Robust $H_{\infty}$ Filtering of Nonhomogeneous Markovian Jump Delay Systems via $N$-Step-Ahead Lyapunov-Krasovskii Functional Approach

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In this paper, we study the robust $H_{\infty}$ filtering problem for a class of nonhomogeneous Markovian jump delay systems (NMJDSs) with an $N$-step-ahead Lyapunov-Krasovskii functional (NALKF) approach. The $N$-step-ahead approach is utilized to reduce the conservatism of robust $H_{\infty}$ filtering. We aim to design filters such that, for all possible time-varying transition probabilities and all admissible parameter uncertainties and time-delays, the filtering error system is mean-square stable with a smaller estimated error and a lower dissipative level. In terms of linear matrix inequalities, sufficient conditions for the solvability of the addressed problem are developed via a moving horizon method. An illustrative example is included to elucidate advantages of the developed results and the practical potential of NALKF approach for NMJDSs.

1. Introduction

Filtering is one of the most fundamental problems in modern control theory and application fields, such as control system synthesis, state estimation, and information fusion. From the signal-processing point of view, useful signals are inevitably contaminated by exogenous noise, which leads to the discrepancy between measured signals and actual signals. Thus it is necessary to minimize inaccuracies caused by noise and estimate the signal as close to the actual one [1]. Filtering methods are rightly aimed at this purpose. The goal of filtering is to estimate signals that are corrupted by noises or unmeasurable or technically difficult to measure.

With the development of Kalman filtering theory [2] for stochastic systems, numerous extended results of such an optimal filtering have been reported with burgeoning research interest. To name a few, robust Kalman filter has been designed for continuous-time delay systems with norm bounded uncertainties [3] and discrete-time uncertain systems [4] by fully making use of the mean value and the variance knowledge of Gaussian noise. Nowadays, a new research trend is to study Kalman filtering for networked systems. The distributed Kalman filtering with the effects of network structures was introduced by a thorough bibliographic review in [5]. Recently, distributed extended Kalman filter with nonlinear consensus estimate [6] or with event-triggered scheme and stability guarantee [7] has been developed. As a typical application example, formation of autonomous multiple vehicles was achieved by employing distributed Kalman filter design. Each vehicle aims to estimate its own state, such as position, velocity, and acceleration, by its locally available measurements, and has limited communication with its neighbors [8]. Fading measurements are also noteworthy phenomena induced by the wireless network, where fading rates during the communication are described by random variables with known statistical properties dependent on or independent of the system mode. It usually results in unpredictable performance degradation for the filters [9]. Tobit Kalman filter [10] and modified extended Kalman filter [11] have been further investigated...
over fading channels with transmission failure or signal fluctuation.

The successful application of Kalman filter in the aerospace and aviation industry has led applications in industry in the 1970s. However, these attempts have shown that there was a serious mismatch between the underlying assumptions of Kalman filtering and the industrial state estimation problem. In terms of engineering application, it is quite costly or difficult to get an accurate system model and engineers are seldom aware of the statistical characteristics of noise that affect the industrial processes. A useful scheme to deal with modeling uncertainty and non-Gaussian noise is $H_\infty$ filtering. The objective is to minimize the $L_2$ gain of the filtering error system from noise inputs to filtering errors. $H_\infty$ filtering can be effective in the case that noise inputs are energy-bounded signals, without any statistical knowledge of the noise, rather than Gaussian white noise [12]. Moreover, it is robust against uncertainties both in the system modeling and exogenous noise.

The development of $H_\infty$ filtering is reviewed in [13–16] and references therein. It is worth mentioning that an $H_\infty$ filtering for a class of special switched systems, i.e., Markovian jump systems (MJSs), attracted much attention of the control community. MJSs can well depict various physical phenomena, such as solar thermal receiver, Samuelson’s multiplier-accelerator model, NASA F-8 test aircraft, and networked control systems [17]. Such systems may run with external environmental changes, actuator fault, and communication time delay, data packet loss, and so on. These random factors often cause jumping phenomenon of system structure or parameters. Aiming at the MJSs, a great deal of research work has emerged (see [17–19] and references therein). The essential difference between MJSs and linear systems is the modes jumping character, which is governed by the transition probabilities (TPs). Therefore, the $H_\infty$ filtering for nonhomogeneous Markovian jump systems (NMJSSs), that is, MJSs with time-varying TPs, starts to penetrate into the research front-line of the filtering. Filtering for discrete-time uncertain MJSs [20], robust $H_\infty$ filtering for continuous-time nonlinear NMJSSs with randomly occurring uncertainties [21], and finite-time $H_\infty$ filtering for nonlinear singular NMJSSs [22] have been extensively studied. Moreover, the $H_\infty$ filtering for time-delayed systems has also attracted great research interests (see [23–25] and references therein). Many effective approaches, such as the famous Lyapunov-Krasovskii functional (LKF) approach and Lyapunov-Razumikhin functional approach, are developed independently for handling time delay [26]. Specifically, robust filtering performance can be achieved by constructing a proper LKF for time delay systems.

