from the point $y_{Q_2}^+$ intersects with curve Γ_{Im} at the point (x_{Q_3}, y_{Q_3}) , where $y_{Q_3} < y_{Q_2}$. After that, y_{Q_3} maps to the point $y_{Q_3}^+$. Then, for the Poincaré map $p(y_{Q_2}^+)$, the following inequality holds:

$$p(y_{Q_2}^+) < y_{Q_2}^+. \quad (43)$$

For case (c_2) , the following inequality is obtained:

$$p(y_T) < y_T. \quad (44)$$

For the lowest impulsive point τ , the following inequality is satisfied:

$$p(\tau) > \tau. \quad (45)$$

Therefore, for case (c_1) , from (43) and (45), it is shown that the Poincaré map has at least one fixed point in the interval $(\tau, y_{Q_2}^+)$ due to continuity of the Poincaré map. For case (c_2) , from (44) and (45), it is shown that Poincaré map has a fixed point in the interval (τ, y_T) ; hence, periodic solution of order-1 exists for system (1). This completes the proof. $\square$

Remark. The fixed point of the Poincaré map can also be found directly from (23). Let y^* be the fixed point; then, we have

$$y^* = -\frac{a}{b}W\left[-\frac{b}{a}y^*\left(\exp\left(-\frac{b}{a}y^* + \frac{A_l}{a}\right)\right)\right] + \tau, \quad (46)$$

i.e.,

$$-\frac{b}{a}(y^* - \tau) = W\left[-\frac{b}{a}y^*\left(\exp\left(-\frac{b}{a}y^* + \frac{A_l}{a}\right)\right)\right]. \quad (47)$$

By applying the definition of Lambert W function, we get

$$\begin{aligned} & -\frac{b}{a}(y^* - \tau) \exp\left(-\frac{b}{a}(y^* - \tau)\right) \\ & = -\frac{b}{a}y^* \exp\left(-\frac{b}{a}y^* + \frac{A_l}{a}\right), \end{aligned} \quad (48)$$

after simple calculation, from above, we can easily obtain the following equation:

$$(y^* - \tau) = y^* \exp\left(\frac{A_l}{a} - \frac{b}{a}\tau\right). \quad (49)$$

This indicates that there is a unique fixed point

$$y^* = \frac{\tau}{1 - \exp(A_l/a - (b/a)\tau)} \quad (50)$$

provided that

$$\begin{aligned} & \exp\left(\frac{A_l}{a} - \frac{b}{a}\tau\right) < 1 \\ & \text{or } A_l < b\tau. \end{aligned} \quad (51)$$

Theorem 7. Assume that $p(y_{P_1}) > y_{P_1}$; then for Case (ii), the fixed point of the Poincaré map $p(y_i^+)$ exists and hence periodic solution of order-1 exists for system (1).

Proof. In Section 2, it is shown that for Case (ii), there is a curve Γ_2 which meets with the curve Γ_{ph} at two points (x_{P_1}, y_{P_1}) and (x_{P_2}, y_{P_2}) and is tangent to the curve Γ_{Im} at point $S = (x_S, y_S)$. Let $p(y_{P_1}) = y_{S^+} = y_{P_1}$; then, clearly the curve $\overline{P_1 S}$ forms a periodic solution of order-1 for system (1).

If $y_{S^+} > y_{P_1}$, then $p(y_{P_1}) = y_{S^+}$; that is, the point y_{S^+} lies above the point y_{P_1} and can be written as

$$p(y_{P_1}) > y_{P_1}. \quad (52)$$

Furthermore, the trajectory starting from the point S^+ intersects with the curve Γ_{Im} at a point S_1 which is less than the point S , i.e., $y_{S_1} < y_S$. Then, it is clear that its impulsive effect (denoted by S_1^+) will be less than S^+ , i.e., $y_{S_1^+} < y_{S^+}$. All these outcomes affirm that for Case (ii), the Poincaré map $p(y_{S^+})$ satisfies the following relationship:

$$p(y_{S^+}) < y_{S^+}. \quad (53)$$

For Case (ii), it is clear from (52) and (53) that the fixed point exists for the Poincaré map in the interval (y_{P_1}, y_{S^+}) and hence periodic solution of order-1 exists for system (1). This completes the proof. $\square$

