Research Article

Global Asymptotic Almost Periodic Synchronization of Clifford-Valued CNNs with Discrete Delays

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In this paper, we are concerned with Clifford-valued cellular neural networks (CNNs) with discrete delays. Since Clifford algebra is a unital associative algebra and its multiplication is noncommutative, to overcome the difficulty of the noncommutativity of the multiplication of Clifford numbers, we first decompose the considered Clifford-valued neural network into $2^{m^2}n$ real-valued systems. Second, based on the Banach fixed point theorem, we establish the existence and uniqueness of almost periodic solutions of the considered neural networks. Then, by designing a novel state-feedback controller and constructing a proper Lyapunov function, we study the global asymptotic synchronization of the considered neural networks. Finally, a numerical example is presented to show the effectiveness and feasibility of our results.

1. Introduction

Clifford algebras, which were invented by mathematician W. K. Clifford, are algebras generated from vector spaces with quadratic forms and are unitary associative algebras. They contain real numbers, complex numbers, quaternions, and several other hypercomplex systems as special cases. Clifford algebras have important applications in a variety of fields, including geometry, theoretical physics, and digital image processing [1]. Clifford-valued neural networks generalize real-valued, complex-valued, quaternion-valued, and octonion-valued neural networks and so on. Since the Clifford-valued neural networks can use multistate activation functions to process multilevel information and require much fewer connection weight parameters of the networks, they have recently been an active research area [2–5]. Due to the noncommutativity of Clifford numbers' multiplication, it brings great difficulties to the research of Clifford-valued neural networks. At present, the results of research on Clifford-valued neural networks are still rare, especially, for the nonautonomous ones.

On the one hand, it is well known that, in the design and implementation of neural networks, the existence of periodic

solutions or almost periodic solutions of nonautonomous neural networks is as important as the existence of equilibrium points of autonomous neural networks. At the same time, we know that, even if all the time-varying coefficients in a neural network are periodic functions (such as connection coefficients, connection weight functions, external inputs, and time delays, etc.), if their periods are incommensurable, then, for such a neural network, it is also impossible to have a periodic solution. Therefore, studying the almost periodicity of neural networks is more practical and important than studying the periodicity of neural networks. At present, the periodicity has been extensively studied as an important dynamic property of various neural networks. However, there has been no paper published on the almost periodicity of Clifford-valued neural networks yet.

On the other hand, the synchronization of nonlinear systems has become an important research topic due to its potential applications in various fields such as secure communication, image encryption, information science, and so on. Particularly, recently, many authors have studied the synchronization problem for various neural network systems [6–15]. For example, the synchronization problem for chaotic memristor-based neural networks with time-varying delays

was studied in [11]. The global asymptotic synchronization problem of nonidentical fractional-order neural networks with Riemann-Liouville derivative was investigated in [12]. The synchronization of an inertial neural network with time-varying delays was investigated in [13]. The finite-time cluster synchronization of coupled fuzzy cellular neural networks with Markovian switching topology, discontinuous activation functions, proportional leakage, and time-varying unbounded delays was studied in [14]. The global exponential almost periodic synchronization of quaternion-valued neural networks with time-varying delays was investigated in [15]. But until now, the results of the synchronization of Clifford-valued neural networks have not been reported.

In summary, it is meaningful to study the almost periodicity and the synchronization problem of Clifford-valued neural networks. Therefore, our main purpose of this paper is to investigate the problem of the existence of almost periodic solutions and global asymptotic synchronization of Clifford-valued CNNs with discrete delays. To the best of our knowledge, this is the first paper to study such a problem for Clifford-valued neural networks with discrete delays. Our methods of this paper can be used to study other types of Clifford-valued neural networks.

The rest of this paper is organized as follows: in Section 2, we introduce some basic concepts, notations, and lemmas and give a model description. In Section 3, we study the existence of almost periodic solutions of Clifford-valued CNNs with discrete delays. In Section 4, we investigate the global asymptotic almost periodic synchronization of Clifford-valued CNNs with discrete delays. In Section 5, we give an example to demonstrate the effectiveness and feasibility of our results. In Section 5, we give a conclusion.

2. Preliminaries and Model Description

The real Clifford algebra over $\mathbb{R}^m$ is defined as

$$\mathcal{A} = \left\{ \sum_{A \subseteq \{1, 2, \dots, m\}} a_A e_A, a_A \in \mathbb{R} \right\}, \quad (1)$$

where $e_A = e_{h_1} e_{h_2} \cdots e_{h_\nu}$ with $A = \{h_1, h_2, \dots, h_\nu\}$, $1 \leq h_1 < h_2 < \cdots < h_\nu \leq m$. Moreover, $e_\emptyset = e_0 = 1$ and $e_{\{h\}} = e_h$, $h = 1, 2, \dots, m$ are called Clifford generators which satisfy the relations $e_i^2 = -1$, $i = 1, 2, \dots, m$, and $e_i e_j + e_j e_i = 0$, $i \neq j$, $i, j = 1, 2, \dots, m$.

For simplicity, when one element is the product of multiple Clifford generators, we will write its subscripts together. For example, $e_4 e_5 e_7 e_9 = e_{4579}$. Denote $\Lambda = \{\emptyset, 1, 2, \dots, A, \dots, 12 \cdots m\}$, and then we have

$$\mathcal{A} = \left\{ \sum_A a_A e_A, a_A \in \mathbb{R} \right\}, \quad (2)$$

where $\sum_A$ is short for $\sum_{A \in \Lambda}$ and $\mathcal{A}$ is isomorphic to $\mathbb{R}^{2^m}$.

For any $x = \sum_A x_A e_A \in \mathcal{A}$, the involution of x is defined as

$$\bar{x} = \sum_A x_A \bar{e}_A, \quad (3)$$

where $\bar{e}_A = (-1)^{\mu[A](\mu[A]+1)/2} e_A$, and if $A = \emptyset$, then $\mu[A] = 0$; if $A = h_1 h_2 \cdots h_\nu \in \Lambda$, then $\mu[A] = \nu$. It is easy to see that $e_A \bar{e}_A = \bar{e}_A e_A = 1$ and for every $x = \sum_A x^A e_A \in \mathcal{A}$, $\bar{x} = \sum_A x^A \bar{e}_A$. Moreover, we have $\overline{xy} = \bar{y} \bar{x}$, $\forall x, y \in \mathcal{A}$.

