Research Article

Robust H_∞ Filtering of Nonhomogeneous Markovian Jump Delay Systems via N -Step-Ahead Lyapunov-Krasovskii Functional Approach

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In this paper, we study the robust H_∞ filtering problem for a class of nonhomogeneous Markovian jump delay systems (NMJDSs) with an N -step-ahead Lyapunov-Krasovskii functional (NALKF) approach. The N -step-ahead approach is utilized to reduce the conservatism of robust H_∞ filtering. We aim to design filters such that, for all possible time-varying transition probabilities and all admissible parameter uncertainties and time-delays, the filtering error system is mean-square stable with a smaller estimated error and a lower dissipative level. In terms of linear matrix inequalities, sufficient conditions for the solvability of the addressed problem are developed via a moving horizon method. An illustrative example is included to elucidate advantages of the developed results and the practical potential of NALKF approach for NMJDSs.

1. Introduction

Filtering is one of the most fundamental problems in modern control theory and application fields, such as control system synthesis, state estimation, and information fusion. From the signal-processing point of view, useful signals are inevitably contaminated by exogenous noise, which leads to the discrepancy between measured signals and actual signals. Thus it is necessary to minimize inaccuracies caused by noise and estimate the signal as close to the actual one [1]. Filtering methods are rightly aimed at this purpose. The goal of filtering is to estimate signals that are corrupted by noises or unmeasurable or technically difficult to measure.

With the development of Kalman filtering theory [2] for stochastic systems, numerous extended results of such an optimal filtering have been reported with burgeoning research interest. To name a few, robust Kalman filter has been designed for continuous-time delay systems with norm bounded uncertainties [3] and discrete-time uncertain systems [4] by fully making use of the mean value and

the variance knowledge of Gaussian noise. Nowadays, a new research trend is to study Kalman filtering for networked systems. The distributed Kalman filtering with the effects of network structures was introduced by a thorough bibliographic review in [5]. Recently, distributed extended Kalman filter with nonlinear consensus estimate [6] or with event-triggered scheme and stability guarantee [7] has been developed. As a typical application example, formation of autonomous multiple vehicles was achieved by employing distributed Kalman filter design. Each vehicle aims to estimate its own state, such as position, velocity, and acceleration, by its locally available measurements, and has limited communication with its neighbors [8]. Fading measurements are also noteworthy phenomena induced by the wireless network, where fading rates during the communication are described by random variables with known statistical properties dependent on or independent of the system mode. It usually results in unpredictable performance degradation for the filters [9]. Tobit Kalman filter [10] and modified extended Kalman filter [11] have been further investigated

over fading channels with transmission failure or signal fluctuation.

The successful application of Kalman filter in the aerospace and aviation industry has led applications in industry in the 1970s. However, these attempts have shown that there was a serious mismatch between the underlying assumptions of Kalman filtering and the industrial state estimation problem. In terms of engineering application, it is quite costly or difficult to get an accurate system model and engineers are seldom aware of the statistical characteristics of noise that affect the industrial processes. A useful scheme to deal with modeling uncertainty and non-Gaussian noise is H_∞ filtering. The objective is to minimize the l_2 gain of the filtering error system from noise inputs to filtering errors. H_∞ filtering can be effective in the case that noise inputs are energy-bounded signals, without any statistical knowledge of the noise, rather than Gaussian white noise [12]. Moreover, it is robust against uncertainties both in the system modeling and exogenous noise.

The development of H_∞ filtering is reviewed in [13–16] and references therein. It is worth mentioning that an H_∞ filtering for a class of special switched systems, i.e., Markovian jump systems (MJSs), attracted much attention of the control community. MJSs can well depict various physical phenomena, such as solar thermal receiver, Samuelson's multiplier-accelerator model, NASA F-8 test aircraft, and networked control systems [17]. Such systems may run with external environmental changes, actuator fault, communication time delay, data packet loss, and so on. These random factors often cause jumping phenomenon of system structure or parameters. Aiming at the H_∞ filtering for MJSs, a great deal of research work has emerged (see [17–19] and references therein). The essential difference between MJSs and linear systems is the modes jumping character, which is governed by the transition probabilities (TPs). Therefore, the H_∞ filtering for nonhomogeneous Markovian jump systems (NMJSs), that is, MJSs with time-varying TPs, starts to penetrate into the research front-line of the filtering. Filtering for discrete-time uncertain NMJSs [20], robust H_∞ filtering for continuous-time nonlinear NMJSs with randomly occurring uncertainties [21], and finite-time H_∞ filtering for nonlinear singular NMJSs [22] have been extensively studied. Moreover, the H_∞ filtering for time-delayed systems has also attracted great research interests (see [23–25] and references therein). Many effective approaches, such as the famous Lyapunov-Krasovskii functional (LKF) approach and Lyapunov-Razumikhin functional approach, are developed independently for handling time delay [26]. Specifically, robust filtering performance can be achieved by constructing a proper LKF for time delay systems.

However, filter design approaches developed for NMJSs in the above-mentioned work are conservative. To reduce the design conservatism, an interesting idea is to construct nonmonotonic Lyapunov functional (NLF) [27]. It has been successfully used for T-S fuzzy model to relax the monotonicity requirement of LF and further reduce the conservatism of the stability criteria, i.e., allowing the LF to increase locally during several sampling periods. Two-sample variation, i.e., $V(x_{k+2}) < V(x_k)$ [28–30], and N -sample variation, i.e.,

$V(x_{k+N}) < V(x_k)$ [31–37], were fully developed for the T-S fuzzy model. Stability analysis and synthesis, robust H_∞ controller design, observer-based fuzzy controller design, and output feedback stabilization have been intensively studied. Some preliminary results for switched systems [38, 39] and NMJSs [40] have also been reported.

Based on the above observations, it can be concluded that the NLF approach has not been fully developed for filtering of time-delayed jump systems. A big challenge is how to deal with noise of future time and delay interval according to the predictive horizon N . In this study, we aim to investigate the H_∞ filtering problem for a class of nonhomogeneous Markovian jump delay systems (NMJDSs) via an N -step-ahead Lyapunov-Krasovskii functional (NALKF) approach. The main contributions and novelties of this paper are summarized as follows:

(i) The NALKF approach is developed to reduce the conservatism of the filtering design by properly constructing a LKF and allow the underlying LKF to increase during the period of N sampling time steps ahead of the current time within each jump mode.

(ii) The linear matrix inequality (LMI) formulation of the sufficient conditions for filtering is obtained by moving the horizon from $k + N - 1$ to $k + 1$ step by step. Due to the predictive horizon, the derivation is not trivial.

(iii) For all possible time-varying TPs and all admissible parameter uncertainties and time-delays, the filtering error system is mean-square stable with a smaller estimated error and a lower dissipative level.

Notations. $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote n -dimensional Euclidean space and set of all $n \times m$ matrices, respectively. $\|\cdot\|_2$ refers to the Euclidean vector norm. $\mathcal{E}\{\cdot \mid \mathcal{F}_k\}$ stands for the conditional mathematical expectation, where $\mathcal{F}_k = \sigma\{(x_0, r_0), \dots, (x_k, r_k)\}$ is the σ -algebra. $\lambda_{\min}(\Theta)$ represents the minimum eigenvalue of matrix Θ . $\text{diag}_n(\dots)$ stands for the block-diagonal matrix with n blocks given by matrices in $(\dots)$. In symmetric matrices, we use $*$ as an ellipsis for the symmetric terms above or below the diagonal. Identity and zero matrices of appropriate dimensions are denoted by $\mathbf{I}$ and $\mathbf{0}$, respectively. $\mathbb{Z}_{[s_1, s_2]} \triangleq \{l \mid l \in \mathbb{Z}, s_1 \leq l \leq s_2\}$, where $\mathbb{Z}$ is the set of integers. $l_{2[0, S]}$ is the space of summable sequences on $\mathbb{Z}_{[0, S]}$, where S may be finite or infinite.

2. System Description and Problem Formulation

Given the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, we consider a discrete-time NMJDS described by the following stochastic difference equation

$$\begin{aligned} x_{k+1} &= (A(r_k) + \delta A(r_k)) x_k \\ &\quad + (A_d(r_k) + \delta A_d(r_k)) x_{k-d} + B(r_k) w_k \\ y_k &= C(r_k) x_k + D(r_k) w_k \\ z_k &= G(r_k) x_k + F(r_k) w_k \\ x_k &= \phi_k, \quad k = -d, -d+1, \dots, 0, \end{aligned} \tag{1}$$

where $k \in \mathbb{Z}_{[0,M]}$, $x_k \in \mathbb{R}^{n_x}$ is the state vector, $y_k \in \mathbb{R}^{n_y}$ is the measured output, $z_k \in \mathbb{R}^{n_z}$ is the output to be estimated, $w_k \in \mathbb{R}^{n_w}$ is the exogenous disturbances satisfying $w_k \in l_{2[0,S]}$, integer $d > 0$ is the time delay, and ϕ_k is a real-valued initial function. When $w_k = 0$, system (1) is usually named as a *disturbance-free* NMJDS.

For each possible value of $r_k = i_0$, we denote matrices associated with the “ i_0 th jump mode” by $\mathcal{A}_{i_0}$, where $\mathcal{A}$ can be replaced by any symbol. Matrices A_{i_0} , A_{d,i_0} , B_{i_0} , C_{i_0} , D_{i_0} , G_{i_0} , and F_{i_0} are known constant matrices which depend on the jump mode r_k . The matrices δA_{i_0} and $\delta A_{d,i_0}$ represent time-varying parameter uncertainties, which are assumed to be norm bounded and can be noted as

$$[\delta A_{i_0} \quad \delta A_{d,i_0}] = E_{i_0} \mathcal{L}_{i_0}(k) [H_{i_0}^a \quad H_{i_0}^d], \quad (2)$$

where E_{i_0} , $H_{i_0}^a$, and $H_{i_0}^d$ are known constant matrices which characterize the structure of uncertainties. Matrices $\mathcal{L}_{i_0}(k)$ are unknown time-varying matrix functions with Lebesgue measurable elements satisfying $\mathcal{L}_{i_0}^T(k) \mathcal{L}_{i_0}(k) \leq I$.

The symbol r_k is a discrete-time, discrete-state Markov chain taking values in $\mathbb{Z}_{[1,\beta]}$ with transition probabilities

$$\Pr(r_{k+1} = i_1 \mid r_k = i_0) = \pi_{i_0 i_1}(k), \quad (3)$$

where $\pi_{i_0 i_1}(k) \geq 0$ and satisfying $\sum_{i_1=1}^{\beta} \pi_{i_0 i_1}(k) = 1$.

Symbols $\pi_{i_0 i_1}(k) \triangleq \sum_{f_0=1}^{\rho} \mu_{f_0}(k) \pi_{i_0 i_1}^{f_0}(f_0 \in \mathbb{Z}_{[1,\rho]})$ are entries of the TP matrix $\Pi(k)$, which is defined as

$$\Pi(k) = \Pi(\mu(k)) \triangleq \sum_{f_0=1}^{\rho} \mu_{f_0}(k) \Pi_{f_0}, \quad (4)$$

where $\mu_{f_0}(k) \geq 0$, $\sum_{f_0=1}^{\rho} \mu_{f_0}(k) = 1$, and Π_{f_0} are given TP matrices. All polytopic indices of indicator function $\mu(k)$ over the time interval $[k, k+N]$ can be boiled down into a set $\mathbb{F}_{[0,N]} \triangleq \{f_0, f_1, \dots, f_N\}$, where $f_0, f_1, \dots, f_N \in \mathbb{Z}_{[1,\rho]}$.

