In this paper, a robust adaptive sliding mode control scheme is developed for attitude and altitude tracking of a quadrotor unmanned aerial vehicle (UAV) system under the simultaneous effect of parametric uncertainties and consistent external disturbance. The underactuated dynamic model of the quadrotor UAV is first built via the Newton–Euler formalism. Considering the nonlinear and strongly coupled characteristics of the quadrotor, the controller is then designed using a sliding mode approach. Meanwhile, additional adaptive laws are proposed to further improve the robustness of the proposed control scheme against the parametric uncertainties of the system. It is proven that the control laws can eliminate the altitude and attitude tracking errors, which are guaranteed to converge to zero asymptotically, even under a strong external disturbance. Finally, numerical simulation and experimental tests are performed, respectively, to verify the effectiveness and robustness of the proposed controller, where its superiority to linear quadratic control and active disturbance rejection control has been demonstrated clearly.
quadrotor system. Rios et al. and Basci et al. [27, 28] develop a continuous SMC approach to deal with the robust output tracking control problem for a quadrotor. In addition, a fractional-order SMC scheme has been suggested to stabilize the attitude of quadrotors [29, 30].

Besides pure SMC, integration of SMC with other control approaches is also used to realize attitude control and to accomplish the task of trajectory tracking for quadrotor systems. More clearly, Yang et al. [31] consider a hybrid state-feedback control scheme that incorporates robust SMC and optimal linear quadratic regulator (LQR) in a hierarchical multiple-layer structure for a small-scale quadrotor. In [32], a robust nonlinear controller that combines SMC and backstepping technique is proposed, where the regular SMC for attitude inner loop is first developed to guarantee fast convergence of attitude angles and backstepping control is then applied to position loop until the desired attitude is obtained.

It is worth pointing out that, in the aforementioned works, parametric uncertainties and external disturbance have not been considered simultaneously under the framework of SMC. As a matter of fact, the parameters of a quadrotor, e.g., the moments of inertia, are difficult to be measured directly. It is also hard to get the exact values of these parameters even if they can be measured in certain ways. Meanwhile, quadrotor UAVs are inevitably subject to external disturbances during flight, such as wind gusts.

To address the problems of parameter perturbation and external disturbances, an integral predictive/nonlinear $H_{\infty}$ controller with robust performance is presented in the case of aerodynamic disturbance as well as uncertain mass and inertial terms [33]. In [34], to compensate for the time-varying parameter uncertainties and external disturbances, a robust guaranteed cost controller and an optimal robust guaranteed cost controller are presented for the set-point tracking of the quadrotor UAV, respectively. A robust attitude stabilization controller is proposed, which consists of a nominal state-feedback controller and a robust compensator, for quadrotor systems under the influences of nonlinear and coupling dynamics, including parametric uncertainties, unmodeled uncertainties, and external disturbances [35]. Besides, in [36], by combining with SMC, a robust backstepping-based approach is investigated for position and attitude tracking of a quadrotor UAV subject to external disturbances and parameter uncertainties, associated with the presence of aerodynamic forces and possible wind force. Moreover, a hierarchical control strategy based on adaptive radial basis function neural networks (RBFNNs) and double-loop integral SMC is presented in [37] for the trajectory tracking of the quadrotor UAV.

In this work, the quadrotor system is considered subject to nonlinear and coupling dynamics, parametric uncertainties, and external disturbances. A novel robust adaptive sliding mode controller (ASMC) is developed to asymptotically reduce the altitude and attitude tracking errors. The main contributions of the paper are summarized as follows: (1) associated with SMC, adaptive laws are designed for simultaneous estimation of all parameters of UAV including mass, arm length, moments of inertia, and coefficients of the quadrotor system. Different from the most parameter-estimation-based controllers that always assume the availability of partial parameters of the UAV and estimate the remaining, the proposed controller shows stronger applicability for different types of quadrotor UAV. (2) The robustness of the proposed controller is guaranteed by the novel learning mechanism that takes a parallel structure. (3) The tracking convergence of the ASMC is verified by rigorous Lyapunov-based analysis, simulations, as well as real experiments.

The remainder of this paper is organized as follows. In Section 2, the dynamics of the quadrotor is presented. The control strategy is exposed in detail in Section 3. In Section 4, both simulation and experiment are conducted to verify the effectiveness of the proposed controller. Section 5 concludes the work.

2. Mathematical Model of Quadrotor UAV

In this section, the mathematical model and control principle of the quadrotor UAV are introduced. The schematic diagram of the quadrotor is shown in Figure 1, where $E$ and $B$ represent the earth and body frames, respectively. In flight experiment of the quadrotor UAV, rotors 1 and 4 are controlled to rotate clockwise, while rotors 2 and 3 are controlled to rotate counterclockwise. It is worth noticing that the left or right motion of the quadrotor UAV will create a roll angle by increasing (decreasing) the speeds of rotors 1 and 3 and decreasing (increasing) the speeds of rotors 2 and 4. Meanwhile, in order to obtain the pitch angle around the $y$-axis, the quadrotor can be controlled by increasing (decreasing) the speeds of rotors 1 and 2 and decreasing (increasing) the speeds of rotors 3 and 4. Furthermore, to rotate the quadrotor around the $z$-axis, it is suggested to increase (decrease) the speeds of rotors 1 and 4 and decrease (increase) the speeds of rotors 2 and 3. In addition, taking-off/landing tasks can be accomplished by increasing or decreasing all the rotors’ speed uniformly.

