Dynamical Analysis of the SEIB Model for Brucellosis Transmission to the Dairy Cows with Immunological Threshold

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As we all know, bacteria is different from virus which with certain types can be killed by the immune cells in the body. The brucellosis, a bacterial disease, can invade the body by indirect transmission from environment, which has not been researched by combining with immune cells. Considering the effects of immune cells, we put a minimum infection dose of brucellosis invading into the dairy cows as an immunological threshold and get a switch model. In this paper, we accomplish a thorough dynamics analysis of a SEIB switch model. On the one hand, we can get a disease-free and bacteria-free steady state and up to three endemic steady states which may be thoroughly analyzed in different cases of a minimum infection dose in a switch model. On the other hand, we calculate the basic reproduction number $R_0$ and know that the disease-free and bacteria-free steady state is a global stability when $R_0 < 1$, and the one of the endemic steady state is a conditionally global stability when $R_0 > 1$. We find that different amounts of $R_0$ may lead to different steady states of brucellosis, and considering the effects of immunology is more serious in mathematics and biology.

1. Introduction

Brucellosis is the zoonotic sex contagion which is named as Mediterranean relax heat, wave heat, or wave form heat [1–3]. It is characterized by extensive host, strong infectivity, and difficulty in radical treatment after infection [4, 5]. Besides, it has a serious harm to economy, society, and public health. Therefore, brucellosis is listed as one of the communicable diseases that must be notified in the World Organization for Animal Health and is classified as second kind of animal diseases in China. Meanwhile, it is as the first zoonotic disease to be controlled in the National Medium and Long Term Plan for Animal Disease Prevention and Control 2011-2020 formulated by the ministry of agriculture. Brucellosis is acute or chronic infectious disease caused by Brucella which is a group of small bulbous gram-negative bacteria [6, 7]. In 1985, the World Health Organization (WHO) divided Brucella into six species and nineteen biological types [7]. They are more popular with Br. melitensis, Br. brovis, and Br. suis in China, among which the most popular Brucella is Br. melitensis and the second is Br. brovis. The most noticeable symptom of brucellosis is miscarriage in cows and orchitis in bulls. In addition, it can cause arthritis in the knee and wrist frequently [2, 3]. The route of spread is through direct contact with diseased animals and indirect infection with bacteria in the environment [8, 9]. And it can be divided into three ways. The first transmitted way is contacting with skin, such as direct contact with droppings, vaginal secretions, and vaginal delivery content of sick animals. It also can be indirectly exposed to the environment and objects contaminated by sick animals. The second infected way is via the digestive tract, like eating food, water, or milk contaminated with pathogens. And the last way of transmission is through the respiratory tract.
Table 1: The description of parameters in original model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>The annual introduction number of dairy cows</td>
<td>year⁻¹</td>
</tr>
<tr>
<td>q</td>
<td>The annual birth rate of dairy cows</td>
<td>year⁻¹</td>
</tr>
<tr>
<td>d</td>
<td>The annual natural elimination rate</td>
<td>year⁻¹</td>
</tr>
<tr>
<td>δ</td>
<td>Clinical outcome rate</td>
<td>year⁻¹</td>
</tr>
<tr>
<td>c</td>
<td>The annual elimination rate for the positive cows</td>
<td>year⁻¹</td>
</tr>
<tr>
<td>m</td>
<td>The annual quantity of Brucella</td>
<td>year⁻¹</td>
</tr>
<tr>
<td>w</td>
<td>The annual natural mortality of Brucella</td>
<td>year⁻¹</td>
</tr>
<tr>
<td>β₁</td>
<td>Cow-to-cow transmission rate</td>
<td>none</td>
</tr>
<tr>
<td>β₂</td>
<td>Brucella-to-cow transmission rate</td>
<td>none</td>
</tr>
<tr>
<td>h</td>
<td>The sterilizing rate in a disinfection</td>
<td>none</td>
</tr>
<tr>
<td>e</td>
<td>The number of disinfections every year</td>
<td>once</td>
</tr>
</tbody>
</table>

When bacteria enter the body, mainly in the liver, spleen, bone marrow, lymphatic tissue, and other cells, those will grow, multiply, and produce endotoxin which can cause damaged tissues. In the meantime, it often releases bacteria and toxins into the blood causing systemic bacteremia and allergic reactions. But the Brucella does not take effect immediately when it enters the body. The bacteria with a small number will be killed by immune cells and only amounts of bacteria over a certain threshold called a minimum infection dose (MID) in environment can do harm to body [10, 11]. Different bacteria have variant thresholds which should be tested by experiment. Some bacteria threshold can be seen in [12]. Also, there were many authors researching the brucellosis, who had been unconscious of the impact on immunization [13–19]. By referring to [20, 21], we will put the pathogenicity of brucellosis and MID into this paper. The authors [12, 20, 21] incorporated a MID, c, into the incidence term α(B), in a cholera transmission model, which was a piecewise continuous function which was zero under the MID and was a Holling – II response curve over that.

\[
\alpha(B) = \begin{cases} 
0 & \text{if } B < c, \\
\frac{a(B - c)}{(B - c) + H} & \text{if } B \geq c.
\end{cases}
\]

(1)

where the parameter c represents MID, H is half-saturation pathogen density, and the description of parameter a is maximum rate of infection. One of typical mathematical models given in [22] is as follows:

\[
\dot{S} = A + q(S + E + I) - \beta_1SI - \beta_2SV - dS,
\]
\[
\dot{E} = \beta_1SI + \beta_2SV - dE - \delta E,
\]
\[
\dot{I} = \delta E - dI - cI,
\]
\[
\dot{B} = m(E + I) - (w + eh)B.
\]

(2)

where all parameters are positive and the description of parameters can be found in Table 1.