If $p(y_{P_1}) < y_{P_1}$, then in this case when the solution of system (1) touches the curve Γ_{Im} , it will move to the interval $[\tau, y_{S^+}]$ after a single impulsive effect. If $\tau \leq y_{P_2}$, then from inequality (39) in Lemma 4, any trajectory initiating from y^+ with $y^+ \in [\tau, y_{P_2}]$ will reach curve Γ_{Im} and after a finite number of pulse actions, it will finally enter into the closed trajectory Γ_2 , and there will be no more pulse actions on that trajectory. If $\tau > y_{P_2}$, then any solution of system (1) will move to the interior of closed trajectory after a single impulsive effect. From these results, it can be concluded that in this case, there does not exist any fixed point for the Poincaré map.

Based on the above results, we now examine the stability of order-1 periodic solution, and we have the following main results.

Theorem 8. Assume that for Case (i), the fixed point y^* of the Poincaré map $p(y_i^+)$ exists. Further,

(a) if $p(y_T) < y_T$, then it is globally stable;

(b) if $p(y_T) > y_T$, then it is globally stable provided that $p^2(y^+) > y^+$ for $y^+ \in [y_T, y^*)$.

Proof. From Theorem 6, we know that for Case (i) the fixed point of the Poincaré map exists. Firstly, we prove the uniqueness of the fixed point. From (18), we know that

$$y_{i+1}^+ = -\frac{a}{b}W\left[-\frac{b}{a}y_i^+\left(\exp\left(-\frac{b}{a}y_i^+ + \frac{A_1}{a}\right)\right)\right] + \tau. \quad (54)$$

We also know that for this case $A_1 < 0$. Therefore, if $\tau < y_1^+ < y_0^+ < a/b$ holds, then from the monotonicities of Lambert W

function and $g(y)$, the following relationship is satisfied for the impulsive point sequence $\{y_k^+\}_{k=0}^\infty$:

$$\tau < y_{k+1}^+ < y_k^+ < y_{k-1}^+ < \cdots < y_2^+ < y_1^+ < y_0^+ < \frac{a}{b}. \quad (55)$$

From the domain of the Poincaré map, it is clear that if $A_1 < 0$, the relationship (55) must be satisfied by all the impulsive points. From which, we conclude that the impulsive point sequence is monotonically decreasing and there must exist a unique constant $y^* \in [\tau, y_0^+]$, such that $\lim_{k \rightarrow \infty} y_k^+ = y^*$. The above discussion shows that the fixed point is unique. The global stability can be examined as follows:

(a) If $p(y_T) < y_T$, then we will show that the fixed point is globally stable. For this, let $y_i^+ \in [0, y^*)$; then $y_i^+ < p(y_i^+) < y^*$. This shows that $p^k(y_i^+)$ for $k \geq 1$ is monotonically increasing and $\lim_{k \rightarrow \infty} p^k(y_i^+) = y^*$. Again, let $y_i^+ \in (y^*, p(a_1 + ab_1)/bb_1 + \tau)$; then there exists two possibilities: (1) for all k , we have $p^k(y_i^+) > y^*$. We know that for this interval $p(y_i^+) < y_i^+$, so the series $p^k(y_i^+)$ is monotonically decreasing and as a result we can write $\lim_{k \rightarrow \infty} p^k(y_i^+) = y^*$; (2) let $p^k(y_i^+) > y^*$ be not true for all k ; then let m_1 be the smallest positive integer such that $p^{m_1}(y_i^+) < y^*$. Therefore, by using the same method for $y_i^+ \in [0, y^*)$, if k increases, then $p^{m_1+k}(y_i^+)$ is also monotonically increases and $\lim_{k \rightarrow \infty} p^{m_1+k}(y_i^+) = y^*$. This proves that the result shown in Case (a) is true.