As usual, the following assumptions are given to make the consequent controller design and analysis rigorous [10, 38]:

(i) The structure of the quadrotor UAV is rigid, and there are no internal forces or deformations
(ii) The structure of the quadrotor UAV is symmetrical
(iii) The center of mass and the origin of body frame coincide
(iv) The earth frame is an inertial frame

The six degrees of freedom of the quadrotor UAV are defined as

$$[x, y, z, \phi, \theta, \varphi] \in \mathbb{R}^6,$$  

where $[x, y, z]$ is the position vector relative to the inertial frame $E$ and $[\phi, \theta, \varphi]$ denotes the vector of the roll angle around the $x$-axis, the pitch angle around the $y$-axis, as well as the yaw angle around the $z$-axis. The rotation matrices around each of the three axes are
The total lift of the quadrotor UAV will be exploited consequently. We first address the translational dynamics. Notice that the total lift force generated by the rotors can be given in the body frame as

\[
F_e^b = \left[ k_x \dot{x}, k_y \dot{y}, k_z \dot{z} \right]^T, 
\]

where \(k_x, k_y, \) and \(k_z\) are the air resistance coefficients in the three directions. Moreover, the effect of gravity of the quadrotor can be described in the earth frame as

\[
F_g^e = [0, 0, mg]^T, 
\]

where \(m\) is the mass of quadrotor. In summary, applying Newton’s second law, the translational dynamics of the quadrotor UAV subjected to external disturbance can be obtained by (2)–(9) \cite{14, 27}.

\[
\ddot{x} = \frac{U_1 (\sin \theta \cos \phi \cos \varphi + \sin \phi \sin \varphi) - k_x \dot{x}}{m} + d_x, 
\]

\[
\ddot{y} = \frac{U_1 (\sin \theta \cos \phi \sin \varphi - \sin \phi \cos \varphi) - k_y \dot{y}}{m} + d_y, 
\]

\[
\ddot{z} = \frac{U_1 (\cos \phi \cos \theta) - k_z \dot{z} - mg}{m} + d_z, 
\]

where \(d_x, d_y, \) and \(d_z\) represent the effect of wind gusts on the translational accelerations in the form of additive external disturbances.

Then, the rotational dynamics of the quadrotor UAV are addressed. Let the vector \([p, q, r]\) represent the quadrotor’s angular velocity in the body frame. The angular motion dynamics of the quadrotor can be expressed as \cite{13, 16}

\[
M_x = J_x \dot{p} + (J_z - J_y) qr, 
\]

\[
M_y = J_y \dot{q} + (J_x - J_z) pr, 
\]

\[
M_z = J_z \dot{r} + (J_y - J_x) pq, 
\]

where \(M_x, M_y, \) and \(M_z\) are the components of the resultant torque acting on the quadrotor in the three directions of \(x, y, \) and \(z\), respectively, and \(J_x, J_y, \) and \(J_z\) are the inertias of the quadrotor around \(x, y, \) and \(z\), respectively.

Meanwhile, let \(M_{lx}, M_{ly}, \) and \(M_{lz}\) denote the components of the lifting torque of the quadrotor in the three directions of \(x, y, \) and \(z\), respectively. They can be expressed as \cite{32}

\[
M_{lx} = LU_2, 
\]

\[
M_{ly} = LU_3, 
\]

\[
M_{lz} = fu_4, 
\]

where \(L\) is the arm length of the quadrotor and \(f\) is the scaling factor from force to moment. In addition, \(U_2, U_3, \) and \(U_4\) are attitude control inputs and defined as \(U_2 \equiv (F_1 + F_3 - F_2 - F_4), \)

\(U_3 \equiv (F_3 + F_4 - F_1 - F_2), \) and \(U_4 \equiv (F_1 + F_4 - F_2 - F_3), \) respectively.

Besides, the quadrotor is subject to gyroscopic torque during flight, whose components \(M_{gx}, M_{gy}, \) and \(M_{gz}\) around the \(x, y, \) and \(z\) axes, respectively, are given by \cite{32}

\[
M_{gx} = J_i q \Omega_{1i}, 
\]

\[
M_{gy} = J_i p \Omega_{1i}, 
\]

\[
M_{gz} = 0, 
\]

In (13), \(J_i\) is the inertia of each propeller and \(\Omega_{1i} \equiv \Omega_2 + \Omega_3 - \Omega_1 - \Omega_4, \) where \(\Omega_i, i = 1, \ldots, 4\) stands for the angular speed of the \(i\)th propeller.
Considering (11)–(13), the angular motion dynamics of the quadrotor can be rewritten in the following form:

\[
\begin{align*}
J_x \ddot{\phi} &= (J_y - J_z) q r - J_r q \Omega_{11} + LU_2, \\
J_y \ddot{\theta} &= (J_z - J_x) p r + J_p \Omega_{11} + LU_3, \\
J_z \ddot{\phi} &= (J_x - J_y) p q + f U_4.
\end{align*}
\] (14)

Compared with the brushless motor, the propeller is light. Hence, it is reasonable to ignore the moment of inertia caused by the propellers here [26]. Notice that the transformation matrix from \([\phi, \theta, \psi]\) to \([p, q, r]\) is given by [16, 17]