In [22], the authors gave a detailed demonstration of the basic reproduction number \( R_0 \) (\( R_0 = \frac{(A(w + eh)\beta_1\delta + Am(d + \delta))}{(d + \delta)(w + eh)(d - q)} \)), which can estimate the occurrence of the epidemic, and made brief dynamical analysis on the global stability.

Basing on model (2) and considering the immunology of Brucella in the environment, we get a switch system with MID (M). If the number of Brucella in environment (B) is less than or equal to the MID, then we can get a globally asymptotically stable disease-free and bacteria-free steady state while the basic reproduction number is less than or equal to 1, an unstable disease-free and bacteria-free equilibrium and a conditionally globally stable endemic steady state when the basic reproduction number is more than 1. Else if \( B > M \), it is going to be a little bit complicated. Combining with the relationship between parameter M and \( M_g \) (which is the formula received from the system), we can attain the three different switch systems. One of them can attain four equilibria and can produce a backward bifurcation [23, 24] and might be two steady states at most under certain parameter conditions.

2. Dynamical Analysis of the Model

2.1. The Dynamical Model. We classify the dairy cows into three compartments: the susceptible compartment S, the exposed compartment E, and the infectious compartment I in Figure 1. And the Brucella in environment is denoted by B.

One of the key differences of (2) is the incidence term \( \alpha(B)S \) of indirect transmission of Brucella to susceptible cow [12], where \( \alpha(B) \) is the pathogen density dependent component. Unlike [12, 20] using Holling – II response curve when the bacteria density is over the MID, a simple linear form is used in this paper. There are chiefly the following reasons. Firstly, the half-saturation pathogen density (H) is difficult to be determined in actual application. Secondly, the Brucella which is discharged to the nature by dairy cows and other livestock is hard to reach saturation. Lastly, that can get more practical conclusions with immunological threshold than [22]. The expression of \( \alpha(B) \) in this paper is defined as

\[
\alpha(B) = \begin{cases} 
0 & \text{if } 0 \leq B \leq M, \\
\frac{a(B - M)}{a(B - M)} & \text{if } B > M.
\end{cases}
\]

(3)
Table 2: The description of new parameters in model (4).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>MID</td>
<td>cell liter(^{-1})</td>
</tr>
<tr>
<td>a</td>
<td>Maximum rate of infection</td>
<td>none</td>
</tr>
<tr>
<td>(\beta)</td>
<td>Indirect transmission rate</td>
<td>none</td>
</tr>
</tbody>
</table>

If the number of bacteria is more than MID, then the Brucella of environment can bring the indirect influence to the dairy cows. At this time, \(a\) is representative of maximum rate of indirect infection, and \(M\) is denoted MID of Brucella, \(B-M\) is representative of bacteria amount which can enter the body and take effect. If the number is less than or equal to MID, then the bacteria of environment have no effect on the susceptible and dairy cows were infected just by the direct transmission from susceptible cow.

So the model is a system of ordinary differential equation as follows:

\[
\begin{align*}
\dot{S} &= A + q(S + E + I) - \beta SI - dS - a(B)S, \\
\dot{E} &= \beta SI + a(B)S - dE - \delta E, \\
\dot{I} &= \delta E - dI - cI, \\
\dot{B} &= m(E + I) - (w + eh)B,
\end{align*}
\]

(4)

where the description of other parameters which are not mentioned in Table 1 can be found in Table 2.

Remark 1. In this paper, we also use the condition \(d > q\) like [22].

2.2. Forward Invariance. The first equation of (4) gives us that \(S(t) \geq 0\), when \(S = 0\). In the same way, we can get \(E(t) \geq 0\) and \(I(t) \geq 0\) for \(t > 0\). Hence, we know that \(S(t) \geq 0, E(t) \geq 0,\) and \(I(t) \geq 0\) for \(t > 0\). As \(\dot{S} + \dot{E} + \dot{I} = A - (d-q)(S+E+I)\), we can get \(0 \leq S + E + I \leq A/(d-q)\) by using the character of \(\dot{S} + \dot{E} + \dot{I}\). Cause \(\dot{S} + \dot{E} + \dot{I} = A - (d-q)(S+E+I)\) is a linear function [12].

In the same way, if \(B = 0\), then \(\dot{B}(t) = 0\). Thus, the inequality \(B(t) \geq 0\) for \(t > 0\) is true. If \(B(0) \in [0, m(E + I)/(w + eh))\), then \(B(t) \in [0, m(E + I)/(w + eh))\) for any \(t > 0\). To sum up, we get the following theorem.

Theorem 2 (feasible region). The set

\[
\Omega = \left\{ (S, E, I, B) : 0 \leq S + E \leq \frac{A}{d-q}, 0 \leq B \leq \frac{mA}{(w+eh)(d-q)}, S \geq 0, E \geq 0, I \geq 0 \right\}
\]

(5)

defines a forward invariant region of system (2).