(b) If $p(y_T) > y_T$, the following three intervals can be taken:

$$\begin{aligned} (1) & y_i^+ \in [y_T, y^*), \\ (2) & y_i^+ \in [0, y_T), \\ (3) & y_i^+ \in \left(y^*, \frac{p(a_1 + ab_1)}{bb_1} + \tau\right). \end{aligned} \quad (56)$$

For case (1), we know that $p(y_i^+)$ is monotonically decreasing in this interval; thus, for all $y_i^+ \in [y_T, y^*)$, we have $p(y_T) \geq p(y_i^+) > y^*$. Also using the second condition $p^2(y_i^+) > y_i^+$, we get $y_i^+ < p^2(y_i^+) < y^*$ for all $y_i^+ \in [y_T, y^*)$. By induction, it is concluded that $p^{2(k-1)}(y_i^+) < p^{2k}(y_i^+) < y^*$ for all $k \geq 1$, which indicates that $p^{2k}(y_i^+)$ is monotonically increasing with $\lim_{k \rightarrow \infty} p^{2k}(y_i^+) = y^*$ and the monotonicity of $p^{2(k-1)}(y_i^+)$ follows as well.

For case (2), as $p(y_i^+)$ is monotonically increasing in this interval, so there must exist $m_2 \geq 1$ such that either $p^{m_2}(y_i^+) \in [y_T, y^*)$ or $p^{m_2}(y_i^+) > y^*$. If $p^{m_2}(y_i^+) \in [y_T, y^*)$, then following case (1), we obtain $\lim_{k \rightarrow \infty} p^{m_2+2k}(y_i^+) = y^*$. For the latter case $p^{m_2}(y_i^+) > y^*$, there must exist a $\tilde{y} \in [y_T, y^*)$ such that $p^{m_2}(y_i^+) = p(\tilde{y})$; then $\lim_{k \rightarrow \infty} p^{m_2+2k}(y_i^+) = y^*$ monotonically.

For case (3), if $y_i^+ \in (y^*, p(a_1 + ab_1)/bb_1 + \tau)$, then there exists a positive integer m_3 such that $p^{m_3}(y_i^+) \in [0, y_T)$ or $[y_T, y^*)$, and in the light of cases (1) and (2), it is seen that the result in case (b) is true. Hence, the proof is completed. $\square$

Theorem 9. Assume that $p(y_{P_1}) > y_{P_1}$; then for Case (ii), the unique fixed point y^* of the Poincaré map is globally stable if $\tau \geq y_{P_1}$ and $p^2(y_i^+) > y_i^+$ for all $y_i^+ \in [y_{P_1}, y^*)$.

Proof. From Theorem 7, we know that for Case (ii), the fixed point of the Poincaré map exists. It is obvious from (39) in Lemma 4 that $p(y_i^+) > y_i^+$ for all $y_i^+ \in (0, y_{P_2})$ and also we know that $p(0) = \tau > 0$ is true. So from these we can say that the fixed point does not exist in the interval $[0, y_{P_2}]$. From Theorem 5, we also know that $p(y_i^+)$ is decreasing on $[y_{P_1}, p(a_1 + ab_1)/bb_1 + \tau)$, so the unique fixed point for $p(y_i^+)$ exists in $[y_{P_1}, p(a_1 + ab_1)/bb_1 + \tau)$.

If $y_i^+ \in [y_{P_1}, y^*)$, then from the monotonicity of Poincaré map in this interval, we get $p(y_{P_1}) \geq p(y_i^+) > y^*$. Moreover, from the condition $p^2(y_i^+) > y_i^+$, we get $y_i^+ < p^2(y_i^+) < y^*$ for all $y_i^+ \in [y_{P_1}, y^*)$. By induction, it can be concluded that $p^{2(k-1)}(y_i^+) < p^{2k}(y_i^+) < y^*$ for all $k \geq 1$, which shows that $p^{2k}(y_i^+)$ is monotonically increasing with $\lim_{k \rightarrow \infty} p^{2k}(y_i^+) = y^*$.

If $\tau \geq y_{P_1}$, then from the monotonicity of Poincaré map, it follows that, for any $y_i^+ \in [0, y_{P_2}] \cup [y^*, p(a_1 + ab_1)/bb_1 + \tau)$, there must exist $j \geq 1$, such that $P^j(y_i^+) \in [y_{P_1}, y^*)$. Then, for any $y_i^+ \in [0, y_{P_2}] \cup [y^*, p(a_1 + ab_1)/bb_1 + \tau)$, $\lim_{k \rightarrow \infty} p^{j+2k}(y_i^+) = y^*$ monotonically. Hence, the results presented in this theorem are true. $\square$

4.2. Nonexistence of Order- k ($k \geq 3$) Periodic Solution. This subsection discusses the conditions under which order-3 periodic solutions do not exist for the system in Cases (i) and (ii).