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
0 & \cos \phi & -\sin \phi \\
\sin \phi \cot \theta & \cos \phi \cot \theta
\end{bmatrix}
\begin{bmatrix}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{bmatrix},
\]
(15)

To ensure flight safety, the attitude angles of the quadrotor are always kept small purposely during the flight. Thus, it follows from (15) that \([\phi, \theta, \psi]^T = [p, q, r]^T\) [39]. In consequence, the rotational dynamics of the quadrotor subjected to external disturbance [14, 27] can be derived from (14):

\[
\begin{align*}
\ddot{\phi} &= \left( J_y - J_z \right) \dot{\phi} r - J_r \dot{\phi} \Omega_{11} + LU_2 + d_\phi, \\
\ddot{\theta} &= \left( J_z - J_x \right) \dot{\theta} p r + J_p \dot{\theta} \Omega_{11} + LU_3 + d_\theta, \\
\ddot{\psi} &= \left( J_x - J_y \right) \dot{\psi} p q + f U_4 + d_\psi,
\end{align*}
\] (16)

where the terms \(d_\phi, d_\theta, \text{ and } d_\psi\) denote the effect of wind gusts on the quadrotor’s angular accelerations in the form of additive external disturbances.

3. Controller Design and Stability Analysis

In this section, a robust ASMC scheme is proposed for the quadrotor UAV system with the goal of tracking the desired altitude and attitude in the presence of parametric uncertainties and consistent external disturbance. The whole control scheme is depicted in Figure 2. It is worth mentioning that although dynamical coupling exists between the translational dynamics (10) and the rotational dynamics (16), \(U_i, i = 1, \ldots, 4\) can be designed separately [26, 40].

**Remark 1.** In the control of the quadrotor, the changes of the states \(\phi\) and \(\theta\) will result in movement in the \(x\) and \(y\) directions, respectively. Therefore, instead of considering the full six degrees of freedom of the UAV, it is sufficient to control the four degrees of freedom (\(\phi, \theta, \psi, \text{ and } z\)) only. In this regard, the considered quadrotor UAV system has a fully actuated dynamic model [38], where four independent control variables are to be designed.
where the altitude controller $U_1$ and the roll, pitch, and yaw controllers $U_i$, $i = 2, \ldots, 4$ will be designed under the framework of ASMC.

To make the subsequent analysis concise and readable, some of the involved disturbance components in (18) are rescaled as $\ddot{d}_x \equiv m \ddot{x}_d$, $\ddot{d}_\psi \equiv (J_f f) \ddot{d}_\psi$, and some of the parameters are redefined as $\beta_1 \equiv \beta_1 k_1$, $\beta_2 \equiv \beta_2 k_2$, $\beta_3 \equiv \beta_3 k_3$, $\beta_4 \equiv (I_y - I_z) / L$, $\beta_5 \equiv (I_x - I_z) / L$, $\beta_6 \equiv (I_z - I_y) / L$, and $\beta_7 \equiv (J_f / I_f)$. Moreover, let $\hat{\beta}_i$ denote the estimation of $\beta_i$, $i = 1, \ldots, 8$. Then, $\hat{\beta}_i - \beta_i$ is the estimation error.

**Assumption 1.** It is assumed that all the disturbances are uniformly bounded, i.e., $|d_x| \leq \eta_x$, $|d_\psi| \leq \eta_\psi$, $|d_\phi| \leq \eta_\phi$, $|\ddot{d}_x| \leq \eta_1$, $|\ddot{d}_\psi| \leq \eta_2$, $|\ddot{d}_\phi| \leq \eta_3$, and $|\ddot{d}_\psi| \leq \eta_4$ with known positive constants $\eta_x, \eta_\psi, \eta_\phi, \eta_1, \eta_2, \eta_3, \eta_4$.

**Assumption 2.** In the design of controller for the quadrotor, to avoid any singularity, we set [32, 33, 41]
\begin{align}
\pi / 2 < x_\phi < \pi / 2, \\
\pi / 2 < x_\theta < \pi / 2.
\end{align}

### 3.2. Altitude Controller Design
We first address $U_1$. Define the altitude error as
\[
e_z = x_{sd} - x_5,
\]
where $x_{sd}$ is the desired altitude, and set the sliding surface as
\[
s_1 = c_1 e_z + \dot{e}_z,
\]
with $c_1$ being a positive constant.

The control law $U_1$ of altitude is designed as
\[
U_1 = \frac{1}{\cos x_\phi \cos x_\theta} \left( \hat{\beta}_1 (c_1 e_z + x_{sd} + g) + \hat{\beta}_2 x_\phi + \eta_1 \text{sgn}(s_1) + k_1 s_1 \right),
\]
where $\text{sgn}(\cdot)$ denotes the signum function, $k_1$ a positive constant, and $\hat{\beta}_j$, $j = 1, 2$ are learned in the following way:
\[
\hat{\beta}_1 = \gamma_1 s_1 (c_1 e_z + x_{sd} + g),
\]
\[
\hat{\beta}_2 = \gamma_2 s_1 x_\phi,
\]
where $\gamma_1$ and $\gamma_2$ are positive learning gains.

The first main result of the paper is summarized as follows.