2.3. Equilibria of System (4)

2.3.1. Equilibria When Bacteria Are Less Than or Equal to MID. When \(0 \leq B \leq M\), then \(a(B) = 0\) and the model is

\[
\begin{align*}
\dot{S} &= A + q(S + E + I) - \beta SI - dS, \\
\dot{E} &= \beta SI - dE - \delta E, \\
\dot{I} &= \delta E - dI - cI, \\
\dot{B} &= m(E + I) - (w + eh)B,
\end{align*}
\]

(6)

We solve the equation and can get \(E_0 = (S_0, 0, 0, 0)\), a disease-free and bacteria-free equilibrium where \(S_0 = A/(d-q)\), and an endemic equilibrium \(E_1 = (S_1, E_1, I_1, B_1)\), where

\[
S_1 = \frac{(d+\delta)(d+c)}{\beta \delta},
\]

\[
E_1 = \frac{(d+c)[\beta \delta A - (d-q)(d+\delta)(d+c)]}{\beta \delta [(d+\delta)(d+c) - q(d+c+\delta)]},
\]

(7)

\[
I_1 = \frac{\beta \delta A - (d-q)(d+\delta)(d+c)}{\beta [(d+\delta)(d+c) - q(d+c+\delta)]},
\]

\[
B_1 = \frac{m(d+c+\delta)[\beta \delta A - (d-q)(d+\delta)(d+c)]}{(w+eh)\beta \delta [(d+\delta)(d+c) - q(d+c+\delta)]}.
\]

Remark 3. \(S_1 > 0, E_1 > 0, I_1 > 0\) and \(B_1 > 0\).
According to the spectral radius theory [25, 26], we can directly obtain the basic reproduction number based on [22] when $\beta_2 = 0$. That is

$$R_0 = \rho \left( FV^{-1} \right) = \frac{\beta_0 \delta A}{(d - q) (d + c) (d + \delta)}. \quad (8)$$

Then we can attain the existence of equilibria: if $R_0 \leq 1$, then there is only one equilibrium $E_0$; if $R_0 > 1$, then there are disease-free and bacteria-free equilibrium $E_0$ and endemic equilibrium $E_1$.

2.3.2. Equilibria When Bacteria Are More Than MID. When $B > M$, the equation is

$$\dot{S} = A + q (S + E + I) - \beta SI - dS - a (B - M) S,$$
$$\dot{E} = \beta SI + a (B - M) S - dE - \delta E,$$
$$\dot{I} = \delta E - dI - cI,$$
$$\dot{B} = m (E + I) - (w + eh) B. \quad (9)$$

Obviously, there is no disease-free and bacteria-free equilibria because of $M \geq 0$. Thus, system (9) implies that

$$E = \frac{(d + c) (w + eh) B}{m (d + c + \delta)},$$
$$I = \frac{\delta (w + eh) B}{m (d + c + \delta)},$$
$$S = \frac{(d + \delta) E}{\beta I + \alpha (B)} = \frac{q (E + I) + A}{\beta I + \alpha (B) + d - q}. \quad (10)$$

The equation of $S$ about $B$ can be written as $F(B) = 0$. Define $F(B) = B^2 (w + eh) [(d + c) (d + \delta) - q (d + c + \delta)] [\beta \delta (w + eh) + a M (d + c + \delta)] + B m (d + c + \delta)\{(d + c) - q (d + c + \delta)\} - (w + eh) [\beta \delta (w + eh) - a M (d + c + \delta)] A + a M m^2 (d + c + \delta)^2. From the expression, we can know that $F(B)$ is a quadratic function, in which the first and third coefficients are positive. Therefore, the existence of solution for $F(B) = 0$ depends on the second coefficient and $\Delta$. If the second coefficient is nonnegative, no matter $\Delta$ is, then $F(B) = 0$ has no positive solution. If the second coefficient is negative, the positive solution depends on the sign of $\Delta$: when $\Delta < 0$, there is no solution; when $\Delta = 0$, there is one positive solution; and when $\Delta > 0$, there are two positive solutions.

Now we simplify them and can get the new second coefficient $b_1$:

$$b_1 := m (d + c + \delta) \cdot \frac{\left[ (w + eh) - a M (d + c + \delta) \right]}{A + b_11}, \quad (11)$$

where

$$b_{11} := -a M (w + eh) \left[ (d + c) (d + \delta) - q (d + c + \delta) \right] + (w + eh) (d - q) (d + c + \delta), \quad (12)$$

and the new $\Delta$ about $A$:

$$\Delta (A) := A^2 m^2 \left[ (d + c + \delta)^2 - \beta \delta (w + eh) - a M (d + c + \delta) \right]^2 + A \{ -2 m^2 (d + c + \delta)^2 (w + eh) \cdot \left[ \beta \delta (w + eh) + a M (d + c + \delta) \right] + (d - q) (d + \delta) \} + m^2 (d + c + \delta)^2 \cdot \left[ -a M (w + eh) \left[ (d + c) (d + \delta) - q (d + c + \delta) \right] + (w + eh) (d - q) (d + c + \delta) \right]^2. \quad (13)$$

We can know $b_1$ is a liner function about $A$ and $\Delta(A)$ is still a quadratic function which the first and third coefficients are positive and the second coefficient is less than zero. Then we calculate the $\Delta(A)$ and can know that the sign of $\Delta(\Delta(A))$ is always positive. So we can solve $\Delta(A) = 0$ and get $A_1, A_2$, where

$$A_1 = (w + eh) \cdot \frac{a M \left[ (d + c) (d + \delta) - q (d + c + \delta) \right] + (d - q) (d + c) (d + \delta) - 2 \sqrt{a M \left[ (d + c) (d + \delta) - q (d + c + \delta) \right] (d - q) (d + c)} (d + \delta)}{\beta \delta (w + eh) + a M (d + c + \delta)},$$
$$A_2 = (w + eh) \cdot \frac{a M \left[ (d + c) (d + \delta) - q (d + c + \delta) \right] + (d - q) (d + c) (d + \delta) + 2 \sqrt{a M \left[ (d + c) (d + \delta) - q (d + c + \delta) \right] (d - q) (d + c)} (d + \delta)}{\beta \delta (w + eh) + a M (d + c + \delta)}. \quad (14)$$

By the analysis with the feature of function, we can get the conclusion as follows.