Theorem 10. Assume that $y_{Q_2^+} > y_T$ and $y_{Q_3^+} \geq y_T$; then in Case (i), periodic solution of order- k ($k \geq 3$) does not exist for system (1).

Proof. Let $y_{Q_2^+} > y_T$ and $y_{Q_3^+} = y_T$; then clearly the curves $\overline{TQ_2}$ and $\overline{Q_2^+Q_3^+}$ formulate an order-2 periodic solution for system (1). If $y_{Q_3^+}$ is strictly greater than y_T , then in this case all the impulsive points will initiate from the interval $[y_{Q_3^+}, y_{Q_2^+}]$ and they must lie above the point Q_3^+ . From the disjointness of trajectories, we get $y_{Q_2^+} > y_{Q_4^+} > y_{Q_5^+} > y_{Q_3^+} > y_T$, where $y_{Q_5^+}$ is the successor point of y_T . The solution initiating from the point $(x_{Q_2^+}, y_{Q_2^+})$ touches the curve Γ_{Im} at the point (x_{Q_3}, y_{Q_3}) . When a single pulse action is applied, the point $(x_{Q_3^+}, y_{Q_3^+})$ is obtained. Similarly, the solution with initial point $(x_{Q_4^+}, y_{Q_4^+})$ gives the vertical coordinate $y_{Q_5^+}$ after single pulse action.

Assuming that a solution is initiating from the point $(x_{Q_5^+}, y_{Q_5^+})$ and letting k times pulse action be applied on it, then the following relation among the impulsive points is obtained:

$$\begin{aligned} P(y_T) = y_{Q_2^+} &> y_{Q_4^+} > y_{Q_6^+} > \cdots > y_{Q_{2n}^+} > y_{Q_{2n+2}^+} > \cdots \\ &> y_{Q_{2n+1}^+} > y_{Q_{2n-1}^+} > \cdots > y_{Q_5^+} > y_{Q_3^+} > y_T. \end{aligned} \quad (57)$$

From (57), it can be concluded that the Poincaré map either exists a unique fixed point y^* in the interval $[Q_3^+, Q_2^+]$, such that

$$\lim_{n \rightarrow \infty} y_{Q_{2n}^+} = \lim_{n \rightarrow \infty} y_{Q_{2n+1}^+} = y^*, \quad (58)$$

or there exists two distinct fixed points y_1^* and y_2^* , such that

$$\begin{aligned} \lim_{n \rightarrow \infty} y_{Q_{2n}^+} &= y_1^*, \\ \lim_{n \rightarrow \infty} y_{Q_{2n+1}^+} &= y_2^*. \end{aligned} \quad (59)$$

Hence, if $y_{Q_2^+} > y_T$ and $y_{Q_3^+} > y_T$, then for Case (i), there does not exist any periodic solution of order- k ($k \geq 3$) for system (1). $\square$

Theorem 11. Assume that $y_{S^+} > y_{P_1}$ and $y_{S_1^+} \geq y_{P_1}$, then in Case (ii), periodic solution of order- k ($k \geq 3$) does not exist for system (1).

Proof. Let $y_{S^+} > y_{P_1}$ and $y_{S_1^+} = y_{P_1}$; then clearly the curves $\widehat{P_1S}$ and $\widehat{S^+S_1^+}$ formulate an order-2 periodic solution for system (1). If $y_{S_1^+}$ is strictly greater than y_{P_1} , then in this case all the impulsive points initiate from the interval $[y_{S_1^+}, y_{S^+}]$ and they must lie above the point $y_{S_1^+}$. From the disjointness of any trajectories, it is easy to get $y_{S^+} > y_{S_2^+} > y_{S_3^+} > y_{S_4^+} > y_{S_1^+} > y_{P_1}$, where $y_{S_1^+}, y_{S_2^+}, y_{S_3^+}, y_{S_4^+}, y_{S^+}$ are, respectively, the successor points of $y_{S^+}, y_{S_1^+}, y_{S_2^+}, y_{S_3^+}$ and y_{P_1} . Assuming that a solution is initiating from the point (x_{S^+}, y_{S^+}) and letting k times pulse action be applied, then the following relation among the impulsive points is obtained:

$$\begin{aligned} P(y_{P_1}) &= y_{S^+} > y_{S_2^+} > y_{S_4^+} > y_{S_6^+} > \cdots > y_{S_{2n}^+} > y_{S_{2n+2}^+} \\ &> \cdots > y_{S_{2n+1}^+} > y_{S_{2n-1}^+} > \cdots > y_{S_5^+} > y_{S_3^+} > y_{S_1^+} \\ &> y_{P_1}. \end{aligned} \quad (60)$$