However, filter design approaches developed for NMJSSs in the above-mentioned work are conservative. To reduce the design conservatism, an interesting idea is to construct nonmonotonic Lyapunov functional (NLF) [27]. It has been successfully used for T-S fuzzy model to relax the monotonicity requirement of LF and further reduce the conservatism of the stability criteria, i.e., allowing the LF to increase locally during several sampling periods. Two-sample variation, i.e., $V(x_{k+2}) < V(x_k)$ [28–30], and $N$-sample variation, i.e., $V(x_{k+N}) < V(x_k)$ [31–37], were fully developed for the T-S fuzzy model. Stability analysis and synthesis, robust $H_\infty$ controller design, observer-based fuzzy controller design, and output feedback stabilization have been intensively studied. Some preliminary results for switched systems [38, 39] and NMJSSs [40] have also been reported.

Based on the above observations, it can be concluded that the NLF approach has not been fully developed for filtering of time-delayed jump systems. A big challenge is how to deal with noise of future time and delay interval according to the predictive horizon $N$. In this study, we aim to investigate the $H_\infty$ filtering problem for a class of nonhomogeneous Markovian jump delay systems (NMJDSs) via an $N$-step-ahead Lyapunov-Krasovskii functional (NALKF) approach. The main contributions and novelties of this paper are summarized as follows:

(i) The NALKF approach is developed to reduce the conservatism of the filtering design by properly constructing a LKF and allow the underlying LKF to increase during the period of $N$ sampling time steps ahead of the current time within each jump mode.

(ii) The linear matrix inequality (LMI) formulation of the sufficient conditions for filtering is obtained by moving the horizon from $k + N - 1$ to $k + 1$ step by step. Due to the predictive horizon, the derivation is not trivial.

(iii) For all possible time-varying TPs and all admissible parameter uncertainties and time-delays, the filtering error system is mean-square stable with a smaller estimated error and a lower dissipative level.

Notations. $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote $n$-dimensional Euclidean space and set of all $n \times m$ matrices, respectively. $\| \cdot \|_2$ refers to the Euclidean vector norm. $\mathcal{F}_k$ stands for the conditional mathematical expectation, where $\mathcal{F}_k = \sigma((x_0, r_0), \ldots, (x_k, r_k))$ is the $\sigma$-algebra. $\lambda_{\min}(\Theta)$ represents the minimum eigenvalue of matrix $\Theta$. $\text{diag}(\cdots)$ stands for the block-diagonal matrix with $n$ blocks given by matrices in ($\cdots$). In symmetric matrices, we use * as an ellipsis for the symmetric terms above or below the diagonal. Identity and zero matrices of appropriate dimensions are denoted by $I$ and $0$, respectively. $Z_{[s_1, s_2]} = \{I \mid I \in Z, s_1 \leq I \leq s_2\}$, where $Z$ is the set of integers, $l_{i,0,S}$ is the space of summable sequences on $Z_{[0,S]}$, where $S$ may be finite or infinite.

2. System Description and Problem Formulation

Given the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, we consider a discrete-time NMJDS described by the following stochastic difference equation

$$
\begin{align*}
x_{k+1} &= (A(r_k) + A_d(r_k)) x_k + (A_d(r_k) + A_s(r_k)) x_{k-d} + B(r_k) w_k \\
y_k &= C(r_k) x_k + D(r_k) w_k \\
z_k &= G(r_k) x_k + F(r_k) w_k \\
x_k &= \phi_k, \quad k = -d, -d + 1, \ldots, 0,
\end{align*}
$$

(i)
where $k \in \mathbb{Z}_{[0,M]}$, $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^n$ is the measured output, $z_k \in \mathbb{R}^n$ is the output to be estimated, $w_k \in \mathbb{R}^n$ is the exogenous disturbances satisfying $w_k \in l_2[0,\infty)$, integer $d > 0$ is the time delay, and $\phi_k$ is a real-valued initial function. When $w_k = 0$, system (1) is usually named as a disturbance-free NMJDS.