**Theorem 1.** Consider the altitude dynamic model of the quadrotor UAV in the state-space form (18) with controllers (22) and adaptive laws (23). Under Assumptions 1 and 2, the altitude output tracking error of the closed-loop system is guaranteed to converge to zero asymptotically, i.e., $e_z \to 0$ as $t \to \infty$.

---

**Proof 1.** Define the following Lyapunov function candidate:
\[
V_z = \frac{1}{2} m s_1^2 + \frac{1}{2} \beta_1^2 + \frac{1}{2} \beta_2^2,
\]
whose time derivative is
\[
\dot{V}_z = m s_1 \dot{s}_1 + \frac{1}{2} \beta_1 \dot{s}_2 + \frac{1}{2} \beta_2 \dot{s}_2.
\]

For the first term on the right-hand side of (25), notice from (20) and (21) that the derivative of $s_1$ is given by
\[
\dot{s}_1 = c_1 \dot{e}_z + \dot{e}_z + k_1 \dot{x}_6 - \ddot{x}_6.
\]

By virtue of (26) and the sixth equation in (18), it follows that
\[
ms_1 \dot{s}_1 = ms_1 (c_1 \dot{e}_z + \ddot{x}_5 - \ddot{x}_6)
\]
\[
= s_1 (m (c_1 \dot{e}_z + \ddot{x}_5) - U_1 \cos x_\phi \cos x_\theta)
\]
\[
- k_1 \dot{x}_6 + mg \dot{d}_x)
\]
\[
= s_1 (m (c_1 \dot{e}_z + \ddot{x}_5 + s_1 mg + k_1 \dot{x}_6)
\]
\[
- s_1 U_1 \cos x_\phi \cos x_\theta - s_1 \ddot{x}_6)
\]
\[
= m (s_1 (c_1 \dot{e}_z + \ddot{x}_5 + g) + k_1 s_1 x_6
\]
\[
- s_1 U_1 \cos x_\phi \cos x_\theta - s_1 \ddot{x}_6).
\]

Further substituting the altitude controller (22) into (27) renders to
\[
ms_1 \dot{s}_1 = \beta_1 (s_1 (c_1 \dot{e}_z + \ddot{x}_5 + g) + \beta_2 (s_1 x_6
\]
\[
- \dot{\beta}_1 (s_1 (c_1 \dot{e}_z + \ddot{x}_5 + g) - \beta_2 (s_1 x_6)
\]
\[
- s_1 \eta_1 \text{sgn}(s_1) - k_1 s_1^2 - s_1 \ddot{x}_6).
\]

Applying (28) into (25) yields
\[
\dot{V}_z = \dot{\beta}_1 (s_1 (c_1 \dot{e}_z + \ddot{x}_5 + g) - \ddot{x}_5)
\]
\[
- k_1 s_1^2 - s_1 \ddot{x}_6 - \frac{1}{\gamma_1} \beta_1 \dot{s}_1 - \frac{1}{\gamma_2} \beta_2 \dot{s}_1.
\]

Since $\dot{\beta}_1 = \frac{1}{\gamma_1} \dot{s}_1$ and $\dot{\beta}_2 = \frac{1}{\gamma_2} \dot{s}_2$, (29) implies that
\[
\dot{V}_z = \beta_1 (s_1 (c_1 \dot{e}_z + \ddot{x}_5 + g) - \ddot{x}_5)
\]
\[
- s_1 \eta_1 \text{sgn}(s_1) - k_1 s_1^2 - s_1 \ddot{x}_6).
\]

Finally, by substituting the adaptive laws (23) into (30) and adopting the relationship $|\ddot{d}_x| \leq \eta_1$,
The tracking error of the closed-loop system is guaranteed to
be zero if
\[ V_z = -s_1 \eta_1 \text{sgn}(s_1) - k_1 s_1^2 - s_1 \ddot{d}_z \]
\[ = -\eta_1 |s_1| - k_1 s_1^2 - s_1 \ddot{d}_z \]  \hspace{1cm} (31)

\[ \leq -k_1 s_1^2 \leq 0, \]
yielding that \( V_z = 0 \) if and only if \( s_1 = 0 \). Define the set of \( V_z = 0 \) as \( \Theta_j \), that is, \( \Theta_j = \{ s_1 \} | V_z = 0 \}. Moreover, \( \Xi_1 \) is defined as the largest invariant set in \( \Theta_j \). Obviously, the largest invariant set \( \Xi_1 \) only includes the point \( s_1 = 0 \). Therefore, the closed-loop system is asymptotically stabilized, i.e., \( s_1 \to 0 \) when \( t \to \infty \) according to LaSalle’s invariance principle [42]. Observing the detailed expression of the sliding surface (21), the asymptotical convergence of \( s_1 \) directly gives that of \( e_z \). \( \square \)

### 3.3. Attitude Controller Design

Now, we are in the position of the attitude controller design, i.e., the control schemes for the roll, pitch as well as yaw channels. We first address the controller of roll channel based on ASMC.