Since the N -step-ahead scenario will be considered, we further denote a chain of values with respect to the current time jump mode r_k to the future jump mode r_{k+N} . Let $r_k \triangleq i_0$, $r_{k+1} \triangleq i_1, \dots, r_{k+N} \triangleq i_N$; then we have a series of value sets for finite-state Markov chains, i.e., $\mathbb{I}_{0,l} \triangleq \{i_0, i_1, \dots, i_l\}$ ($l \in \mathbb{Z}_{[1,N]}$), over the time interval $[k, k+N]$, where $\{i_0, i_1, \dots, i_N\} \in \mathbb{Z}_{[1,\beta]}$.

In this paper, we are interested in designing the following mode-dependent full-order filter for system (1), that is,

$$\begin{aligned} \hat{x}_{k+1} &= A_F(r_k) \hat{x}_k + B_F(r_k) y_k \\ \hat{z}_k &= C_F(r_k) \hat{x}_k + D_F(r_k) y_k, \end{aligned} \quad (5)$$

where $\hat{x}_k \in \mathbb{R}^{n_x}$ and $\hat{z}_k \in \mathbb{R}^{n_z}$ are the estimated state and estimate of the output z_k , respectively. For $r_k = i_0$, matrices A_{F,i_0} , B_{F,i_0} , C_{F,i_0} , and D_{F,i_0} are filter gains to be determined.

By introducing new vectors $\xi_k = [x_k^T \quad \hat{x}_k^T]^T$ and $e_k = z_k - \hat{z}_k$, the resulting filtering error system of system (1) together with filter (5) becomes

$$\begin{aligned} \xi_{k+1} &= \widehat{A}_{i_0} \xi_k + \widehat{A}_{d,i_0} \xi_{k-d} + \widehat{B}_{i_0} w_k \\ e_k &= \widehat{C}_{i_0} \xi_k + \widehat{D}_{i_0} w_k, \end{aligned} \quad (6)$$

where

$$\begin{aligned} \widehat{A}_{i_0} &= \left[\begin{array}{c|c} \widetilde{A}_{i_0} & 0 \\ \hline B_{F,i_0} C_{i_0} & A_{F,i_0} \end{array} \right], \\ \widehat{A}_{d,i_0} &= \left[\begin{array}{c|c} \widetilde{A}_{d,i_0} & 0 \\ \hline 0 & 0 \end{array} \right], \\ \widehat{B}_{i_0} &= \left[\begin{array}{c} B_{i_0} \\ \hline B_{F,i_0} D_{i_0} \end{array} \right], \\ \widehat{C}_{i_0} &= [G_{i_0} - D_{F,i_0} C_{i_0} \mid -C_{F,i_0}], \\ \widehat{D}_{i_0} &= F_{i_0} - D_{F,i_0} D_{i_0}, \\ \widetilde{A}_{i_0} &= A_{i_0} + E_{i_0} \mathcal{L}_{i_0}(k) H_{i_0}^a, \\ \widetilde{A}_{d,i_0} &= A_{d,i_0} + E_{i_0} \mathcal{L}_{i_0}(k) H_{i_0}^d. \end{aligned} \quad (7)$$

In order to deal with the mean-square stability of the filtering error system (6), the following preliminaries are given.

Definition 1. The *disturbance-free* NMJDS (6) is said to be mean-square stable if

$$\lim_{S \rightarrow \infty} \sum_{k=0}^S \mathcal{E} \{ \|\xi_k\|_2^2 \} < \infty \quad (8)$$

for any finite initial condition $\xi_0 \in \mathbb{R}^{2n_x}$ and $r_0 \in \mathbb{Z}_{[1,\beta]}$.

Assumption 2. NMJDS (1) is mean-square stable for all admissible uncertainties (2).

Assumption 2 is made based on the fact that NMJDS (1) is autonomous without control inputs. Therefore, the original system (1) to be estimated must be mean-square stable, which is a prerequisite for the filtering error system (6) to be mean-square stable.

The purpose of this paper is to design H_∞ filter of the form (5) such that the following two conditions are satisfied:

- (i) The filtering error system (6) with $w_k = 0$ is mean-square stable.
- (ii) Under the zero-initial condition, the filtering error e satisfies

$$\sum_{k=0}^S \mathcal{E} \{ \|e_k\|_2^2 \} \leq \gamma^2 \sum_{k=0}^S \|w_k\|_2^2 \quad (9)$$

for any nonzero $w_k \in l_{2[0,S]}$ and a given/optimized l_2 gain bound $\gamma > 0$.

3. H_∞ Filtering Performance Analysis

In this section, we first provide a relaxed H_∞ analysis result for the filtering error system (6) by employing the NALKF approach, which will be used for filter design in the next section.

Lemma 3. *The disturbance-free NMJDS (6) is mean-square stable, if there exist a set of symmetric matrices $M_{i_l}^{f_l}$, $P_{i_l}^{f_l} = (P_{i_l}^{f_l})^T > 0$, for all the $i_l \in \mathbb{Z}_{[1,\beta]}$, $f_l \in \mathbb{Z}_{[1,\rho]}$, $l \in \mathbb{Z}_{[0,N]}$, $q \in \mathbb{Z}_{[1,N-1]}$, and $Q = Q^T > 0$ such that*

$$\Theta_N = \begin{bmatrix} \Theta_{N,11} & \Theta_{N,12} \\ * & \Theta_{N,22} \end{bmatrix} < 0, \quad (10a)$$

$$\Theta_{\mathbb{Z}_{[1,N-1]}} = \begin{bmatrix} \Theta_{\mathbb{Z}_{[1,N-1]},11} & \Theta_{\mathbb{Z}_{[1,N-1]},12} \\ * & \Theta_{\mathbb{Z}_{[1,N-1]},22} \end{bmatrix} < 0, \quad (10b)$$

$$\Theta_0 = \begin{bmatrix} \Theta_{0,11} & \Theta_{0,12} \\ * & \Theta_{0,22} \end{bmatrix} < 0, \quad (10c)$$

where

$$\begin{aligned} \Theta_{N,11} &= -\widetilde{M}_{i_0,i_{N-1}}^{f_{N-2},f_{N-1}} + \widehat{A}_{i_{N-1}}^T \widetilde{P}_{i_0,i_N}^{f_{N-1},f_N} \widehat{A}_{i_{N-1}} + Q, \\ \Theta_{N,12} &= \widehat{A}_{i_{N-1}}^T \widetilde{P}_{i_0,i_N}^{f_{N-1},f_N} \widehat{A}_{d,i_{N-1}}, \\ \Theta_{N,22} &= -Q + \widehat{A}_{d,i_{N-1}}^T \widetilde{P}_{i_0,i_N}^{f_{N-1},f_N} \widehat{A}_{d,i_{N-1}}, \\ \Theta_{\mathbb{Z}_{[1,N-1]},11} &= -\widetilde{M}_{i_0,i_{N-q-1}}^{f_{N-q-2},f_{N-q-1}} \\ &\quad + \widehat{A}_{i_{N-q-1}}^T \widetilde{M}_{i_0,i_{N-q}}^{f_{N-q-1},f_{N-q}} \widehat{A}_{i_{N-q-1}} + Q, \\ \Theta_{\mathbb{Z}_{[1,N-1]},12} &= \widehat{A}_{i_{N-q-1}}^T \widetilde{M}_{i_0,i_{N-q}}^{f_{N-q-1},f_{N-q}} \widehat{A}_{d,i_{N-q-1}}, \\ \Theta_{\mathbb{Z}_{[1,N-1]},22} &= -Q + \widehat{A}_{d,i_{N-q-1}}^T \widetilde{M}_{i_0,i_{N-q}}^{f_{N-q-1},f_{N-q}} \widehat{A}_{d,i_{N-q-1}}, \\ \Theta_{0,11} &= -P_{i_0}^{f_0} + \widehat{A}_{i_0}^T \widetilde{M}_{i_0,i_1}^{f_0,f_1} \widehat{A}_{i_0} + Q, \\ \Theta_{0,12} &= \widehat{A}_{i_0}^T \widetilde{M}_{i_0,i_1}^{f_0,f_1} \widehat{A}_{d,i_0}, \\ \Theta_{0,22} &= -Q + \widehat{A}_{d,i_0}^T \widetilde{M}_{i_0,i_1}^{f_0,f_1} \widehat{A}_{d,i_0}, \\ \widetilde{M}_{i_0,i_{N-1}}^{f_{N-2},f_{N-1}} &= \sum_{i_{N-1}=1}^{\beta} \pi_{i_0,i_{N-1}}^{f_{N-2}} M_{i_{N-1}}^{f_{N-1}}, \\ \widetilde{P}_{i_0,i_N}^{f_{N-1},f_N} &= \sum_{i_N=1}^{\beta} \pi_{i_0,i_N}^{f_{N-1}} P_{i_N}^{f_N}, \\ \widetilde{M}_{i_0,i_{N-q-1}}^{f_{N-q-2},f_{N-q-1}} &= \sum_{i_{N-q-1}=1}^{\beta} \pi_{i_0,i_{N-q-1}}^{f_{N-q-2}} M_{i_{N-q-1}}^{f_{N-q-1}}, \\ \widetilde{M}_{i_0,i_{N-q}}^{f_{N-q-1},f_{N-q}} &= \sum_{i_{N-q}=1}^{\beta} \pi_{i_0,i_{N-q}}^{f_{N-q-1}} M_{i_{N-q}}^{f_{N-q}}, \\ \widetilde{M}_{i_0,i_1}^{f_0,f_1} &= \sum_{i_1=1}^{\beta} \pi_{i_0,i_1}^{f_0} M_{i_1}^{f_1}. \end{aligned} \quad (11)$$

Proof. Consider a LKF at time k as follows

$$\begin{aligned} V(\varphi_k) &\triangleq V(\mu(k), r_k, \xi_k) \\ &\triangleq \xi_k^T \left(\sum_{f_0=1}^{\rho} \mu_{f_0}(k) P_{i_0}^{f_0} \right) \xi_k + \sum_{m=k+1-d}^k \xi_m^T Q \xi_m, \end{aligned} \quad (12)$$

and denote

$$\Omega_{0,l}(k) \triangleq \sum_{f_0=1}^{\rho} \mu_{f_0}(k) \sum_{f_1=1}^{\rho} \mu_{f_1}(k+1) \cdots \sum_{f_l=1}^{\rho} \mu_{f_l}(k+l), \quad (13)$$

$$\eta_k \triangleq \begin{bmatrix} \xi_k^T & \xi_{k-d}^T \end{bmatrix}^T, \quad (14)$$

$$V'(\varphi_k) \triangleq \xi_k^T \left(\sum_{f_0=1}^{\rho} \mu_{f_0}(k) M_{i_0}^{f_0} \right) \xi_k + \sum_{m=k+1-d}^k \xi_m^T Q \xi_m. \quad (15)$$

By virtue of *Definition 1*, we develop the following steps to prove that

$$\Delta_N V(\varphi_k) = \mathcal{E}\{V(\varphi_{k+N}) \mid \mathcal{F}_k\} - V(\varphi_k) < 0 \quad (16)$$

establishes the mean-square stability of *disturbance-free* NMJDS (6).