From (60), we can follow the same way as in Theorem 10 and can prove that there will exist either unique fixed point or two distinct fixed points for the Poincaré map in the interval $[S_1^+, S^+]$. This completes the proof. $\square$

5. Conclusion

The Lotka-Volterra model is a well-known model of predator-prey interaction to be established on sound mathematical principles. It provides the basis of numerous models used today in the analysis of population dynamics. It is discussed in detail that the most important factor in IPM process is the ET or action threshold. In this paper, regarding IPM system, we have taken and analyzed an impulsive dynamical model with action threshold depending on the pest density and its changing rate (ratio-dependent action threshold), which indicates that the action threshold will depend on the both pest and natural enemy densities. The threshold contains two weighted quantities a_1 and b_1 ; i.e., if the quantity b_1 vanishes, then it will depend only on the pest density and in this

FIGURE 3: Complex shapes of the Poincaré map $p(y_i^+)$ and fixed point y^* : (a) order-1 periodic solution, (b) $a_1 = 0.8, b_1 = 0.2, y^* = 1.6040$, (c) $a_1 = 0.6, b_1 = 0.4, y^* = 1.6140$, and (d) $a_1 = 0.4, b_1 = 0.6, y^* = 1.6285$. All other parametric values are fixed as $a = 1, b = 1, c = 0.3, d = 0.3, p = 0.7, \tau = 1.2$, and $A = 0.8$.

case, the action threshold will be transformed into an *ET*, which have been widely modeled and investigated in previous literatures [22–31]. Thus, we conclude that the simplest pest natural enemy model with nonlinear action threshold was proposed and investigated in the present paper, and the main purpose was to provide a complete qualitative study of system (1) and to illustrate how ratio-dependent action threshold affects the dynamics of the system.

Compared with the previous works [22–31], we conclude that the innovation of the model is embodied in the adoption of action threshold determined by the pest density and its changing rate. This will result in complex curves for impulsive and phase sets; i.e., nonlinear impulsive and phase sets have been obtained in this paper, and consequently now analytical techniques for the existence of order-1 periodic solution and its stability have been developed in the present paper.

With the help of Lambert W function, the Poincaré map is formed for the exact phase set. Consequently, the conditions for the existence and stability of the semitrivial periodic solution are provided. The numerical simulation for a semitrivial periodic solution also strengthens our result shown in theory. Using the definition of Poincaré map, it is inspected that under what condition periodic solution of order-1 exists for the system and also the stability condition is studied. The nonexistence of order- k ($k \geq 3$) periodic solutions was investigated, and the condition is given which ensures the nonexistence of such order periodic solutions. Specific conditions which confirm the second-order periodicity of the system are also studied.

The definition domain of the Poincaré map is very complex for system (1), as shown in Figure 3. Figure 3

demonstrates how the shapes of Poincaré map vary with the slight changes in the weighted parameters a_1 and b_1 . The fixed point of the Poincaré map, i.e., periodic solution of order-1, is also affected by the weighted parameters. If the weighted parameter a_1 decreases, the fixed point of the Poincaré map still exists but increases monotonically, and we get its minimum value whenever $b_1 = 0$; i.e., the threshold level only depends on the pest density, as shown in Figure 3.

Our outcomes have provided some fundamental theoretical conclusions that could be of applied interest. The main results of this paper exhibit that the pests can be controlled completely by applying control action for a finite number of times such that the ratio-dependent action threshold is not surpassed, as shown in Figure 4. From which we can see that the impulsive and phase sets are two straight lines once the weighted parameters $a_1 = 1, b_1 = 0$; i.e., the action threshold only depends on the density of pest population (Figure 4(a)). There exists a unique and stable order-1 periodic solution for the parameter values fixed in Figure 4(a). Similarly, if we fixed the weighted parameters as those shown in Figures 4(b) and 4(c), although system (1) still holds a unique and stable order-1 periodic solution, the impulsive and phase sets become two curves and their shapes depend on the magnitudes of a_1 and b_1 . It is interesting to note that if the farmers are more dependent on the growth rate of pest to apply integrated control strategies, then a finite number of IPM measures could successfully realize the control purpose such that the density of pest population is less than the given action threshold, as shown in Figure 4(d).