Define the tracking error of roll angle

\[ e_\phi = x_{\gamma d} - x_\gamma, \]  \hspace{1cm} (32)

and the corresponding sliding surface

\[ s_2 = c_2 e_\phi + \dot{e}_\phi, \]  \hspace{1cm} (33)

where \( x_{\gamma d} \) is the desired roll angle and \( c_2 \) is a positive constant. The control law for \( U_2 \) is designed as

\[ U_2 = \eta_2 \text{sgn}(s_2) + k_2 s_2 + \tilde{\beta}_3 (c_2 \dot{e}_\phi + \ddot{x}_{\gamma d}) - \tilde{\beta}_4 x_{10} x_{12}, \]  \hspace{1cm} (34)

where \( k_2 \) is a positive constant. The parametric adaptive laws for \( \tilde{\beta}_3 \) and \( \tilde{\beta}_4 \) are set to be

\[ \dot{\tilde{\beta}}_3 = \gamma_3 s_2 (c_2 \dot{e}_\phi + \ddot{x}_{\gamma d}), \]  \hspace{1cm} (35)

\[ \dot{\tilde{\beta}}_4 = -\gamma_4 s_2 x_{10} x_{12}, \]

where \( \gamma_3 \) and \( \gamma_4 \) are two positive learning gains.

The second main result of the paper is summarized as follows.

**Theorem 2.** Consider the dynamic model of the quadrotor UAV in the state-space form (18) with controller (34) and adaptive laws (35). Under Assumption 1, the roll output tracking error of the closed-loop system is guaranteed to converge to zero asymptotically, i.e., \( e_\phi \to 0 \) as \( t \to \infty \).

**Proof 2.** Define the following candidate of Lyapunov function

\[ V_\phi = \frac{1}{2} L_s^2 s_2^2 + \frac{1}{2} \tilde{\beta}_3^2 + \frac{1}{2} \tilde{\beta}_4^2. \]  \hspace{1cm} (36)

Differentiating \( V_\phi \) with respect to time yields

\[ \dot{V}_\phi = \frac{I_x}{L_s} s_2 \dot{s}_2 + \frac{1}{\gamma_3} \ddot{s}_2 \tilde{\beta}_3 + \frac{1}{\gamma_4} \ddot{s}_2 \tilde{\beta}_4, \]  \hspace{1cm} (37)

where, by (32) and (33), \( \dot{s}_2 \) satisfies

\[ \dot{s}_2 = c_2 \dot{e}_\phi + \ddot{x}_{\gamma d} = c_2 \dot{e}_\phi + \ddot{x}_{\gamma d} - \dot{x}_\phi. \]  \hspace{1cm} (38)

Using the eighth equation in (18) and (38), the first term on the right-hand side of (37) can be rewritten as

\[ \frac{I_x}{L_s} \dot{s}_2 \dot{s}_2 = \frac{I_x}{L_s} (c_2 \dot{e}_\phi + \ddot{x}_{\gamma d}) \left( U_2 + \frac{I_y - I_z}{L} x_{10} x_{12} + \ddot{d}_z \right) \]

\[ = s_2 \left( \frac{I_x}{L} (c_2 \dot{e}_\phi + \ddot{x}_{\gamma d}) - (U_2 + \frac{I_y - I_z}{L} x_{10} x_{12} + \ddot{d}_z) \right) \]

\[ = s_2 \frac{I_x}{L} (c_2 \dot{e}_\phi + \ddot{x}_{\gamma d}) - s_2 U_2 - \frac{I_y - I_z}{L} x_{10} x_{12} - s_2 \ddot{d}_z \]  \hspace{1cm} (39)

By the definitions of \( \tilde{\beta}_3 \) and \( \tilde{\beta}_4 \) and substituting the roll controller (34) into (39), we have

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>2.33</td>
<td>kg</td>
</tr>
<tr>
<td>( g )</td>
<td>9.8</td>
<td>m/s²</td>
</tr>
<tr>
<td>( I_x )</td>
<td>0.4</td>
<td>m</td>
</tr>
<tr>
<td>( I_y )</td>
<td>0.16</td>
<td>kg/m²</td>
</tr>
<tr>
<td>( I_z )</td>
<td>0.16</td>
<td>kg/m²</td>
</tr>
<tr>
<td>( k_1 )</td>
<td>0.32</td>
<td>kg/m²</td>
</tr>
<tr>
<td>( k_2 )</td>
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<td>N/m</td>
</tr>
<tr>
<td>( k_3 )</td>
<td>0.008</td>
<td>N/m</td>
</tr>
<tr>
<td>( k_4 )</td>
<td>0.012</td>
<td>N/m</td>
</tr>
<tr>
<td>( f )</td>
<td>0.05</td>
<td>m</td>
</tr>
</tbody>
</table>

**Table 1:** The model parameters of the quadrotor UAV in simulation [34].

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1 )</td>
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<td></td>
</tr>
<tr>
<td>( k_1 )</td>
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<td></td>
</tr>
<tr>
<td>( \eta_1 )</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>0.005</td>
<td></td>
</tr>
<tr>
<td>( \beta_3 )</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>( \beta_4 )</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>( \beta_5 )</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>( \beta_6 )</td>
<td>2.2</td>
<td></td>
</tr>
<tr>
<td>( \beta_7 )</td>
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<td></td>
</tr>
<tr>
<td>( \beta_8 )</td>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2:** Controller parameters in the proposed ASMC.
\[
\frac{I_s}{L} \ddot{s}_2 = \beta_3 s_2(c_2 \dot{e}_\phi + \dot{\bar{x}}_{7d}) - \beta_4 s_2 x_{10} x_{12} - \bar{\beta}_3 s_2(c_2 \dot{e}_\phi + \dot{\bar{x}}_{7d}) \\
+ \bar{\beta}_4 s_2 x_{10} x_{12} - s_2 \eta_2 \text{sgn}(s_2) - k_2 s_2^2 - s_2 \ddot{\bar{\phi}} \\
= \bar{\beta}_3(s_2(c_2 \dot{e}_\phi + \dot{\bar{x}}_{7d})) - \bar{\beta}_4 s_2 x_{10} x_{12} - s_2 \eta_2 \text{sgn}(s_2) \\
- k_2 s_2^2 - s_2 \ddot{\bar{\phi}}.
\]