Subtracting and adding a group of conditional mathematical expectations $\mathcal{E}\{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\}$, $\mathcal{E}\{V'(\varphi_{k+N-2}) \mid \mathcal{F}_k\}$, $\dots$, $\mathcal{E}\{V'(\varphi_{k+1}) \mid \mathcal{F}_k\}$ from and to (16), one has

$$\begin{aligned} \Delta_N V(\varphi_k) &= \mathcal{E}\{V(\varphi_{k+N}) \mid \mathcal{F}_k\} - V(\varphi_k) \\ &= \mathcal{E}\{V(\varphi_{k+N}) \mid \mathcal{F}_k\} \\ &\quad - \mathcal{E}\{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\} \\ &\quad + \mathcal{E}\{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\} \\ &\quad - \mathcal{E}\{V'(\varphi_{k+N-2}) \mid \mathcal{F}_k\} + \cdots \\ &\quad + \mathcal{E}\{V'(\varphi_{k+1}) \mid \mathcal{F}_k\} - V(\varphi_k). \end{aligned} \quad (17)$$

Step 1. Bearing in mind that

$$\xi_{k+N} = \widehat{A}_{i_{N-1}} \xi_{k+N-1} + \widehat{A}_{d,i_{N-1}} \xi_{k-d+N-1} \quad (18)$$

one has

$$\begin{aligned} &\mathcal{E}\{V(\varphi_{k+N}) \mid \mathcal{F}_k\} - \mathcal{E}\{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\} \\ &= (\widehat{A}_{i_{N-1}} \xi_{k+N-1} + \widehat{A}_{d,i_{N-1}} \xi_{k-d+N-1})^T \Omega_{0,N-1}(k) \\ &\quad \cdot \widetilde{P}_{i_0,i_N}^{f_{N-1},f_N} (\widehat{A}_{i_{N-1}} \xi_{k+N-1} + \widehat{A}_{d,i_{N-1}} \xi_{k-d+N-1}) \\ &\quad - \xi_{k+N-1}^T \Omega_{0,N-2}(k) \widetilde{M}_{i_0,i_{N-1}}^{f_{N-2},f_{N-1}} \xi_{k+N-1} \\ &\quad + \xi_{k+N-1}^T Q \xi_{k+N-1} - \xi_{k-d+N-1}^T Q \xi_{k-d+N-1} \\ &= \eta_{k+N-1}^T \Omega_{0,N-1}(k) \Theta_N \eta_{k+N-1}. \end{aligned} \quad (19)$$

It can be inferred from (10a) that

$$\mathcal{E} \{V(\varphi_{k+N}) \mid \mathcal{F}_k\} - \mathcal{E} \{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\} < 0. \quad (20)$$

Therefore, we have

$$\begin{aligned} \Delta_N V(\varphi_k) &< \mathcal{E} \{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\} \\ &\quad - \mathcal{E} \{V'(\varphi_{k+N-2}) \mid \mathcal{F}_k\} + \cdots \\ &\quad + \mathcal{E} \{V'(\varphi_{k+1}) \mid \mathcal{F}_k\} - V(\varphi_k). \end{aligned} \quad (21)$$

Step 2. Bearing in mind that

$$\xi_{k+N-1} = \widehat{A}_{i_{N-2}} \xi_{k+N-2} + \widehat{A}_{d,i_{N-2}} \xi_{k-d+N-2} \quad (22)$$

one obtains

$$\begin{aligned} &\mathcal{E} \{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\} - \mathcal{E} \{V'(\varphi_{k+N-2}) \mid \mathcal{F}_k\} \\ &= (\widehat{A}_{i_{N-2}} \xi_{k+N-2} + \widehat{A}_{d,i_{N-2}} \xi_{k-d+N-2})^T \Omega_{0,N-2}(k) \\ &\quad \cdot \widetilde{P}_{i_0,i_{N-1}}^{f_{N-2},f_{N-1}} (\widehat{A}_{i_{N-2}} \xi_{k+N-2} + \widehat{A}_{d,i_{N-2}} \xi_{k-d+N-2}) \\ &\quad - \xi_{k+N-2}^T \Omega_{0,N-3}(k) \widetilde{M}_{i_0,i_{N-2}}^{f_{N-3},f_{N-2}} \xi_{k+N-2} \\ &\quad + \xi_{k+N-2}^T Q \xi_{k+N-2} - \xi_{k-d+N-2}^T Q \xi_{k-d+N-2} \\ &= \eta_{k+N-2}^T \Omega_{0,N-2}(k) \Theta_1 \eta_{k+N-2}. \end{aligned} \quad (23)$$

If (10b) holds for $q = 1$, then it results in

$$\mathcal{E} \{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\} - \mathcal{E} \{V'(\varphi_{k+N-2}) \mid \mathcal{F}_k\} < 0, \quad (24)$$

which further implies

$$\begin{aligned} \Delta_N V(\varphi_k) &< \mathcal{E} \{V'(\varphi_{k+N-2}) \mid \mathcal{F}_k\} \\ &\quad - \mathcal{E} \{V'(\varphi_{k+N-3}) \mid \mathcal{F}_k\} + \cdots \\ &\quad + \mathcal{E} \{V'(\varphi_{k+1}) \mid \mathcal{F}_k\} - V(\varphi_k). \end{aligned} \quad (25)$$

Repeating the above procedure $N - 2$ times yields

$$\Delta_N V(\varphi_k) < \mathcal{E} \{V'(\varphi_{k+1}) \mid \mathcal{F}_k\} - V(\varphi_k). \quad (26)$$

Step 3. Bearing in mind that

$$\xi_{k+1} = \widehat{A}_{i_0} \xi_k + \widehat{A}_{d,i_0} \xi_{k-d} \quad (27)$$

one has

$$\begin{aligned} &\mathcal{E} \{V'(\varphi_{k+1}) \mid \mathcal{F}_k\} - V(\varphi_k) = (\widehat{A}_{i_0} \xi_k + \widehat{A}_{d,i_0} \xi_{k-d})^T \\ &\quad \cdot \Omega_{0,0}(k) \widetilde{M}_{i_0,i_1}^{f_0,f_1} (\widehat{A}_{i_0} \xi_k + \widehat{A}_{d,i_0} \xi_{k-d}) - \xi_k^T P_{i_0}^{f_0} \xi_k \\ &\quad + \xi_k^T Q \xi_k - \xi_{k-d}^T Q \xi_{k-d} = \eta_k^T \Omega_{0,0}(k) \Theta_0 \eta_k. \end{aligned} \quad (28)$$

It can be inferred from (10c) that

$$\mathcal{E} \{V'(\varphi_{k+1}) \mid \mathcal{F}_k\} - V(\varphi_k) < 0, \quad (29)$$

which implies that

$$\eta_k^T \Omega_{0,0}(k) \Theta_0 \eta_k < 0. \quad (30)$$

Step 4. By observing (30), we obtain

$$\Delta_N V(\varphi_k) < \eta_k^T \Omega_{0,0}(k) \Theta_0 \eta_k \leq -\alpha_{i_0} \eta_k^T \eta_k \leq -\alpha \eta_k^T \eta_k \quad (31)$$

where $\alpha_{i_0} = \min_k \{\lambda_{\min}(\Omega_{0,0}(k) \Theta_0)\}$, $\alpha = \min_{i_0 \in \mathbb{Z}_{[1,\beta]}} \alpha_{i_0}$.

Let $L(\varphi(k)) = \sum_{l=0}^{N-1} V(\varphi(k+l))$; then one has

$$\begin{aligned} V(\varphi(k+N)) - V(\varphi(k)) &< 0 \iff \\ L(\varphi(k+1)) - L(\varphi(k)) &< 0. \end{aligned} \quad (32)$$

Therefore, it can be yielded from (31) and (32) that

$$\begin{aligned} &\mathcal{E} \{L(\varphi_{k+1}) \mid \mathcal{F}_0\} - \mathcal{E} \{L(\varphi_k) \mid \mathcal{F}_0\} \leq -\alpha \eta_k^T \eta_k \implies \\ &\mathcal{E} \left\{ \sum_{k=0}^S \Delta_N V(\varphi_k) \mid \mathcal{F}_0 \right\} \\ &= \mathcal{E} \{L(\varphi_{S+1}) \mid \mathcal{F}_0\} - L(\varphi_0) \\ &\leq -\alpha \mathcal{E} \left\{ \sum_{k=0}^S \|\eta_k\|_2^2 \mid \mathcal{F}_0 \right\} \implies \\ &\mathcal{E} \left\{ \sum_{k=0}^S \|\eta_k\|_2^2 \mid \mathcal{F}_0 \right\} \\ &\leq \alpha^{-1} \{L(\varphi_0) - \mathcal{E} \{L(\varphi_{S+1}) \mid \mathcal{F}_0\}\} \leq \alpha^{-1} L(\varphi_0) \implies \\ &\lim_{S \rightarrow \infty} \mathcal{E} \left\{ \sum_{k=0}^S \|\eta_k\|_2^2 \mid \mathcal{F}_0 \right\} \leq \alpha^{-1} L(\varphi_0) < \infty. \end{aligned} \quad (33)$$

This implies, from Definition 1, that the disturbance-free NMJDS (6) is mean-square stable. The proof is completed. $\square$

Remark 4. Compared with the multistep LF approach developed in [34], the proposed NALKF approach used the N -step-ahead TPs $\sum_{i_N=1}^{\beta} \pi_{i_0,i_N}^{i_{N-1}}$ to replace the multiple multiplication $\sum_{i_0=1}^s \pi_{i_0,i_1}^{(l_0)} \sum_{i_1=1}^s \pi_{i_1,i_2}^{(l_1)} \cdots \sum_{i_{N-1}=1}^s \pi_{i_{N-1},i_N}^{(l_{N-1})}$. It will greatly reduce the number of decision variables, especially for the time-delayed scenario.

The following lemma provides sufficient conditions under which the filtering error system, i.e., NMJDS (6), is mean-square stable and the filtering error e_k satisfies the dissipative requirement (9).