For a Clifford-valued function $z = \sum_A z^A e_A : \mathbb{R} \rightarrow \mathcal{A}$, where $z^A : \mathbb{R} \rightarrow \mathbb{R}$, $A \in \Lambda$, its derivative is given by

$$\dot{z}(t) = \sum_A \dot{z}^A(t) e_A. \quad (4)$$

In view of $e_B \bar{e}_A = (-1)^{\mu[A](\mu[A]+1)/2} e_B e_A$, we can write $e_B \bar{e}_A = e_C$ or $e_B \bar{e}_A = -e_C$, where e_C is a Clifford generator of Clifford algebra $\mathcal{A}$. Hence it is possible to find a unique corresponding basis e_C for the given $e_B \bar{e}_A$. Define

$$\mu[B \cdot \bar{A}] = \begin{cases} 0, & \text{if } e_B \bar{e}_A = e_C, \\ 1, & \text{if } e_B \bar{e}_A = -e_C, \end{cases} \quad (5)$$

and then $e_B \bar{e}_A = (-1)^{\mu[B \cdot \bar{A}]} e_C$. In addition, for any $E \in \mathcal{A}$, we can find a unique E^C satisfying $E^{B \cdot \bar{A}} = (-1)^{\mu[B \cdot \bar{A}]} E^C$ for $e_B \bar{e}_A = (-1)^{\mu[B \cdot \bar{A}]} e_C$. Therefore,

$$\begin{aligned} E^{B \cdot \bar{A}} e_B \bar{e}_A &= E^{B \cdot \bar{A}} (-1)^{\mu[B \cdot \bar{A}]} e_C \\ &= (-1)^{\mu[B \cdot \bar{A}]} E^C (-1)^{\mu[B \cdot \bar{A}]} e_C = E^C e_C \end{aligned} \quad (6)$$

and

$$E = \sum_C E^C e_C \in \mathcal{A}. \quad (7)$$

Throughout this paper, $\mathcal{A}^n, \mathbb{R}^{m \times n}, \mathcal{A}^{m \times n}$ represent the n -dimensional real Clifford vector space, the set of all $m \times n$ real matrices, and the set of all $m \times n$ real Clifford matrices. We define the norm of $\mathbb{R}^n$ as $\|x\| = \sum_{i=1}^n |x_i|$. For $A = (a_{ij})_{n \times n}$, denote $\|A\| = \max_{1 \leq i \leq n} \{\sum_{j=1}^n |a_{ij}|\}$ and, for $x = \sum_A x^A e_A \in \mathcal{A}$, denote $\|x\|_{\mathcal{A}} = \sum_A |x^A|$.

In this paper, we consider the following Clifford-valued cellular neural network with discrete delays:

$$\begin{aligned} \dot{X}_i(t) &= -\delta_i(t) X_i(t) + \sum_{j=1}^n \gamma_{ij}(t) f_j(X_j(t)) \\ &\quad + \sum_{j=1}^n \omega_{ij}(t) g_j(X_j(t - \tau_j(t))) + I_i(t), \end{aligned} \quad (8)$$

$t > 0,$

where $i = 1, 2, \dots, n$, and n corresponds to the number of units in the neural network; $X_i(t) \in \mathcal{A}$ denotes the activation of the i th neuron at time t ; $\delta_i(t) > 0$ represents the rate with which the i th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time t ; $\gamma_{ij}(t), \omega_{ij}(t) \in \mathcal{A}$ represent the strengths of connectivity without and with time delay between cells i and j at time t , respectively; $f_j, g_j : \mathcal{A} \rightarrow \mathcal{A}$ are the activation functions of the signal transmission; $I_i(t) \in \mathcal{A}$ is an external

input on the i th unit at time t ; $\tau_j(t) \geq 0$ is the transmission delay at time t .

The initial values of (8) are

$$X_i(t) = \varphi_i(t), \quad t \in [-\tau, 0], \quad i = 1, 2, \dots, n, \quad (9)$$

where $\varphi_i \in C([- \tau, 0], \mathcal{A})$, $\tau = \max_{1 \leq j \leq n} \sup_{t \in \mathbb{R}} \{|\tau_j(t)|\}$.

Let $X(t) = (X_1(t), X_2(t), \dots, X_n(t))^T \in \mathcal{A}^n$, $\Delta(t) = \text{diag}(\delta_1(t), \delta_2(t), \dots, \delta_n(t)) \in \mathbb{R}^{n \times n}$, $\Gamma(t) = (\gamma_{ij}(t))_{n \times n}$, $\Omega(t) = (\omega_{ij}(t))_{n \times n} \in \mathcal{A}^{n \times n}$, $I(t) = (I_1(t), I_2(t), \dots, I_n(t))^T \in \mathcal{A}^n$, $f(X(t)) = (f_1(X_1(t)), f_2(X_2(t)), \dots, f_n(X_n(t)))^T \in \mathcal{A}^n$, and $g(X(t - \tau(t))) = (g_1(X_1(t - \tau_1(t))), g_2(X_2(t - \tau_2(t))), \dots, g_n(X_n(t - \tau_n(t))))^T \in \mathcal{A}^n$. Then (8) can be transformed into the following equation:

$$\begin{aligned} \dot{X}(t) = & -\Delta(t) X(t) + \Gamma(t) f(X(t)) \\ & + \Omega(t) g(X(t - \tau(t))) + I(t), \quad t > 0 \end{aligned} \quad (10)$$

with the initial value:

$$\begin{aligned} X(t) = & \varphi(t), \\ t \in & [-\tau, 0], \quad \varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T. \end{aligned} \quad (11)$$

Definition 1 (see [16]). A function $y \in C(\mathbb{R}, \mathbb{R}^n)$ is said to be almost periodic on $\mathbb{R}$, if, for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$ such that, for any interval with length $l(\varepsilon)$, there is a number $\tau = \tau(\varepsilon)$ in this interval such that $|y(t + \tau) - y(t)| < \varepsilon$.

Denote by $AP(\mathbb{R}, \mathbb{R}^n)$ the set of all almost periodic functions on $\mathbb{R}$; then $AP(\mathbb{R}, \mathbb{R}^n)$ is a Banach space with the supremum norm $\|u\| = \sum_{i=1}^n \sup_{t \in \mathbb{R}} |u_i(t)|$.

Definition 2. A function $f = (f_1, f_2, \dots, f_n)^T : \mathbb{R} \rightarrow \mathcal{A}^n$, where $f_i = \sum_A f_i^A e_A$, $f_i^A : \mathbb{R} \rightarrow \mathbb{R}$, is called almost periodic if every $f_i^A \in AP(\mathbb{R}, \mathbb{R})$, $i = 1, 2, \dots, n$. Denote by $AP(\mathbb{R}, \mathcal{A}^n)$ the set of all such functions.

Consider the following linear system

$$\dot{x}(t) = Q(t) x(t) + f(t), \quad (12)$$

where $Q(t) = \text{diag}\{-q_1(t), -q_2(t), \dots, -q_n(t)\}$, $q_i \in AP(\mathbb{R}, \mathbb{R})$, $f \in AP(\mathbb{R}, \mathbb{R}^n)$.

According to [16], one can easily get the following lemma.

Lemma 3. Let $q_i(t)$ satisfy $\min_{1 \leq i \leq n} \inf_{t \in \mathbb{R}} \{q_i(t)\} > 0$, and then system (12) has a unique almost periodic solution

$$x(t) = \int_{-\infty}^t e^{\int_s^t Q(u) du} f(s) ds. \quad (13)$$

3. The Existence of Almost Periodic Solutions

In this section, we study the existence and uniqueness of almost periodic solutions by the contracting mapping principle.