(40)

Then, combining (40) and (37) yields

\[
\dot{V}_\phi = \bar{\beta}_3 \dot{s}_2(c_2 \dot{e}_\phi + \dot{\bar{x}}_{7d}) - \bar{\beta}_4 s_2 x_{10} x_{12} + \frac{1}{y_3} \dot{\bar{\beta}}_3 + \frac{1}{y_4} \dot{\bar{\beta}}_4 \\
- s_2 \eta_2 \text{sgn}(s_2) - k_2 s_2^2 - s_2 \ddot{\bar{\phi}}.
\]

(41)

Since \( \dot{\bar{\beta}}_3 = -\hat{\beta}_3 \) and \( \dot{\bar{\beta}}_4 = -\hat{\beta}_4 \), it follows from (41) that

\[
\dot{V}_\phi = \bar{\beta}_3 \dot{s}_2(c_2 \dot{e}_\phi + \dot{\bar{x}}_{7d}) - \frac{1}{y_3} \dot{\hat{\beta}}_3 - \bar{\beta}_4 s_2 x_{10} x_{12} + \frac{1}{y_4} \dot{\hat{\beta}}_4 \\
- s_2 \eta_2 \text{sgn}(s_2) - k_2 s_2^2 - s_2 \ddot{\bar{\phi}}.
\]

(42)
Hence, substituting the adaptive laws (35) into (42) and noticing that \( |\phi| \leq \eta \),
\[
\dot{V}_\phi = -s_2 \eta \text{sgn}(s_2) - k_2 s_2^2 - s_2 \ddot{s}_2
\]
\[
= -\eta |s_2| - k_2 s_2^2 - s_2 \ddot{s}_2
\]
\[
\leq -k_2 s_2^2 \leq 0,
\]
indicating that \( \dot{V}_\phi = 0 \) if and only if \( s_2 = 0 \). Define the set of \( V_\phi = 0 \) as \( \Theta_2 \), that is, \( \Theta_2 = \{ (s_2) | V_\phi = 0 \} \). Moreover, \( \Xi_2 \) is defined as the largest invariant set in \( \Theta_2 \). Obviously, the largest invariant set \( \Xi_2 \) only includes the point \( s_2 = 0 \).

Therefore, the closed-loop system is asymptotically stabilized, i.e., \( s_2 \to 0 \) when \( t \to \infty \) according to LaSalle’s invariance principle [42]. Observing the detailed expression of the sliding surface (33), the asymptotical convergence of \( s_2 \) directly gives that of \( e_\phi \).

Parallel to the way in designing the control and adaptive laws for the altitude and roll channels, the controller design for the pitch and yaw channels can be derived similarly and given in the following:
where $c_i$, $\eta_i$, $k_i$, $i = 3, 4$, and $\gamma_j$, $j = 5, \ldots, 8$ are positive constants, $x_{9d}$ and $x_{11d}$ are the desired pitch and yaw angles, respectively, $e_\theta$ and $e_\phi$ are the tracking errors of pitch and yaw angles, and $s_3$ and $s_4$ are the sliding surfaces, defined as $s_3 = c_3e_\theta + \dot{e}_\theta$ and $s_4 = c_4e_\phi + \dot{e}_\phi$, respectively. \hfill \Box

Remark 2. It is worth mentioning that since the signum functions are involved in the controllers, chattering phenomenon could be serious when the proposed ASMC laws (22), (34), (44), and (46) are applied in the quadrotor UAV system. To resolve this, saturation functions are used to replace the signum functions in all four channels [40]. The resultant control performance has been verified via simulation and real experiment.
4. Simulation and Experimental Tests

In this section, the effectiveness and robustness of the proposed ASMC scheme will be verified via simulation and real experiment, where the performance of LQR and ADRC are given simultaneously for comparison.

4.1. Simulation. In this part, numerical simulations of the LQR, the ADRC, and the proposed ASMC are implemented via Matlab/Simulink. The model parameters of the quadrotor and the design parameters of the ASMC are listed in Tables 1 and 2, respectively.

The initial altitude and attitude of the quadrotor UAV are set as 0 m and [0, 0, 0] rad. The desired/reference position and attitude are assumed to be \( z_d = 8 \) m, \( \phi_d = \theta_d = \varphi_d = 0.5 \) rad, and the external disturbances are \( d_x = d_y = d_z = d_\phi = d_\theta = d_\varphi = \sin(t) \).