Lemma 5. *The NMJDS (6) possesses the γ -disturbance attenuation level in (9) for all $w_k \in l_{2[0,S]}$, $w_k \neq 0$, if there exist a set of symmetric matrices $M_{i_l}^{f_l}, P_{i_l}^{f_l} = (P_{i_l}^{f_l})^T > 0$, for all*

the $i_l \in \mathbb{Z}_{[1,\beta]}$, $f_l \in \mathbb{Z}_{[1,\rho]}$, $l \in \mathbb{Z}_{[0,N]}$, $q \in \mathbb{Z}_{[1,N-1]}$, and $Q = Q^T > 0$ such that

$$\Lambda_N = \begin{bmatrix} \Lambda_{N,11} & \Lambda_{N,12} & \Lambda_{N,13} \\ * & \Lambda_{N,22} & \Lambda_{N,23} \\ * & * & \Lambda_{N,33} \end{bmatrix} < 0, \quad (34a)$$

$$\Lambda_{\mathbb{Z}_{[1,N-1]}} = \begin{bmatrix} \Lambda_{\mathbb{Z}_{[1,N-1]},11} & \Lambda_{\mathbb{Z}_{[1,N-1]},12} & \Lambda_{\mathbb{Z}_{[1,N-1]},13} \\ * & \Lambda_{\mathbb{Z}_{[1,N-1]},22} & \Lambda_{\mathbb{Z}_{[1,N-1]},23} \\ * & * & \Lambda_{\mathbb{Z}_{[1,N-1]},33} \end{bmatrix} < 0, \quad (34b)$$

$$\Lambda_0 = \begin{bmatrix} \Lambda_{0,11} & \Lambda_{0,12} & \Lambda_{0,13} \\ * & \Lambda_{0,22} & \Lambda_{0,23} \\ * & * & \Lambda_{0,33} \end{bmatrix} < 0, \quad (34c)$$

where

$$\begin{aligned} \Lambda_{N,11} &= -\widehat{M}_{i_0,i_{N-1}}^{f_{N-2},f_{N-1}} + \widehat{A}_{i_{N-1}}^T \widehat{P}_{i_0,i_N}^{f_{N-1},f_N} \widehat{A}_{i_{N-1}} + Q \\ &\quad + \widehat{C}_{i_{N-1}}^T \widehat{C}_{i_{N-1}}, \\ \Lambda_{N,12} &= \widehat{A}_{i_{N-1}}^T \widehat{P}_{i_0,i_N}^{f_{N-1},f_N} \widehat{A}_{d,i_{N-1}}, \\ \Lambda_{N,13} &= \widehat{A}_{i_{N-1}}^T \widehat{P}_{i_0,i_N}^{f_{N-1},f_N} \widehat{B}_{i_{N-1}} + \widehat{C}_{i_{N-1}}^T \widehat{D}_{i_{N-1}}, \\ \Lambda_{N,22} &= -Q + \widehat{A}_{d,i_{N-1}}^T \widehat{P}_{i_0,i_N}^{f_{N-1},f_N} \widehat{A}_{d,i_{N-1}}, \\ \Lambda_{N,23} &= \widehat{A}_{d,i_{N-1}}^T \widehat{P}_{i_0,i_N}^{f_{N-1},f_N} \widehat{B}_{i_{N-1}}, \\ \Lambda_{N,33} &= -\gamma^2 \mathbf{I} + \widehat{B}_{i_{N-1}}^T \widehat{P}_{i_0,i_N}^{f_{N-1},f_N} \widehat{B}_{i_{N-1}} + \widehat{D}_{i_{N-1}}^T \widehat{D}_{i_{N-1}}, \\ \Lambda_{\mathbb{Z}_{[1,N-1]},11} &= -\widehat{M}_{i_0,i_{N-q-1}}^{f_{N-q-2},f_{N-q-1}} \\ &\quad + \widehat{A}_{i_{N-q-1}}^T \widehat{M}_{i_0,i_{N-q}}^{f_{N-q-1},f_{N-q}} \widehat{A}_{i_{N-q-1}} + Q \\ &\quad + \widehat{C}_{i_{N-q-1}}^T \widehat{C}_{i_{N-q-1}}, \\ \Lambda_{\mathbb{Z}_{[1,N-1]},12} &= \widehat{A}_{i_{N-q-1}}^T \widehat{M}_{i_0,i_{N-q}}^{f_{N-q-1},f_{N-q}} \widehat{A}_{d,i_{N-q-1}}, \\ \Lambda_{\mathbb{Z}_{[1,N-1]},13} &= \widehat{A}_{i_{N-q-1}}^T \widehat{M}_{i_0,i_{N-q}}^{f_{N-q-1},f_{N-q}} \widehat{B}_{i_{N-q-1}} \\ &\quad + \widehat{C}_{i_{N-q-1}}^T \widehat{D}_{i_{N-q-1}}, \\ \Lambda_{\mathbb{Z}_{[1,N-1]},22} &= -Q + \widehat{A}_{d,i_{N-q-1}}^T \widehat{M}_{i_0,i_{N-q}}^{f_{N-q-1},f_{N-q}} \widehat{A}_{d,i_{N-q-1}}, \\ \Lambda_{\mathbb{Z}_{[1,N-1]},23} &= \widehat{A}_{d,i_{N-q-1}}^T \widehat{M}_{i_0,i_{N-q}}^{f_{N-q-1},f_{N-q}} \widehat{B}_{i_{N-q-1}}, \\ \Lambda_{\mathbb{Z}_{[1,N-1]},33} &= -\gamma^2 \mathbf{I} + \widehat{B}_{i_{N-q-1}}^T \widehat{M}_{i_0,i_{N-q}}^{f_{N-q-1},f_{N-q}} \widehat{B}_{i_{N-q-1}} \\ &\quad + \widehat{D}_{i_{N-q-1}}^T \widehat{D}_{i_{N-q-1}}, \end{aligned}$$

$$\begin{aligned} \Lambda_{0,11} &= -P_{i_0}^{f_0} + \widehat{A}_{i_0}^T \widehat{M}_{i_0,i_1}^{f_0,f_1} \widehat{A}_{i_0} + Q + \widehat{C}_{i_0}^T \widehat{C}_{i_0}, \\ \Lambda_{0,12} &= \widehat{A}_{i_0}^T \widehat{M}_{i_0,i_1}^{f_0,f_1} \widehat{A}_{d,i_0}, \\ \Lambda_{0,13} &= \widehat{A}_{i_0}^T \widehat{M}_{i_0,i_1}^{f_0,f_1} \widehat{B}_{i_0} + \widehat{C}_{i_0}^T \widehat{D}_{i_0}, \\ \Lambda_{0,22} &= -Q + \widehat{A}_{d,i_0}^T \widehat{M}_{i_0,i_1}^{f_0,f_1} \widehat{A}_{d,i_0}, \\ \Lambda_{0,23} &= \widehat{A}_{d,i_0}^T \widehat{M}_{i_0,i_1}^{f_0,f_1} \widehat{B}_{i_0}, \\ \Lambda_{0,33} &= -\gamma^2 \mathbf{I} + \widehat{B}_{i_0}^T \widehat{M}_{i_0,i_1}^{f_0,f_1} \widehat{B}_{i_0} + \widehat{D}_{i_0}^T \widehat{D}_{i_0}, \end{aligned} \quad (35)$$

and $\widehat{M}_{i_0,i_{N-1}}^{f_{N-2},f_{N-1}}$, $\widehat{P}_{i_0,i_N}^{f_{N-1},f_N}$, $\widehat{M}_{i_0,i_{N-q-1}}^{f_{N-q-2},f_{N-q-1}}$, $\widehat{M}_{i_0,i_{N-q}}^{f_{N-q-1},f_{N-q}}$, $\widehat{M}_{i_0,i_1}^{f_0,f_1}$ are the same as in Lemma 3.

Proof. Consider a LKF $V(\varphi_k)$ and an auxiliary functional $V'(\varphi_k)$ noted in (12) and (15), respectively. It is clear that inequalities (34a)–(34c) incorporate sufficient conditions presented in Lemma 3 and therefore Lemma 5 implies the mean-square stability of the NMJDS (6). Next, we direct our attention to the H_∞ performance analysis via the following steps.

Define new vectors $\zeta_k \triangleq [\xi_k^T \ \xi_{k-d}^T \ w_k^T]^T$ and $\vartheta(k) \triangleq e_k^T e_k - \gamma^2 w_k^T w_k$. We try to prove that

$$\Delta_N V(\varphi_k) + \sum_{l=0}^{N-1} \vartheta_{k+l} < 0 \quad (36)$$

guarantees H_∞ performance of the NMJDS (6).

Adding and subtracting a group of mathematical items

$$\begin{aligned} & -\mathcal{E}\{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\} + \vartheta_{k+N-1} \\ & -\mathcal{E}\{V'(\varphi_{k+N-2}) \mid \mathcal{F}_k\} + \vartheta_{k+N-2} - \dots \\ & -\mathcal{E}\{V'(\varphi_{k+1}) \mid \mathcal{F}_k\} + \vartheta_{k+1} \end{aligned} \quad (37)$$

to and from $\Delta_N V(\varphi_k)$, one has

$$\begin{aligned} \Delta_N V(\varphi_k) &= \mathcal{E}\{V(\varphi_{k+N}) \mid \mathcal{F}_k\} \\ & -\mathcal{E}\{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\} + \vartheta_{k+N-1} \\ & + \mathcal{E}\{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\} \\ & -\mathcal{E}\{V'(\varphi_{k+N-2}) \mid \mathcal{F}_k\} + \vartheta_{k+N-2} + \dots \\ & + \mathcal{E}\{V'(\varphi_{k+1}) \mid \mathcal{F}_k\} - V(\varphi_k) + \vartheta_k \\ & - \sum_{l=0}^{N-1} \vartheta_{k+l}. \end{aligned} \quad (38)$$

First, following the same lines of the proof of Lemma 3, we know that (34a) implies that

$$\begin{aligned} & \mathcal{E}\{V(\varphi_{k+N}) \mid \mathcal{F}_k\} - \mathcal{E}\{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\} + \vartheta_{k+N-1} \\ & = \zeta_{k+N-1}^T \Omega_{0,N-1}(k) \Lambda_N \zeta_{k+N-1} < 0, \end{aligned} \quad (39)$$

which further yields

$$\begin{aligned} \Delta_N V(\varphi_k) &< \mathcal{E} \{V'(\varphi_{k+N-1}) \mid \mathcal{F}_k\} \\ &\quad - \mathcal{E} \{V'(\varphi_{k+N-2}) \mid \mathcal{F}_k\} + \vartheta_{k+N-2} + \dots \\ &\quad + \mathcal{E} \{V'(\varphi_{k+1}) \mid \mathcal{F}_k\} - V(\varphi_k) + \vartheta_k \\ &\quad - \sum_{l=0}^{N-1} \vartheta_{k+l}. \end{aligned} \quad (40)$$

Then, it can be concluded from (34b) that

$$\begin{aligned} &\mathcal{E} \{V(\varphi_{k+N-q}) \mid \mathcal{F}_k\} - \mathcal{E} \{V'(\varphi_{k+N-q-1}) \mid \mathcal{F}_k\} \\ &\quad + \vartheta_{k+N-q-1} \\ &= \zeta_{k+N-q-1}^T \Omega_{0,N-q-1}(k) \Lambda_{\mathbb{Z}_{[1,N-1]}} \zeta_{k+N-q-1} < 0, \\ &\quad (\forall q \in \mathbb{Z}_{[1,N-1]}) \end{aligned} \quad (41)$$

which implies that

$$\begin{aligned} \Delta_N V(\varphi_k) &< \mathcal{E} \{V'(\varphi_{k+1}) \mid \mathcal{F}_k\} - V(\varphi_k) + \vartheta_k \\ &\quad - \sum_{l=0}^{N-1} \vartheta_{k+l}. \end{aligned} \quad (42)$$

Finally, from (34c), we obtain

$$\begin{aligned} \mathcal{E} \{V'(\varphi_{k+1}) \mid \mathcal{F}_0\} - V(\varphi_k) + \vartheta_k &= \zeta_k^T \Omega_{0,0}(k) \Lambda_0 \zeta_k \\ &< 0, \end{aligned} \quad (43)$$

which directly yields that (36) holds by considering (38), (40), (42), and (43) together.