Firstly, since, for any $W \in \mathcal{A}$, we can find a unique W^C satisfying $W^{B\bar{A}} = (-1)^{n(B\bar{A})} W^C$ for $e_B \bar{e}_A = (-1)^{n(B\bar{A})} e_C$, then, based on $e_A \bar{e}_A = \bar{e}_A e_A = 1$ and $e_B \bar{e}_A e_A = e_B$, we can transform Clifford-valued system (10) into the following real-valued neural network:

$$\begin{aligned} \dot{x}^A(t) = & -\Delta(t) x^A(t) + \sum_B \Gamma^{A\bar{B}}(t) f^B(x(t)) \\ & + \sum_B \Omega^{A\bar{B}}(t) g^B(x^\tau(t)) + I^A(t), \quad A \in \Lambda, \end{aligned} \quad (14)$$

where

$$\begin{aligned} x^A = & (x_1^A, x_2^A, \dots, x_n^A)^T, \\ \sum_A x^A e_A = & X, \\ (-1)^{n(A\bar{B})} \Gamma^C = & \Gamma^{A\bar{B}}, \\ (-1)^{n(A\bar{B})} \Omega^C = & \Omega^{A\bar{B}}, \\ \sum_C \Gamma^C e_C = & \Gamma, \\ \sum_C \Omega^C e_C = & \Omega, \\ e_A \bar{e}_B = & (-1)^{n(A\bar{B})} e_C, \\ \Gamma^{A\bar{B}} = & (\gamma_{ij}^{A\bar{B}})_{n \times n}, \\ \Omega^{A\bar{B}} = & (\omega_{ij}^{A\bar{B}})_{n \times n}, \\ I^A = & (I_1^A, I_2^A, \dots, I_n^A)^T, \\ f^B(x) = & (f_1^B(x_1^{C_1}, x_1^{C_2}, \dots, x_1^{C_{2^m}}), \\ & f_2^B(x_2^{C_1}, x_2^{C_2}, \dots, x_2^{C_{2^m}}), \dots, \\ & f_n^B(x_n^{C_1}, x_n^{C_2}, \dots, x_n^{C_{2^m}}))^T, \\ g^B(x^\tau(t)) = & (g_1^B(x_1^{C_1}(t - \tau_1(t)), x_1^{C_2}(t - \tau_1(t)), \dots, \\ & x_1^{C_{2^m}}(t - \tau_1(t))), g_2^B(x_2^{C_1}(t - \tau_2(t)), x_2^{C_2}(t - \tau_2(t)), \\ & \dots, x_2^{C_{2^m}}(t - \tau_2(t))), \dots, g_n^B(x_n^{C_1}(t - \tau_n(t)), \\ & x_n^{C_2}(t - \tau_n(t)), \dots, x_n^{C_{2^m}}(t - \tau_n(t))))^T. \end{aligned} \quad (15)$$

Remark 4. If $x(t) = (x^0(t), x^1(t), \dots, x^A(t), \dots, x^{1 \cdot 2 \cdot \dots \cdot m}(t))^T$ is a solution of (14), then $X(t) = (X_1, X_2, \dots, X_n)^T = (\sum_A x_1^A(t) e_A, \sum_A x_2^A(t) e_A, \dots, \sum_A x_n^A(t) e_A)^T$ is a solution of (8) and vice versa.

Denote

$$\begin{aligned}
S &= \left((x^0)^T, (x^1)^T, (x^2)^T, \dots, (x^A)^T, \dots, (x^{12\dots m})^T \right)^T \in \mathbb{R}^{2^m n}, \\
\tilde{f}(S) &= \left((f^0(x))^T, (f^1(x))^T, (f^2(x))^T, \dots, (f^A(x))^T, \dots, (f^{12\dots m}(x))^T \right)^T, \\
\tilde{g}(S^\tau) &= \left((g^0(x^\tau))^T, (g^1(x^\tau))^T, (g^2(x^\tau))^T, \dots, (g^A(x^\tau))^T, \dots, (g^{12\dots m}(x^\tau))^T \right)^T, \\
\tilde{I}(t) &= \left((I^0)^T, (I^1)^T, (I^2)^T, \dots, (I^A)^T, \dots, (I^{12\dots m})^T \right)^T, \\
\tilde{\Delta} &= \begin{pmatrix} \Delta & 0 & \dots & 0 \\ 0 & \Delta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta \end{pmatrix}_{2^m n \times 2^m n}, \\
\tilde{\Gamma} &= \begin{pmatrix} \Gamma^0 & \dots & \Gamma^{\bar{A}} & \dots & \Gamma^{\overline{12\dots m}} \\ \Gamma^1 & \dots & \Gamma^{1\cdot\bar{A}} & \dots & \Gamma^{1\cdot\overline{12\dots m}} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \Gamma^{12\dots m} & \dots & \Gamma^{12\dots m\cdot\bar{A}} & \dots & \Gamma^{12\dots m\cdot\overline{12\dots m}} \end{pmatrix}_{2^m n \times 2^m n}, \\
\tilde{\Omega} &= \begin{pmatrix} \Omega^0 & \dots & \Omega^{\bar{A}} & \dots & \Omega^{\overline{12\dots m}} \\ \Omega^1 & \dots & \Omega^{1\cdot\bar{A}} & \dots & \Omega^{1\cdot\overline{12\dots m}} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \Omega^{12\dots m} & \dots & \Omega^{12\dots m\cdot\bar{A}} & \dots & \Omega^{12\dots m\cdot\overline{12\dots m}} \end{pmatrix}_{2^m n \times 2^m n},
\end{aligned} \tag{16}$$

and then (14) can be written as

$$\begin{aligned}
\dot{S}(t) &= -\tilde{\Delta}(t) S(t) + \tilde{\Gamma}(t) \tilde{f}(S(t)) + \tilde{\Omega}(t) \tilde{g}(S^\tau(t)) \\
&\quad + \tilde{I}(t), \quad t > 0
\end{aligned} \tag{17}$$

with the initial value:

$$S(t) = \phi(t), \quad t \in [-\tau, 0]. \tag{18}$$

Set $\mathbb{X} = \{\Phi \mid \Phi = ((\Phi^0)^T, (\Phi^1)^T, \dots, (\Phi^{1\cdot 2\dots m})^T)^T \in AP(\mathbb{R}, \mathbb{R}^{2^m n})\}$ with the norm

$$\begin{aligned}
\|\Phi\|_0 &= \sum_{i=1}^n \sum_A |\Phi_i^A|, \\
|\Phi_i^A| &= \sup_{t \in \mathbb{R}} |\Phi_i^A(t)|,
\end{aligned} \tag{19}$$

$$i = 1, 2, \dots, n,$$

which is a Banach space.