For each possible value of $r_k = i_0$, we denote matrices associated with the “$i_0$th jump mode” by $\mathcal{A}_{i_0}$, where $\mathcal{A}$ can be replaced by any symbol. Matrices $A_{i_0}, A_{d_{i_0}}, B_{i_0}, C_{i_0}, D_{i_0}, G_{i_0}$, and $F_{i_0}$ are known constant matrices which depend on the jump mode $r_k$. The matrices $\delta A_{i_0}$ and $\delta A_{d_{i_0}}$ represent time-varying parameter uncertainties, which are assumed to be norm bounded and can be noted as

$$
[\delta A_{i_0} \delta A_{d_{i_0}}] = E_{i_0} \mathcal{Z}_{i_0} (k) \begin{bmatrix} H_{i_0} & H_{i_0}^d \end{bmatrix},
$$

where $E_{i_0}, H_{i_0}^a$, and $H_{i_0}^d$ are known constant matrices which characterize the structure of uncertainties. Matrices $\mathcal{Z}_{i_0} (k)$ are unknown time-varying matrix functions with Lebesgue measurable elements satisfying $\mathcal{Z}_{i_0}^2 (k) \mathcal{Z}_{i_0} (k) \leq I$.

The symbol $r_k$ is a discrete-time, discrete-state Markov chain taking values in $Z_{[1,\beta]}$ with transition probabilities

$$
Pr \{ r_{k+1} = i_1 \mid r_k = i_0 \} = \pi_{i_0,i_1},
$$

where $\pi_{i_0,i_1} \geq 0$ and satisfying $\sum_{i=1}^\beta \pi_{i_0,i} (k) = 1$.

Symbols $\pi_{i_0,i} (k) \equiv \sum_{f=1}^\rho H_{i_0} (k) H_{i_0}^T (k) \pi_{f,i} (k)$ are entries of the TP matrix $\Pi (k)$, which is defined as

$$
\Pi (k) = \Pi (\mu (k)) \equiv \sum_{f=1}^\rho H_{i_0} (k) \Pi_{i_0,i},
$$

where $\mu (k) \geq 0$, $\sum_{f=1}^\rho H_{i_0} (k) = 1$, and $\Pi_{i_0,i}$ are given TP matrices. All polytopic indices of indicator function $\mu (k)$ over the time interval $[k, k + N]$ can be boiled down into a set $\mathcal{F}_{[0,N]} \equiv \{ f_0, f_1, \ldots, f_N \}$, where $f_0, f_1, \ldots, f_N \in Z_{[1,\beta]}$.

Since the $N$-step-ahead scenario will be considered, we further denote a chain of values with respect to the current time jump mode $r_k$ to the future jump mode $r_k$. Let $r_k \equiv i_0, r_{k+1} \equiv i_1, \ldots, r_{k+N} \equiv i_l$; then we have a series of value sets for finite-state Markov chains, i.e., $l_0 \equiv \{ i_0, i_1, \ldots, i_l \} \in Z_{[1,\beta]}$, over the time interval $[k, k + N]$, where $\{ i_0, i_1, \ldots, i_N \} \in Z_{[1,\beta]}$.

In this paper, we are interested in designing the following mode-dependent full-order filter for system (1), that is,

$$
\tilde{x}_{k+1} = A_F (r_k) \tilde{x}_k + B_F (r_k) y_k,
$$

$$
\tilde{z}_k = C_F (r_k) \tilde{x}_k + D_F (r_k) y_k,
$$

where $\tilde{x}_k \in \mathbb{R}^n$ and $\tilde{z}_k \in \mathbb{R}^n$ are the estimated state and estimate of the output $z_k$, respectively. For $r_k = i_0$, matrices $A_{F,i_0}, B_{F,i_0}, C_{F,i_0}$, and $D_{F,i_0}$ are filter gains to be determined.