**Proposition 4.** The existence of positive solutions for $F(B) = 0$: (i) When $b_{11} \geq 0$: if $0 \leq A < A_2$, then $F(B) = 0$ does not get any positive solution; if $A = A_2$, then $F(B) = 0$ has one positive solution $B_1^+$; if $A > A_2$, then $F(B) = 0$ has two positive solutions $B_1^+, B_2^+$ ($B_1^+ < B_2^+$).
Theorem 5. The existence of internal equilibria about \( B > M \):

\[
F(B) = 0 \text{ has two positive solutions } B_1^+, B_2^+.
\]

where

\[
A_f = \frac{\beta \delta (w + eh) ((d + c)(d + \delta) - q(d + c + \delta)) M + m(d + c + \delta)(d - q)(d + c)(d + \delta)}{\beta \delta m(d + c + \delta)}
\]

\[
A_g = \frac{2\beta \delta (w + eh)^2 + am(w + eh)(d + c + \delta)((d + c)(d + \delta) - q(d + c + \delta)) M + (w + eh)m(d + c + \delta)(d - q)(d + c)(d + \delta)}{m(d + c + \delta)[\beta \delta (w + eh) + ma(d + c + \delta)]}.
\]

Proof. We have known the positive value of \( B \); now we need combine with the inequality \( B > M \). The proof uses character function \( F(B) \).

Firstly, we should know the value of \( F(M) \). When \( B = M \), we can get \( F(M) = \beta \delta (w + eh)/((d + c)(d + \delta) - q(d + c + \delta))M^2 - m(d + c + \delta)((w + eh)(\beta \delta A - (d - q)(d + c)(d + \delta))M, \) and the relationship between \( F(M) \) and 0 is translated into the relationship between \( A \) and \( A_f \):

1. If \( M > 0 \):
   - \( A_f > A_1 \), there is an internal equilibrium \( E_2^+ \);
   - \( A_f > A_2 \), there is an internal equilibrium \( E_2^+ \);
   - \( \text{max}[A_1, A_2] < A < A_f \), there are two internal equilibria \( E_1^+, E_3^+ \);
   - \( A = A_2 \), there is an internal equilibrium \( E_2^+ \);

2. If \( M = 0 \) and \( A > A_2 = A_g \), then there is an internal equilibrium \( E_2^+ \).

Finally, we use the above-mentioned relational expressions and combine with Proposition 4; then, we can obtain the final chart as shown in Table 4.

Because positive solutions of \( F(B) \) is equal to existences of internal equilibria of system. Thus, the existence of internal equilibria about \( B > M \) is proved.

Remark 6. \( M_g = m^2(d + c + \delta)^2(a(d - q)(d + c)(d + \delta)/\beta^2 \delta^2(w + eh)) \) is the criterion when \( A_2 = A_g \).

After knowing internal equilibria when \( B > M \), we want to get the relationship between internal equilibria and \( R_0 \), so we need to change the form above all. We had known the \( R_0 = \beta \delta A / (d - q)(d + c)(d + \delta) \). Thus, we change \( A_2 \) to \( P_2 = \beta \delta A_2 / (d - q)(d + c)(d + \delta) \), \( A_g \) to \( P_g = \beta \delta A_g / (d - q)(d + c)(d + \delta) \), and then \( A = P_f = \beta \delta A_f / (d - q)(d + c)(d + \delta) \). From the expression of these, we get \( R_0 = 1 < P_f \) when \( M > 0 \) and \( R_0 = 1 = P_f \) when \( M = 0 \).
The stability of the equilibria, which will be proved in Section 4, will be demonstrated numerically in Figure 2.

**Theorem 7** (existence of equilibria in Figures 2 and 3). The equilibrium $E_0 = (S_0, 0, 0, 0)$ always exists in $Ω$. Here are the existences of internal equilibria:

(1) If $0 < M < M_y$,

(i) If $P_2 > 1$: when $1 < R_0 < P_2$, there only exists $E_1$: when $R_0 = P_2$, there exist $E_1, E_1^*$; when $P_2 < R_0 < P_f$, there exist $E_1, E_1^*, E_2^*$; when $R_0 = P_f$, there exist $E_1, E_2^*$; when $R_0 > P_f$, there exist $E_2^*$.

(ii) If $P_2 = 1$: when $R_0 = P_2$, there exists $E^*$; when $1 < R_0 < P_f$, there exist $E_1, E_1^*, E_2^*$; when $R_0 = P_f$, there exist $E_1, E_2^*$; when $R_0 > P_f$, there exist $E_2^*$.

(iii) If $P_2 < 1$: when $R_0 = P_2$, there exists $E^*$; when $P_2 < R_0 < 1$, there exist $E_1^*, E_2^*$; when $1 < R_0 < P_f$, there exist $E_1, E_1^*, E_2^*$; when $R_0 = P_f$, there exist $E_1, E_2^*$; when $R_0 > P_f$, there exists $E_2^*$.

(2) If $M > M_y$: when $1 < R_0 < P_f$, there exists $E_1$; when $R_0 > P_f$, there exists $E_2^*$.

(3) If $M = 0$ (in this case, $P_f = 1$): when $R_0 > P_2$, there exists $E_1^*$.