In the following, we assume zero-initial conditions, i.e., $x_{-N} = x_{-N+1} = \dots = x_0 = 0$. Define

$$J_S \triangleq \mathcal{E} \left\{ \sum_{k=0}^S (z_k^T z_k - \gamma^2 w_k^T w_k) \right\}. \quad (44)$$

Inequality (36) results in

$$\begin{aligned} &\sum_{k=-N}^S \left\{ \Delta_N V(\varphi_k) + \sum_{l=0}^{N-1} \vartheta_{k+l} \right\} < 0 \implies \\ &\mathcal{E} \{V(\varphi_{S+N}) \mid \mathcal{F}_S\} \\ &\quad + \mathcal{E} \{V(\varphi_{S+N-1}) \mid \mathcal{F}_{S-1}\} + \dots + \underbrace{\mathcal{E} \{V(\varphi_{S+1}) \mid \mathcal{F}_{-N+S+1}\}}_N \\ &\quad - \mathcal{E} \{V(\varphi_{-N}) \mid \mathcal{F}_{-N}\} \\ &\quad - \underbrace{\mathcal{E} \{V(\varphi_{-N+1}) \mid \mathcal{F}_{-N+1}\} - \dots - \mathcal{E} \{V(\varphi_{-1}) \mid \mathcal{F}_{-1}\}}_N \\ &\quad + \sum_{k=-N}^S (\vartheta_k + \vartheta_{k+1} + \dots + \vartheta_{k+N-1}) < 0. \end{aligned} \quad (45)$$

As $S \rightarrow \infty$, all the negative items tend to zero and we obtain

$$NV(\varphi_\infty) + NJ_\infty < 0 \quad (46)$$

which yields $J_\infty < 0$. Therefore, the dissipative inequality (9) holds for $S > 0$. This completes the proof. $\square$

Remark 6. The N -step-ahead LKF approach is an N -step ahead of the current time k and thus it has a potential to allow the LKF to increase during a finite-time interval $[k, k+N]$. Hence, a more relaxed stability condition and a better H_∞ attenuation level could be obtained for the filtering error system (6). Therefore, a better filtering performance could be further expected.

Remark 7. Although mathematical difficulties lie in the derivation of conditions for the H_∞ analysis (as compared with [32]), when the system goes to future steps, we develop the main results from stability condition step by step.

4. Robust H_∞ Filter Design

According to H_∞ performance analysis for the NMJDS (6), we are now in the position to provide numerical testable conditions to the robust H_∞ filter design. Before giving the main results, the following well known lemma must be recalled.

Lemma 8 (see [41]). *Let matrix $\Psi = \Psi^T$, E and H be real matrices of appropriate dimensions, with $\mathcal{L}(k)$ satisfying $\mathcal{L}(k)^T \mathcal{L}(k) \leq \mathbf{I}$; then*

$$\Psi + E \mathcal{L}(k) H + H^T \mathcal{L}(k)^T E^T < 0 \quad (47)$$

if and only if there exists a positive scalar $\varepsilon > 0$ such that

$$\Psi + \varepsilon^{-1} E E^T + \varepsilon H^T H < 0, \quad (48)$$

or equivalently

$$\begin{bmatrix} \Psi & E & \varepsilon H^T \\ * & -\varepsilon \mathbf{I} & \mathbf{0} \\ * & * & -\varepsilon \mathbf{I} \end{bmatrix} < 0. \quad (49)$$

In the following theorem, the addressed filter design problem is solvable for all possible time-varying transition probabilities and all admissible parameter uncertainties and time-delays if a set of LMIs are feasible.

Theorem 9. *The NMJDS (6) is mean-square stable and the dissipative constraint (9) with an optimized attenuation level*

γ is achieved for all nonzero w_k , if there exist a set of symmetric matrices $M_{i,1}^{f_i}, M_{i,2}^{f_i}, M_{i,3}^{f_i}, X_{i_l}, Y_{i_l}, Z_{i_l}, W_{i_l}, V_{i_l}, P_{i,1}^{f_i} = (P_{i,1}^{f_i})^T > 0, P_{i,2}^{f_i} = (P_{i,2}^{f_i})^T > 0, P_{i,3}^{f_i} = (P_{i,3}^{f_i})^T > 0, Q_1 = Q_1^T > 0, Q_2 = Q_2^T > 0, Q_3 = Q_3^T > 0$ for all the $i_l \in \mathbb{Z}_{[1,\beta]}$, $f_l \in \mathbb{Z}_{[1,\rho]}$, $l \in \mathbb{Z}_{[0,N]}$, $q \in \mathbb{Z}_{[1,N-1]}$, and matrices C_{F,i_l}, D_{F,i_l} , scalar ε such that

$$\Xi_N = \begin{bmatrix} \Xi_{N,11} & \mathbf{0} & \Xi_{N,13} & \Xi_{N,14} & \Xi_{N,15} \\ * & \Xi_{N,22} & \mathbf{0} & \Xi_{N,24} & \Xi_{N,25} \\ * & * & \Xi_{N,33} & \Xi_{N,34} & \mathbf{0} \\ * & * & * & \Xi_{N,44} & \Xi_{N,45} \\ * & * & * & * & \Xi_{N,55} \end{bmatrix} < 0, \quad (50a)$$

$$\Xi_{\mathbb{Z}_{[1,N-1]}} = \begin{bmatrix} \Xi_{\mathbb{Z}_{[1,N-1]},11} & \mathbf{0} & \Xi_{\mathbb{Z}_{[1,N-1]},13} & \Xi_{\mathbb{Z}_{[1,N-1]},14} & \Xi_{\mathbb{Z}_{[1,N-1]},15} \\ * & \Xi_{\mathbb{Z}_{[1,N-1]},22} & \mathbf{0} & \Xi_{\mathbb{Z}_{[1,N-1]},24} & \Xi_{\mathbb{Z}_{[1,N-1]},25} \\ * & * & \Xi_{\mathbb{Z}_{[1,N-1]},33} & \Xi_{\mathbb{Z}_{[1,N-1]},34} & \mathbf{0} \\ * & * & * & \Xi_{\mathbb{Z}_{[1,N-1]},44} & \Xi_{\mathbb{Z}_{[1,N-1]},45} \\ * & * & * & * & \Xi_{\mathbb{Z}_{[1,N-1]},55} \end{bmatrix} < 0, \quad (50b)$$

$$\Xi_0 = \begin{bmatrix} \Xi_{0,11} & \mathbf{0} & \Xi_{0,13} & \Xi_{0,14} & \Xi_{0,15} \\ * & \Xi_{0,22} & \mathbf{0} & \Xi_{0,24} & \Xi_{0,25} \\ * & * & \Xi_{0,33} & \Xi_{0,34} & \mathbf{0} \\ * & * & * & \Xi_{0,44} & \Xi_{0,45} \\ * & * & * & * & \Xi_{0,55} \end{bmatrix} < 0, \quad (50c)$$

where

$$\begin{aligned} \Xi_{N,11} &= \left[\begin{array}{c|c} -\widetilde{M}_{i_0,i_{N-1},1}^{f_{N-2},f_{N-1}} + Q_1 & -\widetilde{M}_{i_0,i_{N-1},2}^{f_{N-2},f_{N-1}} + Q_2 \\ * & -\widetilde{M}_{i_0,i_{N-1},3}^{f_{N-2},f_{N-1}} + Q_3 \end{array} \right], \\ \Xi_{N,13} &= \left[\begin{array}{c|c} \mathbf{0} & G_{i_{N-1}}^T - C_{i_{N-1}}^T D_{F,i_{N-1}}^T \\ \mathbf{0} & -C_{F,i_{N-1}}^T \end{array} \right], \\ \Xi_{N,14} &= \left[\begin{array}{c|c} (X_{i_{N-1}} A_{i_{N-1}})^T + (W_{i_{N-1}} C_{i_{N-1}})^T & (Z_{i_{N-1}} A_{i_{N-1}})^T + (W_{i_{N-1}} C_{i_{N-1}})^T \\ \hline V_{i_{N-1}}^T & V_{i_{N-1}}^T \end{array} \right], \\ \Xi_{N,15} &= \left[\begin{array}{ccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & (H_{i_{N-1}}^a)^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right], \\ \Xi_{N,22} &= \begin{bmatrix} -Q_1 & -Q_2 \\ * & -Q_3 \end{bmatrix}, \\ \Xi_{N,24} &= \left[\begin{array}{c|c} (X_{i_{N-1}} A_{d,i_{N-1}})^T & (Z_{i_{N-1}} A_{d,i_{N-1}})^T \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right], \\ \Xi_{N,25} &= \left[\begin{array}{ccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (H_{i_{N-1}}^d)^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right], \\ \Xi_{N,33} &= \left[\begin{array}{c|c} -\gamma^2 \mathbf{I} & F_{i_{N-1}}^T - D_{i_{N-1}}^T D_{F,i_{N-1}}^T \\ * & -\mathbf{I} \end{array} \right], \\ \Xi_{N,34} &= \left[\begin{array}{c|c} (X_{i_{N-1}} B_{i_{N-1}})^T + (W_{i_{N-1}} D_{i_{N-1}})^T & (Z_{i_{N-1}} B_{i_{N-1}})^T + (W_{i_{N-1}} D_{i_{N-1}})^T \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right], \\ \Xi_{N,44} &= \left[\begin{array}{c|c} -X_{i_{N-1}} - X_{i_{N-1}}^T + \widetilde{P}_{i_0,i_{N-1},1}^{f_{N-1},f_N} & -Y_{i_{N-1}} - Z_{i_{N-1}}^T + \widetilde{P}_{i_0,i_{N-1},2}^{f_{N-1},f_N} \\ * & -Y_{i_{N-1}} - Y_{i_{N-1}}^T + \widetilde{P}_{i_0,i_{N-1},3}^{f_{N-1},f_N} \end{array} \right], \\ \Xi_{N,45} &= \begin{bmatrix} X_{i_{N-1}} E_{i_{N-1}} & \mathbf{0} & X_{i_{N-1}} E_{i_{N-1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ Z_{i_{N-1}} E_{i_{N-1}} & \mathbf{0} & Z_{i_{N-1}} E_{i_{N-1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\ \Xi_{N,55} &= \text{diag}_6(-\varepsilon \mathbf{I}), \\ \Xi_{\mathbb{Z}_{[1,N-1]},11} &= \left[\begin{array}{c|c} -\widetilde{M}_{i_0,i_{N-q-1},1}^{f_{N-q-2},f_{N-q-1}} + Q_1 & -\widetilde{M}_{i_0,i_{N-q-1},2}^{f_{N-q-2},f_{N-q-1}} + Q_2 \\ * & -\widetilde{M}_{i_0,i_{N-q-1},3}^{f_{N-q-2},f_{N-q-1}} + Q_3 \end{array} \right], \\ \Xi_{\mathbb{Z}_{[1,N-1]},13} &= \left[\begin{array}{c|c} \mathbf{0} & G_{i_{N-q-1}}^T - C_{i_{N-q-1}}^T D_{F,i_{N-q-1}}^T \\ \mathbf{0} & -C_{F,i_{N-q-1}}^T \end{array} \right], \\ \Xi_{\mathbb{Z}_{[1,N-1]},14} &= \left[\begin{array}{c|c} (X_{i_{N-q-1}} A_{i_{N-q-1}} + W_{i_{N-q-1}} C_{i_{N-q-1}})^T & (Z_{i_{N-q-1}} A_{i_{N-q-1}} + W_{i_{N-q-1}} C_{i_{N-q-1}})^T \\ \hline V_{i_{N-q-1}}^T & V_{i_{N-q-1}}^T \end{array} \right], \end{aligned}$$