In order to obtain our results, we introduce the following assumptions.

(H₁) For $i, j = 1, 2, \dots, n$, $\delta_i, \tau_j \in AP(\mathbb{R}, \mathbb{R}^+)$, $\gamma_{ij}, \omega_{ij}, I_{ij} \in AP(\mathbb{R}, \mathcal{A})$ and for each $1 \leq i \leq n$, $\min_{1 \leq i \leq n} \{\inf_{t \in \mathbb{R}} C_i(t)\} > 0$.

(H₂) For $i = 1, 2, \dots, n$, $f_i, g_i \in C(\mathcal{A}, \mathcal{A})$ and there exist constants L_i^f, L_i^g such that

$$\begin{aligned}
\|f_i(x) - f_i(y)\|_{\mathcal{A}} &\leq L_i^f \|x - y\|_{\mathcal{A}}, \\
\|g_i(x) - g_i(y)\|_{\mathcal{A}} &\leq L_i^g \|x - y\|_{\mathcal{A}},
\end{aligned} \tag{20}$$

and there exist constants $F, G > 0$ such that $\|f(x)\|_{\mathcal{A}} \leq F, \|g(x)\|_{\mathcal{A}} \leq G$, for any $x, y \in \mathcal{A}$.

Remark 5. By (H₂), we can obtain

$$\begin{aligned}
&\sum_A |f_i^A(y_i^{B_1}, y_i^{B_2}, \dots, y_i^{B_{2^m}}) - f_i^A(x_i^{B_1}, x_i^{B_2}, \dots, x_i^{B_{2^m}})| \\
&\leq L_i^f \sum_B |y_i^B - x_i^B|,
\end{aligned}$$

$$\begin{aligned} & \sum_A \left| g_i^A(y_i^{B_1}, y_i^{B_2}, \dots, y_i^{B_{2^m}}) - g_i^A(x_i^{B_1}, x_i^{B_2}, \dots, x_i^{B_{2^m}}) \right| \\ & \leq L_i^g \sum_B |y_i^B - x_i^B|. \end{aligned} \quad (21)$$

Theorem 6. Let (H_1) and (H_2) hold. Suppose that

$$(H_3) \quad (\tilde{\Gamma}^+ L_f + \tilde{\Omega}^+ L_g)(1/\tilde{\Delta}^-) < 1, \text{ where}$$

$$\begin{aligned} \tilde{\Gamma}^+ &= \sup_{t \in \mathbb{R}} \|\tilde{\Gamma}(t)\|, \\ \tilde{\Omega}^+ &= \sup_{t \in \mathbb{R}} \|\tilde{\Omega}(t)\|, \\ \tilde{\Delta}^- &= \inf_{t \in \mathbb{R}} \|\tilde{\Delta}(t)\|, \\ \tilde{I}^+ &= \sup_{t \in \mathbb{R}} \|\tilde{I}(t)\|, \\ L_f &= \max_{1 \leq i \leq n} \{L_i^f\}, \\ L_g &= \max_{1 \leq i \leq n} \{L_i^g\}. \end{aligned} \quad (22)$$

Then system of (8) has a unique almost periodic solution in the region

$$\begin{aligned} & \mathbb{X}_0 \\ &= \left\{ \Phi \in \mathbb{X} \mid \|\Phi\|_0 \leq (\tilde{\Gamma}^+ F + \tilde{\Omega}^+ G + \tilde{I}^+) \frac{1}{\tilde{\Delta}^-} := R \right\}. \end{aligned} \quad (23)$$

Proof. For $\forall \Phi \in \mathbb{X}$, we consider the following system:

$$\begin{aligned} \dot{S}(t) &= -\tilde{\Delta}(t) S(t) + \tilde{\Gamma}(t) \tilde{f}(\Phi(t)) \\ &+ \tilde{\Omega}(t) \tilde{g}(\Phi(t - \tau(t))) + \tilde{I}(t). \end{aligned} \quad (24)$$

Since (H_1) , it follows from Lemma 3 that system (24) has a unique almost periodic solution

$$\begin{aligned} S(t) &= \int_{-\infty}^t e^{\int_s^t -\tilde{\Delta}(u) du} [\tilde{\Gamma}(s) \tilde{f}(\Phi(s)) \\ &+ \tilde{\Omega}(s) \tilde{g}(\Phi(s - \tau(s))) + \tilde{I}(s)] ds. \end{aligned} \quad (25)$$

We define a mapping $T : \mathbb{X}_0 \rightarrow \mathbb{X}$ as follows:

$$\begin{aligned} (T\Phi)(t) &= \int_{-\infty}^t e^{\int_s^t -\tilde{\Delta}(u) du} [\tilde{\Gamma}(s) \tilde{f}(\Phi(s)) \\ &+ \tilde{\Omega}(s) \tilde{g}(\Phi(s - \tau(s))) + \tilde{I}(s)] ds. \end{aligned} \quad (26)$$

We first prove that $T(\mathbb{X}_0) \subset \mathbb{X}_0$. To this end, for each $\Phi \in \mathbb{X}_0$, we have

$$\begin{aligned} \|(T\Phi)(t)\| &= \left\| \int_{-\infty}^t e^{\int_s^t -\tilde{\Delta}(u) du} [\tilde{\Gamma}(s) \tilde{f}(\Phi(s)) \right. \\ &+ \tilde{\Omega}(s) \tilde{g}(\Phi(s - \tau(s))) + \tilde{I}(s)] ds \Big\| \\ &\leq \int_{-\infty}^t \left\| e^{\int_s^t -\tilde{\Delta}(u) du} \right\| \|\tilde{\Gamma}(s)\| \|\tilde{f}(\Phi(s))\| ds \\ &+ \int_{-\infty}^t \left\| e^{\int_s^t -\tilde{\Delta}(u) du} \right\| \|\tilde{\Omega}(s)\| \|\tilde{g}(\Phi(s - \tau(s)))\| ds \\ &+ \int_{-\infty}^t \left\| e^{\int_s^t -\tilde{\Delta}(u) du} \right\| \|\tilde{I}(s)\| ds \\ &\leq \int_{-\infty}^t e^{-\tilde{\Delta}^-(t-s)} \tilde{\Gamma}^+ \|\tilde{f}(\Phi(s))\| ds \\ &+ \int_{-\infty}^t e^{-\tilde{\Delta}^-(t-s)} \tilde{\Omega}^+ \|\tilde{g}(\Phi(s - \tau(s)))\| ds \\ &+ \int_{-\infty}^t e^{-\tilde{\Delta}^-(t-s)} \tilde{I}^+ ds \leq \tilde{\Gamma}^+ F \int_{-\infty}^t e^{-\tilde{\Delta}^-(t-s)} ds \\ &+ \tilde{\Omega}^+ G \int_{-\infty}^t e^{-\tilde{\Delta}^-(t-s)} ds + \tilde{I}^+ \int_{-\infty}^t e^{-\tilde{\Delta}^-(t-s)} ds \\ &\leq (\tilde{\Gamma}^+ F + \tilde{\Omega}^+ G + \tilde{I}^+) \frac{1}{\tilde{\Delta}^-} = R, \end{aligned} \quad (27)$$

which implies $T\Phi \in \mathbb{X}_0$. Hence, $T(\mathbb{X}_0) \subset \mathbb{X}_0$.