Theorem 7 describes the existence of equilibria under different parameters conditions (Table 5), where disease-free and bacteria-free equilibrium $E_0$ always exists. Besides, we give the stability of the equilibria, which will be proved in next part. We just give the stability of $E_0$ and $E_1$; others are too difficult to give the strict mathematical proofs because of lacking exact expression. Therefore, the local stability of $E_1^*$ and $E_2^*$ will be demonstrated numerically in Section 4.

### 3. Stability of Equilibria

#### 3.1. Local Stability of $E_0$ and $E_1$. We calculate the Jacobian to analyze the local stability of each of the equilibria. When $0 ≤ B ≤ M$, the Jacobian is

$$J(S, E, I, B) =
\begin{pmatrix}
q - d - βI & q & q - βS & 0 \\
βI & -(d + δ) & βS & 0 \\
0 & δ & -(d + c) & 0 \\
0 & m & m & -(w + eh)
\end{pmatrix}.$$  

Now considering $E_0$, we use $λ$ for eigenvalues. We can compute

$$\det(λI - J_{E_0}) = [λ - (q - d)](λ + w + eh)$$

$$· [(λ + d + δ)(λ + d + c) - βδS_0].$$

Note $h(λ) = (λ + d + δ)(λ + d + c) - βδS_0$. We can find that $λ_1 = -(d - q) < 0, λ_3 = -(w + eh) < 0$. Now we just consider the eigenvalues of $h(λ)$. The Routh–Hurwitz coefficients of the $h(λ)$ are

$$Δ_1 = d + c + d + δ > 0,$$

$$Δ_2 = (d + c + d + δ)[(d + δ)(d + c) - βδS_0].$$

### Table 4

<table>
<thead>
<tr>
<th>$F(B)$</th>
<th>$F(M)$</th>
<th>$g(M)$</th>
<th>Conclusions</th>
<th>Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M &gt; 0, A &gt; A_2$</td>
<td>$M &gt; 0, A &gt; A_f$</td>
<td>$M &gt; 0, A &gt; A_y$</td>
<td>$M &gt; 0, A = A_2 &gt; A_y$</td>
<td>$B^*$</td>
</tr>
<tr>
<td>$M ≥ 0, A = A_2$</td>
<td>$M = 0$</td>
<td>$M ≥ 0, A &gt; A_y$</td>
<td>$M = 0, A &gt; A_2 = A_y$</td>
<td>$B^*_1$</td>
</tr>
<tr>
<td>$M ≥ 0, A &gt; A_2$</td>
<td>$M &gt; 0, A = A_f$</td>
<td>$M ≥ 0, A &gt; A_y$</td>
<td>$M &gt; 0, A = A_f &gt; A_y$</td>
<td>$B^*_2$</td>
</tr>
<tr>
<td>$M &gt; 0, A &lt; A_f$</td>
<td>$M ≥ 0, A &gt; A_y$</td>
<td>$M &gt; 0, max{A_2, A_y} &lt; A &lt; A_f$</td>
<td>$B^<em>_1, B^</em>_2$</td>
<td></td>
</tr>
</tbody>
</table>

### Figure 2

Density of the infected dairy cows as a function of the basic reproduction number with $0 < M < M_y$. The common values of (a), (b), and (c) used for the parameters are as follows: $w = 3, d = 2, c = 0.8, δ = 4, q = 1, β = 0.2$, and $d = 2$. The different values are (a) $m = 0.1, M = 0.1$; (b) $m = 0.0932, M = 0.0468$; and (c) $m = 0.05, M = 0.0135$. The forward bifurcation occurs at $R_0 = 1$ in all the pictures and the backward bifurcation occurs at $R_0 = 1.2101$ in picture (a), at $R_0 = 1.1056$ in picture (b) and at $R_0 = 1.0567$ in picture (c).
Table 5: Existence of endemic equilibria of model (4).

<table>
<thead>
<tr>
<th>Condition 1</th>
<th>Condition 2</th>
<th>Condition 3</th>
<th>Endemic equilibria</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; M &lt; M_g$</td>
<td>$P_2 &gt; 1$</td>
<td>$1 &lt; R_0 &lt; P_2$</td>
<td>$E_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_0 = P_2$</td>
<td>$E_1, E^*_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_2 &lt; R_0 &lt; P_f$</td>
<td>$E_1, E^<em>_1, E^</em>_2$</td>
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<td></td>
<td></td>
<td>$R_0 = P_f$</td>
<td>$E_1, E^*_2$</td>
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<tr>
<td></td>
<td></td>
<td>$R_0 &gt; P_f$</td>
<td>$E^*_2$</td>
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<tr>
<td></td>
<td>$P_2 = 1$</td>
<td>$1 &lt; R_0 &lt; P_f$</td>
<td>$E_1, E^<em>_1, E^</em>_2$</td>
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<td></td>
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<td>$R_0 = P_f$</td>
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<td></td>
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<td>$R_0 &gt; P_f$</td>
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<tr>
<td></td>
<td>$P_2 &lt; 1$</td>
<td>$R_0 = P_2$</td>
<td>$E^*_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_2 &lt; R_0 &lt; 1$</td>
<td>$E^<em>_1, E^</em>_2$</td>
</tr>
<tr>
<td>$M \geq M_g$</td>
<td>—</td>
<td>$1 &lt; R_0 \leq P_f$</td>
<td>$E_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_0 &gt; P_f$</td>
<td>$E^*_2$</td>
</tr>
<tr>
<td>$M = 0$</td>
<td>—</td>
<td>$R_0 &gt; P_f$</td>
<td>$E_2^*$</td>
</tr>
</tbody>
</table>

Figure 3: The relationship between the basic reproduction number and infected dairy cows while $M \geq M_g$ and $M = 0$, where (2) and (3) of Theorem 9 holds. The common values of (a) and (b) used for the parameters are as follows: $w = 3, d = 2, c = 0.8, \delta = 4, q = 1, \beta = 0.2, a = 2$, and $m = 0.1$. The different values of (a) and (b) are (a) $M = 0.5$, (b) $M = 0$. There is a forward bifurcation at $R_0 = 1$ in (a). The bifurcation at $R_0 = 0.6383$ in (b) is still forward.