Figure 4 also reveals another important fact that the impulsive and phase sets not only significantly change with

FIGURE 4: The existence and stability of order-1 periodic solution of system (1) with $a = 1, b = 1, c = 0.3, d = 0.3, A = 1, p = 0.5, \tau = 0.8$ and different weighted parameters a_1 and b_1 .

the change of the weight parameters a_1 and b_1 , but also depend on the interaction between the pest and its natural enemy. As mentioned before, we can fully consider the more general pest and natural enemy systems, and there is no doubt that this will result in a complex impulsive or phase set, which is crucial for determining the Poincaré map and analyzing the dynamics.

Appendix

This appendix contains certain basic definitions and lemmas, which are useful for the other sections of the paper. We first give the definition of the planar impulsive semidynamical systems with state-dependent feedback control.

Definition A.1. Let P, Q, α, β be continuous functions from R^2 into R and $\mathcal{M} \subset R^2$ shows the impulsive set; then

$$\left\{ \begin{array}{l} \frac{dx(t)}{dt} = P(x, y), \\ \frac{dy(t)}{dt} = Q(x, y), \\ x^+ = x + a(x, y), \\ y^+ = y + b(x, y), \end{array} \right\} \quad \begin{array}{l} (x, y) \notin \mathcal{M}, \\ (x, y) \in \mathcal{M} \end{array} \quad (\text{A.1})$$

is called generalized planar impulsive semidynamical systems, where $(x, y) \in R^2$, and for simplicity, we denote $x^+ = x(t^+)$ and $y^+ = y(t^+)$.

For each point $z = (x, y) \in \mathcal{M}$, the map $I : R^2 \rightarrow R^2$ is defined by

$$I(z) = z^+ = (x + a(x, y), y + b(x, y)) = (x^+, y^+) \in R^2, \quad (\text{A.2})$$

where z^+ is the impulsive point of z .

Let $\mathcal{N} = I(\mathcal{M})$ be the phase set such that $\mathcal{N} \cap \mathcal{M} = \emptyset$. System (1) is an impulsive semidynamical system, where maximum impulsive set

$$\mathcal{M} = \left\{ (x, y) \mid \frac{A}{(a_1 + ab_1)} \leq x \leq x_s, 0 \leq y \leq y_s \right\}, \quad (\text{A.3})$$

is a closed subset of R^2 and continuous function

$$I : (x, y) \in \mathcal{M} \rightarrow (x^+, y^+) = ((1 - p)x, y + \tau) \in R^2. \quad (\text{A.4})$$

The phase set

$$\begin{aligned} \mathcal{N} &= I(\mathcal{M}) \\ &= \{(x^+, y^+) \in R^2 \mid x^+ = (1 - p)x, y^+ = y + \tau\}. \end{aligned} \quad (\text{A.5})$$

If there is no specific restriction on the initial point, it is always assumed that the initial point $(x_0^+, y_0^+) \in \mathcal{N}$. Note that the impulsive set $\mathcal{M}$ represents the control set and the phase set $\mathcal{N}$ denotes the image of the control set after applying the impulsive perturbations.

In the following lines, we provide some definitions regarding impulsive semidynamical systems [32–39]. Let

(X, Π, R) (or (X, Π)) be a semidynamical system, where X is a metric space; R_+ exhibits the set of nonnegative real numbers. If $z \in X$, then the function $\Pi_z : R \rightarrow X$ defined by $\Pi_z(t) = \Pi(z, t)$ is clearly continuous such that $\Pi(z, 0) = z$, and $\Pi(\Pi(z, t), s) = \Pi(z, t + s)$ for all $z \in X$ and $t, s \in R_+$. The set $C^+(z) = \{\Pi(z, t) \mid t \in R\}$ is known as the positive orbit of z . For any subset $\mathcal{M}$ of X , let $\mathcal{M}^+(z) = C^+(z) \cap \mathcal{M} - \{z\}$ and $\mathcal{M}^-(z) = G(z) \cap \mathcal{M} - \{z\}$, where $G(z) = \cup\{G(z, t) \mid t \in R\}$ and $G(z, t) = \{w \in X \mid \Pi(w, t) = z\}$ is the attainable set of z at $t \in R_+$. At last, we set $\mathcal{M}(z) = \mathcal{M}^+(z) \cup \mathcal{M}^-(z)$.