The time histories of the sliding surfaces are illustrated in Figure 3, revealing the fast convergence of all the sliding surfaces. Figure 4 shows the resulting tracking error profiles for the altitude and attitude control of the quadrotor. It can be seen that, despite the disturbances acting on the system, the errors converge within 5 seconds and become very small after that when applying the proposed ASMC to the quadrotor. The convergence speed and disturbance rejection ability for each channel could be optimized by tuning the magnitudes of \( c_i \), \( i = 1, \ldots, 4 \) accordingly. Figure 5 shows the time responses of the LQR, the ADRC, and the proposed ASMC (22), (34), (44), and (46) with saturation function. Since the chattering phenomenon is greatly suppressed, the proposed controller can be easily applied in a real quadrotor UAV system.

Comparison with LQR and ADRC has been carried out and the simulation results are presented in Figures 6–9. In Figure 6, we can see that LQR induces a larger overshoot although it has a faster tracking speed for the given reference signal. Meanwhile, the tracking speed of the ASMC is significantly faster than that of the ADRC. In Figures 7–9, the states \( \phi, \theta, \) and \( \varphi \) are with smaller vibration when applying the ASMC to the quadrotor system. To sum up, the proposed robust ASMC scheme possesses a better tracking performance for the quadrotor even in the presence of parametric uncertainty and external disturbances.

4.2. Experimental Test. In experiments, the proposed control strategy is applied to a quadrotor UAV platform provided by Beijing Links Tech Co., Ltd., as shown in Figure 10. This platform uses a personal computer (PC), a 51.5 cm \( \times \) 51.5 cm test bench, an external DC power module, a power adapter, an USB to RS485 converter, a wind speed sensor, and an AR Drone 2.0 quadrotor system with 1 GHz 32 bit ARM Cortex A8 processor, 3-axis gyroscope, 3-axis accelerometer, 3-axis magnetometer, barometer, ultrasonic sensor, 4 brushless motors, 4 motor controllers, and a Wi-Fi wireless module. To ensure the test safety, the quadrotor is fixed on the test platform. Meanwhile, a large and powerful electric fan is used to generate wind disturbance and acts on the quadrotor.

The controllers are first implemented in Matlab/Simulink. Then, the code generation technique is used to convert the Simulink model into C language code, which is downloaded to the processor via Wi-Fi. It is worth noting...
that the altitude controller is not verified in experiment, since the altitude of the quadrotor is fixed.

To evaluate the control performance of the three control schemes, including ASMC, LQR, and ADRC, two flight tests with different wind speeds of external disturbance are carried out on the quadrotor platform. Notice that the initial attitudes of the quadrotor could be different in the actual tests, since the attitude of the quadrotor cannot be measured when the quadrotor is in the stationary state.

Case 1. The quadrotor is expected to achieve the reference tracking of roll and pitch angles of 0 rad under the wind speed of 1.6 m/s.

The tracking performance is demonstrated in Figures 11 and 12, where the oscillation generated by the ADRC is more severe than that by the ASMC and the LQR. In Figure 11, the tracking error of the ASMC is obviously smaller than that of the LQR. However, the ASMC and the LQR provide roughly equal control effects in Figure 12. Furthermore, to quantitatively evaluate the attitude angle tracking performance achieved by the proposed ASMC, LQR, and ADRC, two performance indices, i.e., the maximum value of the absolute value of the tracking error (MAE) and the absolute value of the mean tracking error (AME) are used. As listed in Tables 3 and 4, the MAE of the ASMC is smaller than that of the LQR and the ADRC. Meanwhile, the AMEs of the ADRC are
clearly the smallest in roll and pitch channels, since the tracking results of roll and pitch have an almost equal amplitude fluctuation above and below zero. In addition, the tracking accuracy obtained from the ASMC is superior to that of LQR in terms of AME.

Case 2. The quadrotor is expected to achieve the reference tracking of roll and pitch angles of 0 rad under the wind speed of 2.8 m/s.

Figures 13 and 14 present the evolution of the attitude of the quadrotor in time. As can be seen from Figure 13, the tracking result of the ADRC has the largest oscillation. The tracking error of the ASMC is significantly smaller than that of the LQR when the time exceeds 30 s. Furthermore, it can be observed from Figure 14 that the ADRC produces the largest overshoot at the beginning of the flight test and that the ASMC has the smallest steady-state error. As a matter of fact, as shown in Tables 5 and 6, ASMC achieves the best tracking performance in roll and pitch channels in terms of the MAE.

Table 3: Comparison of the roll tracking performance under the wind speed of 1.6 m/s.

<table>
<thead>
<tr>
<th>Performance index</th>
<th>ASMC (rad)</th>
<th>LQR (rad)</th>
<th>ADRC (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAE (rad)</td>
<td>0.0359</td>
<td>0.0668</td>
<td>0.0615</td>
</tr>
<tr>
<td>AME (rad)</td>
<td>0.0105</td>
<td>0.0375</td>
<td>0.0015</td>
</tr>
</tbody>
</table>

Table 4: Comparison of the pitch tracking performance under the wind speed of 1.6 m/s.

<table>
<thead>
<tr>
<th>Performance index</th>
<th>ASMC (rad)</th>
<th>LQR (rad)</th>
<th>ADRC (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAE (rad)</td>
<td>0.0163</td>
<td>0.0253</td>
<td>0.0484</td>
</tr>
<tr>
<td>AME (rad)</td>
<td>0.0029</td>
<td>0.0047</td>
<td>4.4182 × 10⁻⁴</td>
</tr>
</tbody>
</table>

Table 5: Comparison of the roll tracking performance under the wind speed of 2.8 m/s.