If $R_0 < 1$, then $\Delta_2 > 0$; if $R_0 > 1$, then $\Delta_2 < 0$. Thus the eigenvalues of $h(\lambda)$ are negative when $R_0 < 1$. So the disease-free and bacteria-free equilibria $E_0$ are locally asymptotically stable while $R_0 < 1$. If $R_0 > 1$, then $E_0$ is a saddle-point equilibrium.

Now considering $E_1$. The characteristic polynomial of Jacobian of $E_1$ is

$$\det(\lambda I - J_{E_1}) = a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0,$$

(19)

where

$$a_0 = 1,$$

$$a_1 = \frac{\beta \delta A - q(d - q)(d + c + \delta)}{(d + \delta)(d + c) - q(d + c + \delta)} + \frac{\beta \delta A - [q(d - q)(d + c + \delta)](d + \delta + d + \delta)}{(d + \delta)(d + c) - q(d + c + \delta)} + (d + c + d + \delta),$$

(20)

$$a_2 = \frac{\beta \delta A}{(d + \delta)(d + c) - q(d + c + \delta)},$$

$$a_3 = \beta \delta A - (d - q)(d + \delta)(d + c).$$
The Routh–Hurwitz coefficients of the det($\lambda I - J_{E_1}$) are

$$\Delta_1 = a_1,$$
$$\Delta_2 = a_1a_2 - a_3,$$
$$\Delta_3 = a_3(a_1a_2 - a_3).$$  \tag{21}

where $a_1 > (\beta \delta A - q(d-q)(d+c+\delta))/((d+\delta)(d+c)-q(d+c+\delta)) > (\beta \delta A - (d-q)(d+\delta)(d+c))/((d+\delta)(d+c)-q(d+c+\delta)) > 0$ and $a_3 > 0$, when $R_0 > 1$.

As for the condition $a_1a_2 - a_3$, we have the following expression:

$$a_1a_2 - a_3 = h_1 + h_2 + h_3,$$  \tag{22}

where

$$h_1 = \left[\beta \delta A - q(d-q)(d+c+\delta)\right]\left[\beta \delta A - q(d-q)(d+c+\delta)\right]\left(d+c+d+\delta\right) - \left[q \beta \delta A - q(d-q)(d+c)(d+\delta)\right] \left[(d+\delta)(d+c) - q(d+c+\delta)\right]^2,$$
$$h_2 = \left[\beta \delta A - q(d-q)(d+c+\delta)\right] (d+c+d+\delta)^2 - (d+c+d+\delta) \left[q \beta \delta A - q(d-q)(d+c)(d+\delta)\right],$$
$$h_3 = -\left[\beta \delta A - (d-q)(d+\delta)(d+c)\right].$$  \tag{23}

Now, we magnify them:

$$h_1 > \left[(d-q)(d+c)(d+\delta) - q(d-q)(d+c+\delta)\right] \left[\beta \delta A - q(d-q)(d+c+\delta)\right] (d+c+d+\delta) - \left[q \beta \delta A - q(d-q)(d+c)(d+\delta)\right] \left[(d+\delta)(d+c) - q(d+c+\delta)\right]^2,$$
$$h_2 > \left[\beta \delta A - q(d-q)(d+c+\delta)\right] (d+c+d+\delta)^2 - (d+c+d+\delta) \left[q \beta \delta A - q(d-q)(d+c)(d+\delta)\right],$$
$$h_3 > -\left[\beta \delta A - (d-q)(d+\delta)(d+c)\right].$$  \tag{24}

We know that $d > q$, so $(d+c)(d+\delta) > q(d+c+\delta)$ and then $a_1a_2 - a_3 > 0$. Hence, the Routh–Hurwitz conditions are satisfied when $R_0 > 1$ and $E_1$ is locally asymptotically stable.

**Theorem 8** (local stability). System (4) has equilibria as follows:

When $0 \leq B \leq M$ and $R_0 < 1$, there is only one equilibrium $E_0$ which is locally asymptotically stable. When $0 \leq B \leq M$ and $R_0 > 1$, $E_0$ is unstable and a unique endemic equilibrium $E_1$ exists which is locally asymptotically stable.

We study the stability of equilibria in order to explain more epidemiological implications. Taking Figure 2(a) as an example, we can get the following biological significance. If $E_0$ is stable ($R_0 < 1$), the brucellosis will fade as time goes by. If $E_1$ is stable ($1 < R_0 < P_2$), disease will maintain the level of $I_1$ all the time. With the scope of $R_0$ ($P_2 < R_0 < P_f$), disease will continue at low-level state ($I_1$) when the current status is in the attraction domain of $E_1$ and will be going to stay at high-level state ($I_1^+$) while the current status exactly situate in the attraction domain of $E_1^+$, which will be demonstrated...
3.2. Global Stability of $E_0$ and $E_1$. In this section, we will demonstrate the global stability of the endemic equilibrium of system (4) by using the Lyapunov function [27, 28].