$$\begin{aligned}
\Xi_{\mathbb{Z}_{[1,N-1]},15} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \left(H_{i_{N-q-1}}^a\right)^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\
\Xi_{\mathbb{Z}_{[1,N-1]},22} &= \begin{bmatrix} -Q_1 & -Q_2 \\ * & -Q_3 \end{bmatrix}, \\
\Xi_{\mathbb{Z}_{[1,N-1]},24} &= \left[\frac{\left(X_{i_{N-q-1}} A_{d,i_{N-q-1}}\right)^T}{\mathbf{0}} \middle| \frac{\left(Z_{i_{N-q-1}} A_{d,i_{N-q-1}}\right)^T}{\mathbf{0}} \right], \\
\Xi_{\mathbb{Z}_{[1,N-1]},25} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \left(H_{i_{N-q-1}}^d\right)^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\
\Xi_{\mathbb{Z}_{[1,N-1]},33} &= \left[\frac{-\gamma^2 \mathbf{I} \mid F_{i_{N-q-1}}^T \quad -D_{i_{N-q-1}}^T \quad D_{F,i_{N-q-1}}^T}{* \mid -\mathbf{I}} \right], \\
\Xi_{\mathbb{Z}_{[1,N-1]},34} &= \left[\frac{\left(X_{i_{N-q-1}} B_{i_{N-q-1}} + W_{i_{N-q-1}} D_{i_{N-q-1}}\right)^T}{\mathbf{0}} \middle| \frac{\left(Z_{i_{N-q-1}} B_{i_{N-q-1}} + W_{i_{N-q-1}} D_{i_{N-q-1}}\right)^T}{\mathbf{0}} \right], \\
\Xi_{\mathbb{Z}_{[1,N-1]},44} &= \left[\frac{-X_{i_{N-q-1}} - X_{i_{N-q-1}}^T + \widetilde{M}_{i_0,i_{N-q-1}}^{f_{N-q-1},f_{N-q}}}{*} \middle| \frac{-Y_{i_{N-q-1}} - Z_{i_{N-q-1}}^T + \widetilde{M}_{i_0,i_{N-q-1}}^{f_{N-q-1},f_{N-q}}}{-Y_{i_{N-q-1}} - Y_{i_{N-q-1}}^T + \widetilde{M}_{i_0,i_{N-q-1}}^{f_{N-q-1},f_{N-q}}} \right], \\
\Xi_{\mathbb{Z}_{[1,N-1]},45} &= \begin{bmatrix} X_{i_{N-q-1}} E_{i_{N-q-1}} & \mathbf{0} & X_{i_{N-q-1}} E_{i_{N-q-1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ Z_{i_{N-q-1}} E_{i_{N-q-1}} & \mathbf{0} & Z_{i_{N-q-1}} E_{i_{N-q-1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\
\Xi_{\mathbb{Z}_{[1,N-1]},55} &= \text{diag}_6(-\varepsilon \mathbf{I}), \\
\Xi_{0,11} &= \left[\frac{-P_{i_0,1}^{f_0} + Q_1}{*} \middle| \frac{-P_{i_0,2}^{f_0} + Q_2}{-P_{i_0,3}^{f_0} + Q_3} \right], \\
\Xi_{0,13} &= \begin{bmatrix} \mathbf{0} \mid G_{i_0}^T - C_{i_0}^T D_{F,i_0}^T \\ \mathbf{0} \mid -C_{F,i_0}^T \end{bmatrix}, \\
\Xi_{0,14} &= \left[\frac{\left(X_{i_0} A_{i_0}\right)^T + \left(W_{i_0} C_{i_0}\right)^T}{V_{i_0}^T} \middle| \frac{\left(Z_{i_0} A_{i_0}\right)^T + \left(W_{i_0} C_{i_0}\right)^T}{V_{i_0}^T} \right], \\
\Xi_{0,15} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \left(H_{i_0}^a\right)^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\
\Xi_{0,22} &= \begin{bmatrix} -Q_1 & -Q_2 \\ * & -Q_3 \end{bmatrix}, \\
\Xi_{0,25} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \left(H_{i_0}^d\right)^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\
\Xi_{0,33} &= \left[\frac{-\gamma^2 \mathbf{I} \mid F_{i_0}^T - D_{i_0}^T D_{F,i_0}^T}{* \mid -\mathbf{I}} \right], \\
\Xi_{N,34} &= \left[\frac{\left(X_{i_0} B_{i_0}\right)^T + \left(W_{i_0} D_{i_0}\right)^T}{\mathbf{0}} \middle| \frac{\left(Z_{i_0} B_{i_0}\right)^T + \left(W_{i_0} D_{i_0}\right)^T}{\mathbf{0}} \right], \\
\Xi_{N,44} &= \left[\frac{-X_{i_0} - X_{i_0}^T + \widetilde{M}_{i_0,i_1}^{f_0,f_1}}{*} \middle| \frac{-Y_{i_0} - Z_{i_0}^T + \widetilde{M}_{i_0,i_1}^{f_0,f_1}}{-Y_{i_0} - Y_{i_0}^T + \widetilde{M}_{i_0,i_1}^{f_0,f_1}} \right], \\
\Xi_{N,45} &= \begin{bmatrix} X_{i_0} E_{i_0} & \mathbf{0} & X_{i_0} E_{i_0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ Z_{i_0} E_{i_0} & \mathbf{0} & Z_{i_0} E_{i_0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\
\Xi_{N,55} &= \text{diag}_6(-\varepsilon \mathbf{I}),
\end{aligned}$$

and $\widetilde{M}_{i_0, i_{N-1}}^{f_{N-2}, f_{N-1}}, \widetilde{P}_{i_0, i_N}^{f_{N-1}, f_N}, \widetilde{M}_{i_0, i_{N-q-1}}^{f_{N-q-2}, f_{N-q-1}}, \widetilde{M}_{i_0, i_{N-q}}^{f_{N-q-1}, f_{N-q}}, \widetilde{M}_{i_0, i_1}^{f_0, f_1}$ are the same as in Lemma 3. Moreover, if (50a)–(50c) are true, the desired filtering parameters of the form (5) can be obtained as

$$\begin{aligned} A_{F, i_0} &= Y_{i_0}^{-1} V_{i_0}, \\ B_{F, i_0} &= Y_{i_0}^{-1} W_{i_0}, \\ C_{F, i_0} &, \\ D_{F, i_0} &. \end{aligned} \quad (52)$$

Proof. Noting that $-R_{i_{N-1}}(\widetilde{P}_{i_0, i_N}^{f_{N-1}, f_N})^{-1} R_{i_{N-1}}^T \leq -R_{i_{N-1}} - R_{i_{N-1}}^T + \widetilde{P}_{i_0, i_N}^{f_{N-1}, f_N}$, applying the well known Schur complement [41] to (34a), and performing a congruence transformation $\text{diag}\{\mathbf{I}, \mathbf{I}, \mathbf{I}, \mathbf{I}, R_{i_{N-1}}\}, \text{diag}\{\mathbf{I}, \mathbf{I}, \mathbf{I}, \mathbf{I}, R_{i_{N-1}}^T\}$ to (34a), we have the following.

$$\begin{bmatrix} -\widetilde{M}_{i_0, i_{N-1}}^{f_{N-2}, f_{N-1}} + Q & \mathbf{0} & \mathbf{0} & \widetilde{C}_{i_{N-1}}^T & (R_{i_{N-1}} \widehat{A}_{i_{N-1}})^T \\ * & -Q & \mathbf{0} & \mathbf{0} & (R_{i_{N-1}} \widehat{A}_{d, i_{N-1}})^T \\ * & * & -\gamma^2 \mathbf{I} & \widehat{D}_{i_{N-1}}^T & (R_{i_{N-1}} \widehat{B}_{i_{N-1}})^T \\ * & * & * & -\mathbf{I} & \mathbf{0} \\ * & * & * & * & -R_{i_{N-1}} - R_{i_{N-1}}^T + \widetilde{P}_{i_0, i_N}^{f_{N-1}, f_N} \end{bmatrix} < 0 \quad (53)$$

Letting

$$\begin{aligned} V_{i_{N-1}} &= Y_{i_{N-1}} A_{F, i_{N-1}}, \\ W_{i_{N-1}} &= Y_{i_{N-1}} B_{F, i_{N-1}}, \\ \widetilde{M}_{i_0, i_{N-1}}^{f_{N-2}, f_{N-1}} &= \begin{bmatrix} \widetilde{M}_{i_0, i_{N-1}, 1}^{f_{N-2}, f_{N-1}} & \widetilde{M}_{i_0, i_{N-1}, 2}^{f_{N-2}, f_{N-1}} \\ * & \widetilde{M}_{i_0, i_{N-1}, 3}^{f_{N-2}, f_{N-1}} \end{bmatrix}, \\ \widetilde{P}_{i_0, i_N}^{f_{N-1}, f_N} &= \begin{bmatrix} \widetilde{P}_{i_0, i_N, 1}^{f_{N-1}, f_N} & \widetilde{P}_{i_0, i_N, 2}^{f_{N-1}, f_N} \\ * & \widetilde{P}_{i_0, i_N, 3}^{f_{N-1}, f_N} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} R_{i_{N-1}} &= \begin{bmatrix} X_{i_{N-1}} & Y_{i_{N-1}} \\ Z_{i_{N-1}} & Y_{i_{N-1}} \end{bmatrix}, \\ Q &= \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix}, \end{aligned} \quad (54)$$

and substituting (54) into (53), one has

$$\Upsilon_N = \left[\begin{array}{c|c|c|c} \Xi_{N,11} & \mathbf{0} & \Xi_{N,13} & \Upsilon_{N,14} \\ * & \Xi_{N,22} & \mathbf{0} & \Upsilon_{N,24} \\ * & * & \Xi_{N,33} & \Xi_{N,34} \\ * & * & * & \Xi_{N,44} \end{array} \right] < 0, \quad (55)$$

where

$$\begin{aligned} \Upsilon_{N,14} &= \left[\frac{(X_{i_{N-1}} \widetilde{A}_{i_{N-1}})^T + (W_{i_{N-1}} C_{i_{N-1}})^T}{V_{i_{N-1}}^T} \middle| \frac{(Z_{i_{N-1}} \widetilde{A}_{i_{N-1}})^T + (W_{i_{N-1}} C_{i_{N-1}})^T}{V_{i_{N-1}}^T} \right], \\ \Upsilon_{N,24} &= \left[\frac{(X_{i_{N-1}} \widetilde{A}_{d, i_{N-1}})^T}{\mathbf{0}} \middle| \frac{(Z_{i_{N-1}} \widetilde{A}_{d, i_{N-1}})^T}{\mathbf{0}} \right]. \end{aligned} \quad (56)$$