Then, we prove that T is a contracting mapping. In fact, for every $\Phi, \Psi \in \mathbb{X}_0$, we have

$$\begin{aligned} \|T\Phi - T\Psi\| &\leq \int_{-\infty}^t \left\| e^{\int_s^t -\tilde{\Delta}(u) du} \right\| \|\tilde{\Gamma}(s)\| \|\tilde{f}(\Phi(s)) \\ &- \tilde{f}(\Psi(s))\| ds + \int_{-\infty}^t \left\| e^{\int_s^t -\tilde{\Delta}(u) du} \right\| \|\tilde{\Omega}(s)\| \\ &\cdot \|\tilde{g}(\Phi(s - \tau(s))) - \tilde{g}(\Psi(s - \tau(s)))\| ds \\ &\leq \int_{-\infty}^t e^{-\tilde{\Delta}^-(t-s)} \tilde{\Gamma}^+ \|\tilde{f}(\Phi(s)) - \tilde{f}(\Psi(s))\| ds \\ &+ \int_{-\infty}^t e^{-\tilde{\Delta}^-(t-s)} \tilde{\Omega}^+ \|\tilde{g}(\Phi(s - \tau(s))) \\ &- \tilde{g}(\Psi(s - \tau(s)))\| ds \\ &\leq \tilde{\Gamma}^+ L_f \sum_{i=1}^n \sum_B |\Phi_i^B - \Psi_i^B| \int_{-\infty}^t e^{-\tilde{\Delta}^-(t-s)} ds \\ &+ \tilde{\Omega}^+ L_g \sum_{i=1}^n \sum_B |\Phi_i^B - \Psi_i^B| \int_{-\infty}^t e^{-\tilde{\Delta}^-(t-s)} ds \leq (\tilde{\Gamma}^+ L_f \\ &+ \tilde{\Omega}^+ L_g) \frac{1}{\tilde{\Delta}^-} \|\Phi - \Psi\|. \end{aligned} \quad (28)$$

Hence, $T : \mathbb{X}_0 \rightarrow \mathbb{X}_0$ is a contracting mapping. Therefore, by the Banach fixed point theorem, there exists a unique point

$\Phi^* \in \mathbb{X}_0$ such that $T\Phi^* = \Phi^*$; that is, system (14) has a unique almost periodic solution. In view of Remark 4, we know that system (8) has a unique almost periodic solution. The proof is complete. $\square$

4. Almost Periodic Synchronization

In this section, we will investigate the global asymptotic synchronization problem of Clifford-valued CNNs with discrete delays. To this end, we consider the system (8) as the drive system and design the response system as

$$\begin{aligned} \dot{y}_i(t) &= -\delta_i(t) y_i(t) + \sum_{j=1}^n \gamma_{ij}(t) f_j(y_j(t)) \\ &\quad + \sum_{j=1}^n \omega_{ij}(t) g_j(y_j(t - \tau_j(t))) + I_i(t) \\ &\quad + U_i(t), \\ y_i(t) &= \rho_i(t), \quad t \in [-\tau, 0], \quad i = 1, 2, \dots, n, \end{aligned} \quad (29)$$

where $\rho_i \in C([-\tau, 0], \mathcal{A})$, $Y_i(t) \in \mathcal{A}$ represents the state of the response system, $U_i(t)$ is a state-feedback controller, and the rest notation is the same as those in system (8).

Let $z_i^A(t) = y_i^A(t) - x_i^A(t)$; then, from (8) and (29), we obtain the following error system:

$$\begin{aligned} \dot{z}_i(t) &= -\delta_i(t) z_i(t) + \sum_{j=1}^n \gamma_{ij}(t) \\ &\quad \cdot (f_j(y_j(t)) - f_j(x_j(t))) + \sum_{j=1}^n \omega_{ij}(t) \\ &\quad \cdot [g_j(y_j(t - \tau_j(t))) - (g_j(x_j(t - \tau_j(t))))] \\ &\quad + U_i(t), \quad t > 0, \\ z_i(t) &= \varrho_i(t), \quad t \in [-\tau, 0], \quad i = 1, 2, \dots, n, \end{aligned} \quad (30)$$

where $\varrho_i(t) = \rho_i(t) - \varphi_i(t)$. To realize the global asymptotic synchronization of the drive-response system, we choose the following state-feedback controller

$$U_i(t) = \theta_i(t) z_i(t) + \sum_{j=1}^n \pi_{ij}(t) h_j(z_j(t)), \quad (31)$$

where $i = 1, 2, \dots, n$, $\theta_i : \mathbb{R} \rightarrow \mathbb{R}^+$, $\pi_{ij} : \mathbb{R} \rightarrow \mathcal{A}$, $h_j : \mathcal{A} \rightarrow \mathcal{A}$. By using a similar approach of transforming system (8) into system (14) and adopting the similar notation there,

system (30) can be transformed into the following real-valued system:

$$\begin{aligned} \dot{z}_i^A(t) &= -\delta_i(t) z_i^A(t) + \sum_{j=1}^n \sum_B \gamma_{ij}^{A\bar{B}}(t) \\ &\quad \cdot (f_j^B(y_j(t)) - f_j^B(x_j(t))) \\ &\quad + \sum_{j=1}^n \sum_B \omega_{ij}^{A\bar{B}}(t) \\ &\quad \cdot [g_j^B(y_j(t - \tau_{ij}(t))) - (g_j^B(x_j(t - \tau_{ij}(t))))], \quad (32) \\ &\quad t > 0, \\ &\quad + \theta_i(t) z_i^A(t) + \sum_{j=1}^n \sum_B \pi_{ij}^{A\bar{B}}(t) h_j^B(z_j(t)), \\ z_i^A(t) &= \varrho_i^A(t), \\ &\quad t \in [-\tau, 0], \quad t \in [-\tau, 0], \quad i = 1, 2, \dots, n. \end{aligned}$$

Remark 7. Under the premise that system (8) has an almost periodic solution, the synchronization of almost periodic system (8) and almost periodic system (29) is called the almost periodic synchronization.

Theorem 8. Let (H_1) – (H_3) hold. Suppose the following.

(H_4) For $i, j = 1, 2, \dots, n$, $\theta_i \in AP(\mathbb{R}, \mathbb{R})$ and $\pi_{ij} \in AP(\mathbb{R}, \mathcal{A})$.