**Theorem 9.** If $0 \leq B \leq M$ and $R_0 < 1$, then $E_0$ is globally asymptotically stable in $\Omega$.

**Proof.** As we can see, $B$ is not acting on the other subclasses. So we can define the Lyapunov function

$$V(S, E, I, B) = S - S_0 - S_0 \ln \frac{S}{S_0} + E + \frac{d + \delta}{\delta} I. \quad (25)$$

Then the derivative of $V$ along system (6) is

$$\frac{dV}{dt} = \left(1 - \frac{S_0}{S}\right) \left[A - (d - q) S + q (E + I) - \beta SI\right]$$

$$+ \left[\beta SI - (d + \delta) E\right] + \frac{d + \delta}{\delta} \left[\delta E - (d + c) I\right]$$

$$= \left(1 - \frac{S_0}{S}\right) \left[A - (d - q) S + q (E + I)\right] + \beta S_0 I$$

$$- (d + \delta) E + \frac{d + \delta}{\delta} \left[\delta E - (d + c) I\right]$$

$$= \left(1 - \frac{S_0}{S}\right) \left[A - (d - q) S + q (E + I)\right]$$

$$+ \frac{\beta S_0 (d - q) (d + c) (d + \delta)}{\delta (d - q)}.$$

According to the definition of the invariant region, we can know that $S \leq S_0$. So $dV/dt < 0$ is true and $dV/dt = 0$ holds only if $S = S_0$, when $R_0 < 1$. $dV/dt = 0$ holds only if $S = S_0$ when $R_0 = 1$. Thus the disease-free and bacteria-free equilibrium $E_0$ is globally asymptotically stable by LaSalle’s Invariance Principle [29].

**Theorem 10.** If $0 \leq B \leq M$, $R_0 > 1$, and $q = 0$, then $E_1$ is a conditionally globally asymptotically stable in $\Omega$.

**Proof.** In this proof, we use the same way to demonstrate the stability when $q = 0$. Similarly, $B$ is not acting on the other subclasses, so we define the Lyapunov function as follows:

$$V(S, E, I, B) = S - S^1 - S^1 \ln \frac{S}{S_1} + E - E^1 - E^1 \ln \frac{E}{E_1}$$

$$+ \frac{d + \delta}{\delta} \left(I - I^1 - I^1 \ln \frac{I}{I_1}\right). \quad (27)$$

Then the derivative of $V$ along system (6) is

$$\frac{dV}{dt} = \left(1 - \frac{S_1}{S}\right) \left[A - dS - \beta SI\right]$$

$$+ \left(1 - \frac{E_1}{E}\right) \left[\beta SI - (d + \delta) E\right]$$

$$+ \frac{d + \delta}{\delta} \left(1 - \frac{I_1}{I}\right) \left[\delta E - (d + c) I\right]. \quad (28)$$

Let $s = S/S^1, e = E/E^1$, and $i = I/I^1$. Then we can get

$$A - dS^1 - \beta S^1 I^1 = 0,$$

$$\beta S^1 I^1 - (d + \delta) E^1 = 0,$$

$$\delta E^1 - (d + c) I^1 = 0.$$

Combining with them, we can obtain that

$$\frac{dV}{dt} = S^1 \left(2d - ds - \frac{d}{s}\right) + \beta S^1 I^1 \left(1 - \frac{1}{s} + \frac{si}{e} - \frac{si}{i}\right)$$

$$+ E^1 \left[(d + \delta) - \frac{(d + \delta) e}{i}\right]$$

$$+ I^1 \left[- \frac{(d + \delta) (d + c) i}{\delta} + \frac{(d + c) (d + \delta)}{\delta}\right]$$

$$= dS^1 \left(2 - s - \frac{1}{s}\right) + \beta S^1 I^1 \left(3 - \frac{1}{s} + \frac{si}{e} - \frac{e}{i}\right). \quad (30)$$

Combining with inequality of arithmetic and geometric means, we can know $2 - s - 1/s \leq 0$ and $(3 - 1/s - si/e - e/i) \leq 0$. Thus,

$$\frac{dV}{dt} \leq 0. \quad (31)$$

According to the definition of the invariant region, only if $s = e = i = 1$ means that $S = S^1, E = E^1$ and $I = I^1$, then $dV/dt = 0$. Similarly with Theorem 9, we can get the endemic equilibrium $E_1$, which is globally asymptotically stable by LaSalle’s Invariance Principle in $\Omega$ when $q = 0$.

**4. Numerical Results**

In an epidemic model, the basic reproduction number $R_0$ is calculated and verified to be a threshold for the dynamics of the disease. The ultimate objective is to control the disease by making the basic reproduction number $R_0$. In the previous analysis of this paper, we analyze the different equilibria in variant ranges of $R_0$. Now, we should know the final infectious size of dairy cows depending on the $R_0$.

In Figure 4(a), only if $R_0 < \min\{1, R_0\}$ (Figures 2(a), 2(b), 2(c), 3(a), and 3(b)), we can know that the number of infected dairy cows will obtain the same disease-free and bacteria-free equilibrium $E_0$. No matter how much the initial values are and the relationship about $M, M_{max},$ and $0$ is. In biology, this means only if the basic reproduction number demands the formula,
Figure 4: Time series of the infected dairy cows. The common values of (a) and (b) used for the parameters are as follows: \( w = 3, d = 2, c = 0.8, \delta = 4, q = 1, \beta = 0.2, a = 2, m = 0.2 \). The different values of (a) and (b) are (a) \( A = 10 \) and (b) \( A = 50 \). The solid line or \( S_1 \) means that \( I \) is entering system (6); the dotted line or \( S_2 \) indicates that \( I \) is going into system (9).