<table>
<thead>
<tr>
<th>Performance index</th>
<th>ASMC (rad)</th>
<th>LQR (rad)</th>
<th>ADRC (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAE (rad)</td>
<td>0.0477</td>
<td>0.0682</td>
<td>0.1206</td>
</tr>
<tr>
<td>AME (rad)</td>
<td>0.0256</td>
<td>0.0441</td>
<td>9.555 × 10⁻⁴</td>
</tr>
</tbody>
</table>
Table 6: Comparison of the pitch tracking performance under the wind speed of 2.8 m/s.

<table>
<thead>
<tr>
<th>Performance index</th>
<th>ASMC</th>
<th>LQR</th>
<th>ADRC</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAE (rad)</td>
<td>0.0194</td>
<td>0.0594</td>
<td>0.0642</td>
</tr>
<tr>
<td>AME (rad)</td>
<td>0.0054</td>
<td>0.0255</td>
<td>0.0015</td>
</tr>
</tbody>
</table>

From the above experimental results and quantitative analysis, it can be concluded that the proposed ASMC control scheme is capable of providing remarkable control performance under the simultaneous effect of parametric uncertainties and external disturbance.

5. Conclusion

A robust ASMC scheme is developed for a quadrotor UAV, which guarantees the asymptotic convergence of the tracking error in altitude and attitude. Adaptive laws are designed for estimation of unknown parameters involved in SMC. A rigorous proof of stability of the proposed control algorithm is accomplished through Lyapunov analysis. Numerical simulation and real experiment have been carried out to verify the effectiveness of the proposed controller. It reveals that the ASMC has a better tracking performance than the LQR or the ADRC in terms of the MAE. Future research will investigate the formation control of the quadrotor UAVs using a multi-agent system approach [43-45].

Appendix

Altitude and attitude controllers of the quadrotor based on LQR are given by

\[ U_1 = \frac{m}{\cos \phi \cos \theta} (g - A_1 x_z + B_1 z_d), \]  
(A.1)

where \( z_d \) is the desired altitude and \( g = 9.8, A_1 = [3.1623, 2.5097], B_1 = 3.1623, \) and \( x_z = [z, \dot{z}]^T \).

\[ U_2 = \frac{J_x}{L} \left( - \frac{J_y - J_z}{J_x} \dot{\theta} \dot{\phi} - A_x x_y + B_x \dot{\phi}_d \right), \]  
(A.2)

\[ U_3 = \frac{J_x}{L} \left( - \frac{J_z - J_y}{J_y} \dot{\phi} \dot{\phi} - A_2 x_{\dot{y}} + B_2 \dot{\phi}_d \right), \]

\[ U_4 = \frac{J_x}{f} \left( - \frac{J_y - J_z}{J_z} \dot{\theta} \dot{\phi} - A_2 x_{\phi} + B_2 \dot{\phi}_d \right), \]

where \( \phi_d, \theta_d, \) and \( \phi_d \) are the desired roll, pitch, and yaw angles, respectively, and \( A_2 = [44.7214, 9.4574], B_2 = 44.7214, x_\phi = [\phi, \dot{\phi}]^T, x_\theta = [\theta, \dot{\theta}]^T, \) and \( x_y = [\varphi, \dot{\varphi}]^T \).

The ADRC scheme can be expressed by

\[ U_i = \beta_{11} (v_{1i} - z_{3i}) + \beta_{2i} (v_{2i} - z_{2i}) - \frac{1}{b_{0i}} z_{3i}, \]  
(A.3)

\[ \ddot{z}_{3i} = z_{3i} - \beta_{011} (z_{1i} - y_{0i}), \]  
(A.4)

\[ \dot{v}_{1i} = v_{2i}, \]  
(A.5)

\[ \dot{v}_{2i} = f_{\text{han}} (v_{1i} - v_{0i}, v_{2i}, r_i, h_i), \]

\[ d_i = r_i h_i^2, \]

\[ a_{0i} = h_i v_{2i}, \]

\[ y_i = (v_{1i} - v_{0i}) + a_{0i}, \]

\[ a_{1i} = \sqrt{d_i (d_i + 8 |y_i|)}, \]

\[ a_{2i} = a_{0i} + \frac{\text{sgn} (y_i) (a_{1i} - d_i)}{2}, \]

\[ a_i = (a_{0i} + y_i) f_{\text{sg}} (y_i, d_i) + a_{1i} (1 - f_{\text{sg}} (y_i, d_i)), \]

\[ \text{fhan} = -r_i a_{0i} \frac{1}{d_i} f_{\text{sg}} (a_i, d_i) - r_i \text{sgn} (a_i) (1 - f_{\text{sg}} (a_i, d_i)), \]  
(A.6)

where \( i = 1, \ldots, 4, v_{0i}, \) and \( y_{0i} \) are the reference signal and output of the controlled plant of the corresponding channel, respectively, and \( f_{\text{sg}} (n_1, n_2) \) is described as

\[ f_{\text{sg}} (n_1, n_2) = \frac{\text{sgn} (n_1 + n_2) - \text{sgn} (n_1 - n_2)}{2} \]  
(A.7)

The design parameters of altitude and attitude controller based on ADRC are listed in Table 7.

Data Availability

All data in the manuscript are available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References