By taking (2) into account, one further has

$$\Upsilon_N = \Psi_N + \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots \\ \hline X_{i_{N-1}} E_{i_{N-1}} & \mathbf{0} & X_{i_{N-1}} E_{i_{N-1}} \\ \hline X_{i_{N-1}} E_{i_{N-1}} & \mathbf{0} & X_{i_{N-1}} E_{i_{N-1}} \end{array} \right] \mathcal{L}_{i_{N-1}}(k)$$

$$\cdot \left[\begin{array}{c|c|c|c|c} H_{i_{N-1}}^a & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & H_{i_{N-1}}^d & \mathbf{0} & \cdots & \mathbf{0} \end{array} \right] \quad (57)$$

where

$$\Psi_N = \left[\begin{array}{c|c|c|c} \Xi_{N,11} & \mathbf{0} & \Xi_{N,13} & \Psi_{N,14} \\ * & \Xi_{N,22} & \mathbf{0} & \Psi_{N,24} \\ * & * & \Xi_{N,33} & \Xi_{N,34} \\ * & * & * & \Xi_{N,44} \end{array} \right] < 0, \quad (58)$$

$$\Psi_{N,14} = \left[\begin{array}{c|c} \frac{(X_{i_{N-1}} A_{i_{N-1}})^T + (W_{i_{N-1}} C_{i_{N-1}})^T}{V_{i_{N-1}}^T} & \frac{(Z_{i_{N-1}} A_{i_{N-1}})^T + (W_{i_{N-1}} C_{i_{N-1}})^T}{V_{i_{N-1}}^T} \end{array} \right],$$

$$\Psi_{N,24} = \left[\begin{array}{c|c} \frac{(X_{i_{N-1}} A_{d,i_{N-1}})^T}{\mathbf{0}} & \frac{(Z_{i_{N-1}} A_{d,i_{N-1}})^T}{\mathbf{0}} \end{array} \right].$$

By employing Schur complement and *Lemma 8* to (57), we know that $Y_N < 0$ implies $\Xi_N < 0$. Therefore, it can be concluded that condition (34a) can be transformed into an LMI formulation (50a) with designed filter gains. Following the same lines of the proof of (50a), we can prove that the filtering gain matrices obtained by *Theorem 9* can achieve a bounded filtering error (9). This completes the proof. $\square$

Remark 10. The NALKF approach also has potentials to be extended to solve other filtering or control problems of time-delayed systems, such as sampled-data synchronization control [42, 43], time-varying delay analysis [44, 45], and H_∞ repetitive-control [46]. However, it should be noted that the NALKF approach can only be developed for discrete-time systems. For the continuous-time case, the higher derivative of the Lyapunov functional should be considered, such as $\min(\dot{V}(t), \ddot{V}(t)) < 0$ or $\dot{V}(t) + \ddot{V}(t) < 0$. Moreover, for time-delayed systems, a trade-off must be considered between the conservatism reduction and computational burden.

5. Illustrative Example

In this section, we provide an illustrative example to verify the effectiveness and applicability of the proposed NALKF method.

Consider NMJDS (1) with parameters as follows

$$A_1 = \begin{bmatrix} 0.5 & -0.6 \\ -0.4 & -0.2 \end{bmatrix},$$

$$A_{d,1} = \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & 0.15 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.15 & 0.3 \end{bmatrix},$$

$$A_{d,2} = \begin{bmatrix} 0.38 & 0 \\ 0.01 & 0.16 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix},$$

$$D_1 = 0.4,$$

$$C_2 = \begin{bmatrix} 0.3 \\ -0.4 \end{bmatrix},$$

$$D_2 = -0.5,$$

$$G_1 = [0.7 \ 0.3],$$

$$F_1 = 0.2,$$

$$G_2 = [0.2 \ 0.4],$$

$$F_2 = -0.1,$$

$$E = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$H_1^a = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.04 \end{bmatrix},$$

$$H_2^a = \begin{bmatrix} 0.02 & 0 \\ 1 & 0.05 \end{bmatrix},$$

$$H_1^d = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix},$$

$$H_2^d = \begin{bmatrix} 0.32 & 0 \\ 1 & 0.68 \end{bmatrix},$$

$$\mathcal{L}_1(k) = \mathcal{L}_2(k) = \begin{bmatrix} \sin(k) & 0 \\ 0 & \sin(k) \end{bmatrix}.$$

(59)

The TP matrix is assumed to be time-varying in a polytope given by its vertices

$$\Pi_1 = \begin{bmatrix} 0.1 & 0.9 \\ 0.2 & 0.8 \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}.$$

(60)

The indicator functions are chosen as

$$\begin{aligned}\mu_1(k) &= \frac{1 - x_{k,1}}{2}, \\ \mu_2(k) &= \frac{1 + x_{k,1}}{2}.\end{aligned}\quad (61)$$

Here $x_{k,1}$ represents the first state of x_k . The initial state, initial estimated state, initial z_k , and initial $\hat{z}_k$ are taken as $x_0 = \hat{x}_0 = [0 \ 0]^T$, $z_0 = \hat{z}_0 = [0 \ 0]^T$. The initial operating mode and time delay are assumed as $r_0 = 1$ and $d = 4$. The exogenous disturbance w_k is taken as $0.2e^{-0.5k}$.

In this case, we would like to provide an optimal H_∞ performance for designing the filter; that is, we are interested in the optimization problem

$$\begin{aligned}\min \quad & \gamma \\ \text{subject to} \quad & (50a)-(50c)\end{aligned}\quad (62)$$

under zero-initial condition. Solving (50a)-(50c) by using the LMI Toolbox, the robust H_∞ filter gains obtained via two-step-ahead approach are listed as

$$\begin{aligned}A_{F,1} &= \begin{bmatrix} -0.0214 & 0.2915 \\ -0.0636 & -0.6562 \end{bmatrix}, \\ A_{F,2} &= \begin{bmatrix} 1.2150 & 0.3716 \\ -0.1274 & 0.1425 \end{bmatrix}, \\ B_{F,1} &= \begin{bmatrix} -3.2391 \\ -0.9403 \end{bmatrix}, \\ B_{F,2} &= \begin{bmatrix} 0.1913 \\ 1.5089 \end{bmatrix}, \\ C_{F,1} &= [-0.6039 \quad -0.4670], \\ C_{F,2} &= [-0.1541 \quad -0.4506], \\ D_{F,1} &= -0.5960, \\ D_{F,2} &= -0.1448.\end{aligned}\quad (63)$$

Similarly, the robust H_∞ filter gains obtained via three-step-ahead approach are listed as

$$\begin{aligned}A_{F,1} &= \begin{bmatrix} -0.3931 & 0.3412 \\ 0.7966 & -0.9720 \end{bmatrix}, \\ A_{F,2} &= \begin{bmatrix} 0.4723 & 0.3298 \\ 0.4882 & 0.3683 \end{bmatrix}, \\ B_{F,1} &= \begin{bmatrix} -7.3667 \\ 6.3261 \end{bmatrix},\end{aligned}$$

$$\begin{aligned}B_{F,2} &= \begin{bmatrix} 0.6209 \\ 0.6633 \end{bmatrix}, \\ C_{F,1} &= [-0.5337 \quad -0.3707], \\ C_{F,2} &= [-0.3412 \quad -0.0273], \\ D_{F,1} &= -0.9465, \\ D_{F,2} &= 0.3611.\end{aligned}\quad (64)$$

It must be pointed out that the conventional filtering approach, such as in [18], can be viewed as a special case of the NALKF approach, i.e., the one-step-ahead approach. The filtering gains can be obtained as

$$\begin{aligned}A_{F,1} &= \begin{bmatrix} 0.0531 & -0.0076 \\ -0.0727 & -0.5311 \end{bmatrix}, \\ A_{F,2} &= \begin{bmatrix} 0.1398 & 1.7352 \\ 0.0533 & 0.7602 \end{bmatrix}, \\ B_{F,1} &= \begin{bmatrix} -9.8681 \\ 4.2420 \end{bmatrix}, \\ B_{F,2} &= \begin{bmatrix} -1.3763 \\ -0.5301 \end{bmatrix}, \\ C_{F,1} &= [0.0761 \quad -1.0295], \\ C_{F,2} &= [-0.0433 \quad -0.6545], \\ D_{F,1} &= -6.9897, \\ D_{F,2} &= -0.5292.\end{aligned}\quad (65)$$

We also get the minimum value of γ and the number of decision variables of NALKF approach, which are shown in Table 1. Obviously, with the increase of the predictive step, we obtain better H_∞ performance. However, it must be pointed out that the number of decision variables of the developed NALKF approach is not increased with the predictive step because the knowledge of the multistep TPs $\pi_{i_0, i_{N-q}}$ is included.

The jumping mode path shown in Figure 1 from time step 0 to time step 50 is generated randomly by employing TP matrices Π_1, Π_2 and indicator functions $\mu_1(k), \mu_2(k)$. The number of iterations is chosen as 50 and each iteration unit length is taken as 1.

The comparison of the estimated error under jumping mode path in Figure 1 is demonstrated in Figure 2, where the red line, green line, and blue line represent the error under conventional filtering approach, two-step-ahead approach, and three-step-ahead approach, respectively. It also can be concluded that with the increase of the predictive step, the estimated error becomes smaller.

Moreover, the estimate of the output $\hat{z}_k$ under jumping mode path in Figure 1 is also drawn in Figure 3, where the black line is the output z_k . It can be observed that the

TABLE 1: Minimum value of γ and number of the decision variables.

a	minimum value of γ	number of decision variables
conventional approach	8.7815	81
two-step approach	1.8374	117
three-step approach	1.6383	117

FIGURE 1: Sample path of the nonhomogeneous Markovian chain.

FIGURE 3: Estimate of z_k .

FIGURE 2: Estimated error.

FIGURE 4: Monte Carlo simulation of estimated error.

three-step-ahead approach (blue line) has the best tracking performance, which also justifies the results of Figure 2.

Finally, a Monte Carlo simulation of the estimated error, which is generated under 20 different jumping mode paths, is given in Figure 4. Such a simulation uses the two-step-ahead approach. It can be seen that the overall estimation quality is good. The extreme case is less than 25%. Therefore, it justifies the effectiveness of the developed approach from another side.