Figure 5: The common values of (a) and (b) used for the parameters are as follows: \( w = 3, d = 2, c = 0.8, \delta = 4, q = 1, \beta = 0.2, a = 2, m = 0.1, M = 0.1, \) and \( A = 25 \). Different colors are representative of different initial values, of which the dot line means that it enters system (6) and the full line means that it goes into system (9) as in Figure 4.

the final infectious size of dairy cows is tending to zero. It is to say, the disease will vanish without any measures after a certain time.

Figure 4(b) demonstrates that when \( R_0 > \max(P_2, P_f) \), the infected dairy cows will obtain the endemic equilibrium \( E^*_2 \). In the meantime, different values of \( M \) have different equilibria: when \( M = 0.5, I_2^* = 15.3531 \); when \( M = 0.1, I_2^* = 15.9432 \); and when \( M = 0, I_2^* = 16.0625 \). This means that only if the basic reproduction number satisfies the above formula, brucellosis will never disappear and it would become epidemic disease. In addition, the more size of MID will lead to the less final infected dairy cows. It might be easy to understand that when the MID is heightened, the bacteria invading the body are diminished by immunology theory [10].

Figures 5 and 7 show that when \( \min(1, P_2) < R_0 < P_f \), the final size of infectious cows will be in bistable state. Figure 5 comes from Figure 2(a) when \( P_2 < R_0 < P_f \) and Figure 7
complexity 11 0 20 40 60 80 0 10 20 0 20 40 60 80

Figure 6: The tendency of $S$, $E$, and $I$.

5. Conclusion and Discussion

Brucellosis is becoming more and more popular in recent years, since it has formed serious endemic disease and it cannot be thoroughly removed in northeast and northwest China. In this paper, we take account of immune influence of Brucella just like researching of cholera [12]. We build the new brucellosis model based on [22]. Compared with other studies, we are more concerned with the effects of immunology on Brucella. We just change the transmission rate from constant $\beta_2$ to the piecewise continuous function between the research [12] and our paper. But the mathematical results we obtain are more complicated and more biological. The original work obtained the basic reproduction number and got two equilibria and then simply analyzed the stability state. In this paper, we take immunological threshold into consideration; then we get the whole mathematical analysis of equilibria in each case and do the locally and globally stable analysis about some equilibria. The MID of Brucella has not been tested by experiment, so we get the serious discussion

Figure 7: The common values of (a) and (b) used for the parameters are as follows: $w = 3, d = 2, c = 0.8, \delta = 4, q = 1, \beta = 0.2, a = 2, m = 0.05, M = 0.014$, and $A = 22$.

Figure 8: The tendency of $S$, $E$, and $I$.

comes from Figure 2(c) when $1 < R_0 < P_f$, in which Figures 5(a) and 7(a) are time series plots of the number of Brucella and Figures 5(b) and 7(b) are time series plots of the number of infected dairy cows. Figures 6 and 8 show tendency of $S$, $E$, $I$. From them we can see that when $R_0$ meets the relational expression, different initial values will eventually lead to two equilibria; one is $E_1$ and the other is $E_2^+$ ($E_1 < E_2^+$). In
with $M > 0$ and $M = 0$: when $0 < M < M_g$ (the formation of $M_g$ is given in this paper), we get the forward bifurcation occurring at $R_0 = 1$ and backward bifurcation appearing at $R_0 = P_2$; when $M \geq M_g$, it just has forward bifurcation at $R_0 = 1$; when $M = 0$, the model is exactly same as in [22] and produces the forward bifurcation at $R_0 = P_2$. By mathematical analysis of them, we can get the global stability $E_0$ when $R_0 < 1$ and a global stability $E_1$ when $R_0 > 1$. The character of $E_{12}$ is given by numerical simulations. From that, we know it will be in bistable state when the basic reproduction number demands $\max\{1, P_3 < R_0 < P_4\}$. We make the basic reproduction number as the measuring standard in order to give suggestions of the brucellosis prevention [30, 31]. While the basic reproduction number is less than 1, brucellosis is becoming a disease-free and bacteria-free stable state. After a certain time, brucellosis will be removed. And while the reproduction number is more than 1 by recording to different relationship about $M$, $M_g$, and 0, the brucellosis will be endemic disease which will hold a stable state with variant values of infected dairy cows. Using the analysis of this paper, we can reduce the number of infected dairy cows to prevent the spreading of epidemic disease, which can even obtain zero by adjusting the size of the basic reproduction number.

Refs. [20, 21] using Holling–II functional response both show that MID is vital for the spread of cholera. Reducing the pathogen density to under the MID in an aquatic reservoir would be more effective measure to control cholera. Similarly, in order to control and prevent the spreading of brucellosis, we can place an emphasis on the incident rate because of the existence of MID. If we can adjust the initial value of Brucella to be below the MID and constrain growth and reproduction conditions of Brucella in environment, then we can obtain the incidence rate of indirect transmission of Brucella to susceptible cow to be zero all the time and the ultimate scale of brucellosis will decrease and even disappear.

In further planning, the incidence term $\alpha(B)$ can be used in the formation by a Holling–II response which is more reasonable in biology than the linear form in this paper. In addition, we have not yet captured and processed the real data about brucellosis of dairy cows to get more effective conclusions. That is what we are going to do next. Furthermore, the real spreading of brucellosis is via multiple zones rather than one zone and the multizone patchy model can describe the transmission in multiple regions. Hence, with considering the network models [32, 33] or reaction-diffusion equations [34–38], conclusions can accord well with practice even more.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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