Definition A.2. A trajectory Π_z in $(X, \Pi, \mathcal{M}, I)$ is said to be periodic of period T_k and order k if there exist nonnegative integers $m \geq 0$ and $k \geq 1$ such that k is the smallest integer for which $z_m^+ = z_{m+k}^+$ and $T_k = \sum_{i=m}^{m+k-1} \Phi(z_i) = \sum_{i=m}^{m+k-1} s_i$.

Here, we recall a lemma from [38, 39], so that the paper will be self-contained.

Lemma A.3. The T -periodic solution $(x, y) = (\xi(t), \eta(t))$ of system

$$\frac{dx}{dt} = P(x, y),$$

$$\frac{dy}{dt} = Q(x, y),$$

$$\text{if } \phi(x, y) \neq 0,$$

$$x^+ = x + a(x, y),$$

$$y^+ = y + b(x, y),$$

$$\text{if } \phi(x, y) = 0$$

(A.6)

is orbitally asymptotically stable and enjoys the property of asymptotic phase if the Floquet multiplier μ_2 satisfies the condition $|\mu_2| < 1$, where

$$\mu_2$$

$$= \prod_{k=1}^q \Delta_k \exp \left(\int_0^T \left[\frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t)) \right] dt \right) \quad (\text{A.7})$$

with

$$\Delta_k = \frac{P_+((\partial b / \partial y)(\partial \phi / \partial x) - (\partial b / \partial x)(\partial \phi / \partial y) + \partial \phi / \partial x) + Q_+((\partial a / \partial x)(\partial \phi / \partial y) - (\partial a / \partial y)(\partial \phi / \partial x) + \partial \phi / \partial y)}{P(\partial \phi / \partial x) + Q(\partial \phi / \partial y)}, \quad (\text{A.8})$$

and ϕ is continuously differentiable with respect to x, y . $(x, y) \notin \mathcal{M}$ can also be denoted by $\phi(x, y) \neq 0$. $P, Q, \partial a / \partial x, \partial a / \partial y, \partial b / \partial x, \partial b / \partial y, \partial \phi / \partial x$ and $\partial \phi / \partial y$ are calculated at the point $(\xi(t_k), \eta(t_k))$, $P_+ = P(\xi(t_k^+), \eta(t_k^+))$ and $Q_+ = Q(\xi(t_k^+), \eta(t_k^+))$, and t_k ($k, q \in N$, N is nonnegative integers) is the time of the k -th jump.

We offer another most significant definition of the Lambert W function. The Lambert W function [40], also known as the omega function, is the inverse function of $\text{LambertW}(z)\exp(\text{LambertW}(z))$. The Lambert W function has a lot of applications in pure and applied mathematics. Here, we give the proper definition of the Lambert W function. We have used the properties of Lambert W function to achieve our goal.

Definition A.4. The Lambert W function is a multivalued inverse function of $z \mapsto ze^z$, satisfying the following equations:

$$\text{LambertW}(z) \exp(\text{LambertW}(z)) = z. \quad (\text{A.9})$$

From (A.9), it follows that

$$\text{LambertW}'(z) = \frac{\text{LambertW}(z)}{z(1 + \text{LambertW}(z))}. \quad (\text{A.10})$$

The function $z \exp(z)$ has the derivative $(z + 1)\exp(z)$ which is positive if $z > -1$. Let $\text{LambertW}(0, z) = \text{LambertW}(z)$ be the inverse function of $z \exp(z)$ which

is restricted on the interval $[-1, \infty)$. In the same way, let $\text{LambertW}(-1, z)$ be the inverse function of $z \exp(z)$ restricted on the interval $(-\infty, -1]$. For this study, both $\text{LambertW}(0, z)$ and $\text{LambertW}(-1, z)$ will be employed only for $z \in [-\exp(-1), 0)$ due to our practical problem. For simplicity, we denote the Lambert W function as W function in the main text.

Data Availability

There were no data used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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