(H_5) For $i = 1, 2, \dots, n$, h_i satisfy the idea that there exist constants L_i^h such that $\|h_i(x) - h_i(y)\|_{\mathcal{A}} \leq L_i^h \|x - y\|_{\mathcal{A}}$ for all $x, y \in \mathcal{A}$, and $h_i(\mathbf{0}) = \mathbf{0}$.

(H_6) For $t \in \mathbb{R}$, $i = 1, 2, \dots, n$, $\tau_i \in C^1(\mathbb{R}, \mathbb{R}^+)$ with $1 - \tau_i'(t) > 0$ and

$$\begin{aligned} \Theta_i &:= \delta_i^- - \widehat{\theta}_i - \sum_A \sum_B \sum_{j=1}^n \gamma_{ji}^{A\bar{B}} L_i^f - \sum_A \sum_B \sum_{j=1}^n \frac{\omega_{ji}^{A\bar{B}} L_i^g}{1 - \widehat{\tau}_i'} \\ &\quad - \sum_A \sum_B \sum_{j=1}^n \pi_{ji}^{A\bar{B}} L_i^h > 0, \end{aligned} \quad (33)$$

where

$$\widehat{\theta}_i = \sup_{t \in \mathbb{R}} \theta_i(t),$$

$$\widehat{M} = \sup_{t \in \mathbb{R}} \|\overline{M}(t)\|,$$

$$\widehat{\gamma}_{ji}^{A\bar{B}} = \sup_{t \in \mathbb{R}} \|\gamma_{ji}^{A\bar{B}}(t)\|,$$

$$\widehat{\tau}_i' = \sup_{t \in \mathbb{R}} \tau_i'(t),$$

$$\widehat{m}_{ji}^{A\bar{B}} = \sup_{t \in \mathbb{R}} \|m_{ji}^{A\bar{B}}(t)\|,$$

$$\widehat{\omega}_{ji}^{A\bar{B}} = \sup_{t \in \mathbb{R}} \|\omega_{ji}^{A\bar{B}}(t)\|,$$

$$\delta_i^- = \inf_{t \in \mathbb{R}} \delta_i(t),$$

$$L_h = \max_{1 \leq i \leq n} \{L_i^h\}.$$

(34)

Then the drive system (8) and the response system (29) implement global asymptotical almost periodic synchronization.

Proof. By Theorem 6, system (8) has an almost periodic solution. In order to show that system (8) and system (29) are globally asymptotically synchronized, we consider a Lyapunov function

$$V(t) = V_1(t) + V_2(t), \quad (35)$$

where

$$V_1(t) = \sum_{i=1}^n \sum_A |z_i^A(t)|, \quad (36)$$

$$V_2 = \sum_A \sum_B \sum_{i=1}^n \sum_{j=1}^n \hat{\omega}_{ji}^{A\bar{B}} \frac{L_i^g}{1 - \hat{\tau}_i'} \int_{t-\tau_i(t)}^t \sum_Q |z_i^Q(s)| ds.$$

Calculating the derivatives of V_1 and V_2 along the solutions of system (32), we have

$$\begin{aligned} D^+V_1(t) &= \sum_{i=1}^n \sum_A \dot{z}_i^A(t) \operatorname{sign}\{z_i^A(t)\} \\ &\leq -\sum_{i=1}^n \sum_A \delta_i(t) |z_i^A(t)| + \sum_{i=1}^n \sum_A \sum_B \sum_{j=1}^n |\gamma_{ij}^{A\bar{B}}(t)| |f_j^B(y_j(t)) - f_j^B(x_j(t))| \\ &\quad + \sum_{i=1}^n \sum_A \sum_B \sum_{j=1}^n |\omega_{ij}^{A\bar{B}}(t)| |g_j^B(y_j(t - \tau_j(t)) - (g_j^B(x_j(t - \tau_j(t))))| + \sum_{i=1}^n \sum_A \theta_i(t) |z_i^A(t)| \\ &\quad + \sum_{i=1}^n \sum_A \sum_B \sum_{j=1}^n |\pi_{ij}^{A\bar{B}}(t)| |h_j^B(z_j(t))| \\ &\leq \sum_{i=1}^n \sum_A |z_i^A(t)| (-\delta_i^- + \hat{\theta}_i) + \sum_A \sum_B \sum_{i=1}^n \sum_{j=1}^n \hat{\gamma}_{ji}^{A\bar{B}} L_i^f \sum_Q |z_i^Q(t)| + \sum_A \sum_B \sum_{i=1}^n \sum_{j=1}^n \hat{\omega}_{ji}^{A\bar{B}} L_i^g \sum_Q |z_i^Q(t - \tau_i(t))| \\ &\quad + \sum_A \sum_B \sum_{i=1}^n \sum_{j=1}^n \hat{\pi}_{ji}^{A\bar{B}} L_i^h \sum_Q |z_i^Q(t)| \end{aligned} \quad (37)$$

and

$$\begin{aligned} D^+V_2(t) &= \sum_A \sum_B \sum_{i=1}^n \sum_{j=1}^n \frac{\hat{\omega}_{ji}^{A\bar{B}} L_i^g}{1 - \hat{\tau}_i'} \sum_Q |z_i^Q(t)| \\ &\quad - \sum_A \sum_B \sum_{i=1}^n \sum_{j=1}^n \frac{\hat{\omega}_{ji}^{A\bar{B}} L_i^g (1 - \tau_i'(t))}{1 - \hat{\tau}_i'} \\ &\quad \cdot \sum_Q |z_i^Q(t - \tau_i(t))|, \\ &\leq \sum_A \sum_B \sum_{i=1}^n \sum_{j=1}^n \frac{\hat{\omega}_{ji}^{A\bar{B}} L_i^g}{1 - \hat{\tau}_i'} \sum_Q |z_i^Q(t)| \\ &\quad - \sum_A \sum_B \sum_{i=1}^n \sum_{j=1}^n \hat{\omega}_{ji}^{A\bar{B}} L_i^g \sum_Q |z_i^Q(t - \tau_i(t))|. \end{aligned} \quad (38)$$

Hence,

$$\begin{aligned} D^+V(t) &= D^+V_1(t) + D^+V_2(t) \leq \sum_{i=1}^n \sum_A |z_i^A(t)| (-\delta_i^- \\ &\quad + \hat{\theta}_i) + \sum_A \sum_B \sum_{i=1}^n \sum_{j=1}^n \left(\hat{\gamma}_{ji}^{A\bar{B}} L_i^f + \frac{\hat{\omega}_{ji}^{A\bar{B}} L_i^g}{1 - \hat{\tau}_i'} \right) \end{aligned}$$

$$\begin{aligned} &\cdot \sum_Q |z_i^Q(t)| + \sum_A \sum_B \sum_{i=1}^n \sum_{j=1}^n \hat{\pi}_{ji}^{A\bar{B}} L_i^h \sum_Q |z_i^Q(t)| \\ &= \sum_{i=1}^n \sum_Q |z_i^Q(t)| \left[-\delta_i^- + \hat{\theta}_i + \sum_A \sum_B \sum_{j=1}^n \hat{\gamma}_{ji}^{A\bar{B}} L_i^f \right. \\ &\quad \left. + \sum_A \sum_B \sum_{j=1}^n \frac{\hat{\omega}_{ji}^{A\bar{B}} L_i^g}{1 - \hat{\tau}_i'} + \sum_A \sum_B \sum_{j=1}^n \hat{\pi}_{ji}^{A\bar{B}} L_i^h \right] \leq -V_1(t) \\ &\cdot \min_{1 \leq i \leq n} \{\Theta_i\}. \end{aligned} \quad (39)$$