6. Conclusion

The robust H_∞ filtering problem has been considered in this paper for nonhomogeneous Markovian jump delay systems via N -step-ahead Lyapunov-Krasovskii functional approach. The robust H_∞ filter has been designed in terms of a feasible optimization problem subject to LMI constraints, which guarantees that the filtering error system is mean-square stable and the filtering error satisfies smaller H_∞ dissipative level for all possible time-varying transition probabilities and all admissible parameter uncertainties and time-delays. The

N -step-ahead approach can be extended to deal with the robust H_∞ dynamic output feedback control problem.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] X.-H. Chang, *Robust Output Feedback H_∞ Control and Filtering for Uncertain Linear Systems*, Springer, London, UK, 2014.
- [2] R. E. Kalman, "A new approach to linear filtering and prediction problems," *Journal of Fluids Engineering*, vol. 82, no. 1, pp. 35–45, 1960.
- [3] M. S. Mahmoud, N. F. Al-Muthairi, and S. Bingulac, "Robust Kalman filtering for continuous time-lag systems," *Systems & Control Letters*, vol. 38, no. 4–5, pp. 309–319, 1999.
- [4] X. Zhu, Y. C. Soh, and L. Xie, "Design and analysis of discrete-time robust Kalman filters," *Automatica*, vol. 38, no. 6, pp. 1069–1077, 2002.
- [5] M. S. Mahmoud and H. M. Khalid, "Distributed Kalman filtering: a bibliographic review," *IET Control Theory & Applications*, vol. 7, no. 4, pp. 483–501, 2013.
- [6] W. Li, Y. Jia, and J. Du, "Distributed extended Kalman filter with nonlinear consensus estimate," *Journal of The Franklin Institute*, vol. 354, no. 17, pp. 7983–7995, 2017.
- [7] G. Battistelli, L. Chisci, and D. Selvi, "A distributed Kalman filter with event-triggered communication and guaranteed stability," *Automatica*, vol. 93, pp. 75–82, 2018.
- [8] D. Viegas, P. Batista, P. Oliveira, and C. Silvestre, "Discrete-time distributed Kalman filter design for formations of autonomous vehicles," *Control Engineering Practice*, vol. 75, pp. 55–68, 2018.
- [9] S. Dey, A. S. Leong, and J. S. Evans, "Kalman filtering with faded measurements," *Automatica*, vol. 45, no. 10, pp. 2223–2233, 2009.
- [10] H. Geng, Z. Wang, Y. Liang, Y. Cheng, and F. E. Alsaadi, "Tobit Kalman filter with fading measurements," *Signal Processing*, vol. 140, pp. 60–68, 2017.
- [11] X. Liu, L. Li, Z. Li, H. H. C. Iu, and T. Fernando, "Stochastic stability of modified extended Kalman filter over fading channels with transmission failure and signal fluctuation," *Signal Processing*, vol. 138, pp. 220–232, 2017.
- [12] H. Gao and X. Li, *Robust Filtering for Uncertain Systems-A Parameter-Dependent Approach*, Springer, London, UK, 2014.
- [13] X. Zhang and Q. Han, " H_∞ filtering for networked systems," *Automatica*, vol. 49, no. 5, pp. 1428–1435, 2013.
- [14] X. Q. Xiao, L. Zhou, and G. P. Liu, "Event-triggered H_∞ filtering of continuous-time switched linear systems," *Signal Processing*, vol. 141, pp. 343–349, 2017.
- [15] X. Xiao, J. H. Park, and L. Zhou, "Event-triggered H_∞ filtering of discrete-time switched linear systems," *ISA Transactions*, vol. 77, pp. 112–121, 2018.
- [16] J. Wang, F. Li, and Y. Sun, "Asynchronous l_2 - l_∞ filtering for networked fuzzy systems with Markov jump parameters over a finite-time interval," *IET Control Theory & Applications*, vol. 10, no. 17, pp. 2175–2185, 2016.
- [17] L. Zhang, T. Yang, P. Shi, and Y. Zhu, *Analysis and Design of Markov Jump Systems with Complex Transition Probabilities*, Springer, Switzerland, 2016.
- [18] H. Liu, Y. Ding, and J. Cheng, "New results on H_∞ filtering for Markov jump systems with uncertain transition rates," *ISA Transactions*, vol. 69, pp. 43–50, 2017.
- [19] M. Sathishkumar, R. Sakthivel, C. Wang, B. Kaviarasan, and S. Marshal Anthoni, "Non-fragile filtering for singular Markovian jump systems with missing measurements," *Signal Processing*, vol. 142, pp. 125–136, 2018.
- [20] Y. Yin, P. Shi, F. Liu, and K. L. Teo, "Filtering for discrete-time nonhomogeneous Markov jump systems with uncertainties," *Information Sciences*, vol. 259, pp. 118–127, 2014.
- [21] M. Hua, L. Zhang, F. Yao, J. Ni, W. Dai, and Y. Cheng, "Robust H_∞ filtering for continuous-time nonhomogeneous Markov jump nonlinear systems with randomly occurring uncertainties," *Signal Processing*, vol. 148, pp. 250–259, 2018.
- [22] J. Wang, S. Ma, C. Zhang, and M. Fu, "Finite-time H_∞ filtering for nonlinear singular systems with nonhomogeneous Markov jumps," *IEEE Transactions on Cybernetics*, vol. 49, no. 6, pp. 2133–2143, 2018.
- [23] S. Zhou and G. Feng, " H_∞ filtering for discrete-time systems with randomly varying sensor delays," *Automatica*, vol. 44, no. 7, pp. 1918–1922, 2008.
- [24] Y. Zhang, G. Cheng, and C. Liu, "Finite-time unbiased H_∞ filtering for discrete jump time-delay systems," *Applied Mathematical Modelling: Simulation and Computation for Engineering and Environmental Systems*, vol. 38, no. 13, pp. 3339–3349, 2014.
- [25] F. Wang, W.-W. Che, and H.-L. Xu, " H_∞ filtering for uncertain systems with time-delay and randomly occurred sensor nonlinearities," *Neurocomputing*, vol. 174, pp. 571–576, 2016.
- [26] Y.-Y. Cao, Z. Lin, and T. Hu, "Stability analysis of linear time-delay systems subject to input saturation," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 49, no. 2, pp. 233–240, 2002.
- [27] A. A. Ahmadi and P. A. Parrilo, "Non-monotonic Lyapunov functions for stability of discrete time nonlinear and switched systems," in *Proceedings of the 47th IEEE Conference on Decision and Control, CDC '08*, pp. 614–621, Mexico, 2008.
- [28] S. F. Derakhshan and A. Fatehi, "Non-monotonic Lyapunov functions for stability analysis and stabilization of discrete time Takagi-Sugeno fuzzy systems," *International Journal of Innovative Computing, Information and Control*, vol. 10, no. 4, pp. 1567–1586, 2014.
- [29] S. F. Derakhshan, A. Fatehi, and M. G. Sharabiany, "Nonmonotonic observer-based fuzzy controller designs for discrete time T-S fuzzy systems via LMI," *IEEE Transactions on Cybernetics*, vol. 44, no. 12, pp. 2557–2567, 2014.
- [30] S. F. Derakhshan and A. Fatehi, "Non-monotonic robust H_2 fuzzy observer-based control for discrete time nonlinear systems with parametric uncertainties," *International Journal of Systems Science*, vol. 46, no. 12, pp. 2134–2149, 2015.
- [31] A. Kruszewski, T. M. Guerra, and S. Labiod, "Stabilization of Takagi-Sugeno discrete models: towards a unification of the

- results,” in *Proceedings of the IEEE International Fuzzy Systems Conference*, pp. 1–6, 2007.
- [32] A. Kruszewski, R. Wang, and T. M. Guerra, “Non-quadratic stabilization conditions for a class of uncertain nonlinear discrete time T-S fuzzy models: a new approach,” *IEEE Transactions on Automatic Control*, vol. 53, no. 2, pp. 606–611, 2008.
 - [33] T. M. Guerra, A. Kruszewski, and S. Lauber, “Discrete Takagi-Sugeno models for control: where are we,” *Annual Reviews in Control*, vol. 33, no. 1, pp. 37–47, 2009.
 - [34] T. M. Guerra, A. Kruszewski, and M. Bernal, “Control law proposition for the stabilization of discrete Takagi-Sugeno models,” *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 3, pp. 724–731, 2009.
 - [35] Y.-J. Chen, M. Tanaka, K. Inoue et al., “A non-monotonically decreasing relaxation approach of Lyapunov functions to guaranteed cost control for discrete fuzzy systems,” *IET Control Theory & Applications*, vol. 8, no. 16, pp. 1716–1722, 2014.
 - [36] A. Nasiri, S. K. Nguang, A. Swain, and D. J. Almkhles, “Robust output feedback controller design of discrete-time Takagi-Sugeno fuzzy systems: a non-monotonic Lyapunov approach,” *IET Control Theory & Applications*, vol. 10, no. 5, pp. 545–553, 2016.
 - [37] A. Nasiri, S. K. Nguang, A. Swain, and D. J. Almkhles, “Reducing conservatism in an H_{∞} robust state-feedback control design of T-S fuzzy systems: a non-monotonic approach,” *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 1, pp. 386–390, 2018.
 - [38] J. Wen, S. K. Nguang, P. Shi, and X. D. Zhao, “Stability and H_{∞} control of discrete-time switched systems via one-step ahead Lyapunov function approach,” *IET Control Theory & Applications*, vol. 12, no. 8, pp. 1141–1147, 2018.
 - [39] Y. Xie, J. Wen, S. K. Nguang, and L. Peng, “Stability, l_2 -gain and robust H_{∞} control for switched systems via N-step ahead Lyapunov function approach,” *IEEE Access*, vol. 5, no. 1, pp. 26400–26408, 2017.
 - [40] J. Wen, S. K. Nguang, P. Shi, and A. Nasiri, “Robust H_{∞} control of discrete-time non-homogenous Markovian jump systems via multistep Lyapunov function approach,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 47, no. 7, pp. 1439–1450, 2017.
 - [41] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics, Philadelphia, Pa, USA, 1994.
 - [42] S.-P. Xiao, H.-H. Lian, K. L. Teo, H.-B. Zeng, and X.-H. Zhang, “A new Lyapunov functional approach to sampled-data synchronization control for delayed neural networks,” *Journal of The Franklin Institute*, vol. 355, no. 17, pp. 8857–8873, 2018.
 - [43] H. Lian, S. Xiao, Z. Wang, X. Zhang, and H. Xiao, “Further results on sampled-data synchronization control for chaotic neural networks with actuator saturation,” *Neurocomputing*, vol. 346, pp. 30–37, 2019.
 - [44] X.-M. Zhang, Q.-L. Han, Z. Wang, and B.-L. Zhang, “Neuronal state estimation for neural networks with two additive time-varying delay components,” *IEEE Transactions on Cybernetics*, vol. 47, no. 10, pp. 3184–3194, 2017.
 - [45] X.-M. Zhang, Q.-L. Han, and J. Wang, “Admissible delay upper bounds for global asymptotic stability of neural networks with time-varying delays,” *IEEE Transactions on Neural Networks and Learning Systems*, vol. 29, no. 11, pp. 5319–5329, 2018.
 - [46] L. Zhou, J. She, and S. Zhou, “Robust H_{∞} control of an observer-based repetitive-control system,” *Journal of The Franklin Institute*, vol. 355, no. 12, pp. 4952–4969, 2018.

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