Integrating the above inequality over interval $[t_0, t]$, we get

$$V(t) + \min_{1 \leq i \leq n} \{\Theta_i\} \int_{t_0}^t V_1(s) ds \leq V(t_0) < +\infty, \quad (40)$$

$t \geq t_0.$

Thus,

$$\lim_{t \rightarrow +\infty} \sup_{s \geq t_0} \int_{t_0}^s V_1(s) ds \leq V(t_0) \leq \frac{V(t_0)}{\min_{1 \leq i \leq n} \{\Theta_i\}} < +\infty. \quad (41)$$

Therefore, $\lim_{t \rightarrow +\infty} V_1(t) = 0$. By the definition of $V_1(t)$, we obtain that $\lim_{t \rightarrow +\infty} |y_i^B(t) - x_i^B(t)| = 0$, for all $B \in \Lambda$. Hence, the drive system (8) and the response system (29) are globally asymptotically synchronized. The proof is complete. $\square$

5. A Numerical Example

In this section, we present an example to illustrate our results.

Example 1. For $m = 2, n = 2$, take the following Clifford-valued CNN as the drive system:

$$\begin{aligned} \dot{x}(t) &= -\Delta(t)x(t) + \Gamma(t)f(x(t)) \\ &\quad + \Omega(t)g(x(t - \tau(t))) + I(t), \quad t > 0, \\ x(t) &= \varphi(t), \quad t \in [-\tau, 0] \end{aligned} \quad (42)$$

and the response system is designed as

$$\begin{aligned} \dot{y}(t) &= -\Delta(t)y(t) + \Gamma(t)f(y(t)) \\ &\quad + \Omega(t)g(y(t - \tau(t))) + I(t) + U(t), \\ &\quad t > 0, \end{aligned} \quad (43)$$

$$x(t) = \phi(t), \quad t \in [-\tau, 0],$$

where

$$\begin{aligned} \Delta(t) &= \begin{pmatrix} 2 + \sin \sqrt{2}t & 0 \\ 0 & 4 + \sin t \end{pmatrix}, \\ \Gamma(t) &= \begin{pmatrix} 0.2 \sin t + \sin te_1 & 0.1 + 0.3 \cos te_2 + 0.6e_{12} \\ 0.05 - 0.2 \cos \sqrt{5}te_2 + 0.4e_{12} & 0.1 + 0.2e_1 + 0.05 \sin \sqrt{3}te_{12} \end{pmatrix}, \\ \Omega(t) &= \begin{pmatrix} 0.3 \sin \sqrt{3}t + 0.01 \sin te_1 & 0.1 + 0.02 \cos \sqrt{3}te_2 + 0.3e_{12} \\ 0.05 - 0.2 \sin te_2 + 0.05e_{12} & 0.2 + 0.2e_1 + 0.05 \sin \sqrt{3}te_{12} \end{pmatrix}, \\ f_j(x_j) &= \frac{1}{65} \sin x_j^0 - \frac{1}{70} \sin \sqrt{2}x_j^2 e_2, \\ g_j(x_j) &= \frac{1}{58} \cos \sqrt{3}x_j^2 e_2 - \frac{1}{55} \sin \sqrt{6}x_j^{12} e_{12}, \\ U(t) &= \theta_i(t)e(t) + \Pi(t)h(z(t - \sigma(t))), \\ h_j(z_j) &= \frac{1}{75} \sin z_j^0 - \frac{1}{68} \sin \sqrt{2}z_j^2 e_2, \\ \theta(t) &= \begin{pmatrix} 0.01 + 0.03 \sin 2t & 0 \\ 0 & 0.1 + 0.05 \cos \sqrt{2}t \end{pmatrix}, \\ \begin{pmatrix} I_1(t) \\ I_2(t) \end{pmatrix} &= \begin{pmatrix} \frac{1}{10} \sin x_1^0(t) + \frac{1}{15} \sin \sqrt{3}x_1^{12}(t)e_{12} \\ \frac{1}{10} \sin x_2^0(t) + \frac{1}{15} \sin \sqrt{3}x_2^{12}(t)e_{12} \end{pmatrix}, \\ \begin{pmatrix} \tau_1(t) \\ \tau_2(t) \end{pmatrix} &= \begin{pmatrix} 0.1 \sin 3.2t + 0.2 \\ 0.1 \sin 3.2t + 0.3 \end{pmatrix}, \\ \Pi(t) &= \begin{pmatrix} 0.2 \sin \sqrt{2}t + 0.01 \sin te_1 & 0.1 + 0.3 \cos te_2 + 0.6e_{12} \\ 0.05 - 0.2 \cos \sqrt{5}te_2 + 0.3e_{12} & 0.1 + 0.2e_1 + 0.05 \sin \sqrt{3}te_{12} \end{pmatrix}. \end{aligned} \quad (44)$$

According to their definitions, we have

$$\begin{aligned}
 \Gamma^0(t) &= \begin{pmatrix} 0.2 \sin t & 0.1 \\ 0.05 & 0.1 \end{pmatrix}, \\
 \Gamma^1(t) &= \begin{pmatrix} \sin t & 0 \\ 0 & 0.2 \end{pmatrix}, \\
 \Gamma^2(t) &= \begin{pmatrix} 0 & 0.3 \cos t \\ -0.2 \cos \sqrt{5}t & 0 \end{pmatrix}, \\
 \Gamma^{12}(t) &= \begin{pmatrix} 0 & 0.6 \\ 0.4 & 0.05 \sin \sqrt{3}t \end{pmatrix}, \\
 \Omega^0(t) &= \begin{pmatrix} 0.3 \sin \sqrt{3}t & 0.1 \\ 0.05 & 0.2 \end{pmatrix}, \\
 \Omega^1(t) &= \begin{pmatrix} 0.01 \sin t & 0 \\ 0 & 0.2 \end{pmatrix}, \\
 \Omega^2(t) &= \begin{pmatrix} 0 & 0.02 \cos \sqrt{3}t \\ -0.2 \sin t & 0 \end{pmatrix}, \\
 \Omega^{12}(t) &= \begin{pmatrix} 0 & 0.3 \\ 0.05 & 0.05 \sin \sqrt{3}t \end{pmatrix}, \\
 \Pi^0(t) &= \begin{pmatrix} 0.2 \sin \sqrt{2}t & 0.1 \\ 0.05 & 0.1 \end{pmatrix}, \\
 \Pi^1(t) &= \begin{pmatrix} 0.01 \sin t & 0 \\ 0 & 0.2 \end{pmatrix}, \\
 \Pi^2(t) &= \begin{pmatrix} 0 & 0.3 \cos t \\ -0.2 \cos \sqrt{5}t & 0 \end{pmatrix}, \\
 \Pi^{12}(t) &= \begin{pmatrix} 0 & 0.6 \\ 0.3 & 0.05 \sin \sqrt{3}t \end{pmatrix},
 \end{aligned} \tag{45}$$

and

$$\begin{aligned}
 \tilde{\Gamma} &= \begin{pmatrix} \Gamma^0 & \Gamma^1 & \Gamma^2 & \Gamma^{12} \\ \Gamma^1 & \Gamma^{1\cdot\bar{1}} & \Gamma^{1\cdot\bar{2}} & \Gamma^{1\cdot\bar{12}} \\ \Gamma^2 & \Gamma^{2\cdot\bar{1}} & \Gamma^{2\cdot\bar{2}} & \Gamma^{2\cdot\bar{12}} \\ \Gamma^{12} & \Gamma^{12\cdot\bar{1}} & \Gamma^{12\cdot\bar{2}} & \Gamma^{12\cdot\bar{12}} \end{pmatrix} \\
 &= \begin{pmatrix} \Gamma^0 & -\Gamma^1 & -\Gamma^2 & -\Gamma^{12} \\ \Gamma^1 & \Gamma^0 & -\Gamma^{12} & \Gamma^2 \\ \Gamma^2 & \Gamma^{12} & \Gamma^0 & -\Gamma^1 \\ \Gamma^{12} & -\Gamma^2 & \Gamma^1 & \Gamma^0 \end{pmatrix},
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\Omega} &= \begin{pmatrix} \Omega^0 & \Omega^1 & \Omega^2 & \Omega^{12} \\ \Omega^1 & \Omega^{1\cdot\bar{1}} & \Omega^{1\cdot\bar{2}} & \Omega^{1\cdot\bar{12}} \\ \Omega^2 & \Omega^{2\cdot\bar{1}} & \Omega^{2\cdot\bar{2}} & \Omega^{2\cdot\bar{12}} \\ \Omega^{12} & \Omega^{12\cdot\bar{1}} & \Omega^{12\cdot\bar{2}} & \Omega^{12\cdot\bar{12}} \end{pmatrix} \\
 &= \begin{pmatrix} \Omega^0 & -\Omega^1 & -\Omega^2 & -\Omega^{12} \\ \Omega^1 & \Omega^0 & -\Omega^{12} & \Omega^2 \\ \Omega^2 & \Omega^{12} & \Omega^0 & -\Omega^1 \\ \Omega^{12} & -\Omega^2 & \Omega^1 & \Omega^0 \end{pmatrix}, \\
 \tilde{\Pi} &= \begin{pmatrix} \Pi^0 & \Pi^1 & \Pi^2 & \Pi^{12} \\ \Pi^1 & \Pi^{1\cdot\bar{1}} & \Pi^{1\cdot\bar{2}} & \Pi^{1\cdot\bar{12}} \\ \Pi^2 & \Pi^{2\cdot\bar{1}} & \Pi^{2\cdot\bar{2}} & \Pi^{2\cdot\bar{12}} \\ \Pi^{12} & \Pi^{12\cdot\bar{1}} & \Pi^{12\cdot\bar{2}} & \Pi^{12\cdot\bar{12}} \end{pmatrix} \\
 &= \begin{pmatrix} \Pi^0 & -\Pi^1 & -\Pi^2 & -\Pi^{12} \\ \Pi^1 & \Pi^0 & -\Pi^{12} & \Pi^2 \\ \Pi^2 & \Pi^{12} & \Pi^0 & -\Pi^1 \\ \Pi^{12} & -\Pi^2 & \Pi^1 & \Pi^0 \end{pmatrix}.
 \end{aligned} \tag{46}$$

By calculating, we obtain $\tilde{\Delta}^- = 1$, $\tilde{\Gamma}^+ = 1.6$, $L_1^f = L_2^f = 1/65$, $L_1^g = L_2^g = 1/55$, $\tilde{\Omega}^+ = 0.75$, $L_1^h = L_2^h = 1/68$, $(\gamma_{ji}^{A\bar{B}})^+ = 0.6$, $(\omega_{ji}^{A\bar{B}})^+ = 0.3$, $(\pi_{ji}^{A\bar{B}})^+ = 0.6$, $\hat{\theta}_1 = 0.04$, $\hat{\theta}_2 = 0.15$, $c_1^- = 1$, $c_2^- = 3$, $\hat{\tau}_i' = 0.32$, $i, j = 1, 2$.

It is easy to get the following inequalities:

$$\begin{aligned}
 &(\tilde{\Gamma}^+ L_f + \tilde{\Omega}^+ L_g) \frac{1}{\tilde{\Delta}^-} < 1, \\
 &\min_{1 \leq i \leq 2} \left\{ -\delta_i^- + \hat{\theta}_i + \sum_A \sum_B \sum_{j=1}^2 (\gamma_{ji}^{A\bar{B}})^+ L_i^f \right. \\
 &\quad \left. + \sum_A \sum_B \sum_{j=1}^2 \frac{(\omega_{ji}^{A\bar{B}})^+ L_i^g}{1 - \hat{\tau}_i'} + \sum_A \sum_B \sum_{j=1}^2 (\pi_{ji}^{A\bar{B}})^+ L_i^h \right\} \\
 &\approx -0.015 < 0.
 \end{aligned} \tag{47}$$

Therefore, all the conditions of Theorem 8 are satisfied. Hence, by Theorem 8, the drive system (42) and the response system (43) implement global asymptotic almost periodic synchronization (see Figures 1–5).

6. Conclusion

In this paper, we studied the problem of the global asymptotic almost periodic synchronization for a class of Clifford-valued CNNs with discrete delays by decomposing the considered

FIGURE 1: The states of $x^0(t)$, $x^1(t)$, $x^2(t)$, and $x^{12}(t)$ in (42) without any controller.

FIGURE 2: The states of four parts of $x_1(t)$ and $x_2(t)$ in 2-dimensional space without control.

FIGURE 3: The states of $y^0(t)$, $y^1(t)$, $y^2(t)$, and $y^{12}(t)$ in (43) under controller $U(t)$.

FIGURE 4: The states of four parts of $y_1(t)$ and $y_2(t)$ in 2-dimensional space under controller $U(t)$.

FIGURE 5: Synchronization errors $z(t) = y(t) - x(t)$.

neural networks into $2^{m^2}n$ real-valued systems and by designing a new state-feedback controller. Our results are new and our methods can be used to study other types of Clifford-valued neural networks.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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