By introducing new vectors $\xi_k = [\tilde{x}_k^T \tilde{z}_k^T]^T$ and $\xi_k = z_k - \tilde{z}_k$, the resulting filtering error system of system (1) together with filter (5) becomes

$$
\xi_{k+1} = \bar{A}_1 \xi_k + \bar{A}_{d_{i_0}} \xi_{k-d} + \bar{B}_n w_k,
$$

$$
e_k = \bar{C}_n \xi_k + \bar{D}_n w_k,
$$

where

$$\bar{A}_1 = \begin{bmatrix} \bar{A}_{i_0} & 0 \\ B_{F,i_0} C_{i_0} & A_{F,i_0} \end{bmatrix},
$$

$$\bar{A}_{d_{i_0}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

$$\bar{B}_n = \begin{bmatrix} B_{i_0} \\ B_{F,i_0} D_{i_0} \end{bmatrix},
$$

$$\bar{C}_n = \begin{bmatrix} G_{i_0} - D_{F,i_0} C_{i_0} & -C_{F,i_0} \end{bmatrix},
$$

$$\bar{D}_n = F_{i_0} - D_{F,i_0} D_{i_0},
$$

$$\bar{A}_i = A_{i_0} + E_{i_0} \mathcal{Z}_{i_0} (k) H_{i_0}^a,
$$

$$\bar{A}_{d_{i_0}} = A_{d_{i_0}} + E_{i_0} \mathcal{Z}_{i_0} (k) H_{i_0}^d.
$$

In order to deal with the mean-square stability of the filtering error system (6), the following preliminaries are given.

**Definition 1.** The disturbance-free NMJDS (6) is said to be mean-square stable if

$$
\lim_{S \to -\infty} \sum_{k=0}^S \mathbb{E}\{ \| \xi_k \|_2^2 \} < \infty
$$

for any finite initial condition $\xi_0 \in \mathbb{R}^{2n}$, and $r_0 \in Z_{[1,\beta]}$.

**Assumption 2.** NMJDS (1) is mean-square stable for all admissible uncertainties (2).

**Assumption 2** is made based on the fact that NMJDS (1) is autonomous without control inputs. Therefore, the original system (1) to be estimated must be mean-square stable, which is a prerequisite for the filtering error system (6) to be mean-square stable.

The purpose of this paper is to design $H_{\infty}$ filter of the form (5) such that the following two conditions are satisfied:

(i) The filtering error system (6) with $w_k = 0$ is mean-square stable.

(ii) Under the zero-initial condition, the filtering error $e_k$ satisfies

$$
\sum_{k=0}^S \mathbb{E}\{ \| e_k \|_2^2 \} \leq \gamma^2 \sum_{k=0}^S \| w_k \|_2^2
$$

for any nonzero $w_k \in l_2[0,\infty)$ and a given/optimized $l_2$ gain bound $\gamma > 0$.

### 3. $H_{\infty}$ Filtering Performance Analysis

In this section, we first provide a relaxed $H_{\infty}$ analysis result for the filtering error system (6) by employing the NALKF approach, which will be used for filter design in the next section.
Lemma 3. The disturbance-free NMJDS (6) is mean-square stable, if there exist a set of symmetric matrices $M_{ij}^f$, $P_{ij}^f = (P_{ij}^f)^T > 0$, for all the $i_j \in \mathbb{Z}_{[1,N]}^q$, $f_j \in \mathbb{Z}_{[1,N]}^q$, $l \in \mathbb{Z}_{[0,N]}^q$, $q \in \mathbb{Z}_{[N-1]}^q$, and $Q = Q^T > 0$ such that

$$
\Theta_N = \begin{bmatrix} \Theta_{N,11} & \Theta_{N,12} \\ \Theta_{N,12}^T & \Theta_{N,22} \end{bmatrix} < 0,
$$

(10a)

$$
\Theta_{Z,[n-1]} = \begin{bmatrix} \Theta_{Z,[n-1],11} & \Theta_{Z,[n-1],12} \\ \Theta_{Z,[n-1],12}^T & \Theta_{Z,[n-1],22} \end{bmatrix} < 0,
$$

(10b)

$$
\Theta_0 = \begin{bmatrix} \Theta_{0,11} & \Theta_{0,12} \\ \Theta_{0,12}^T & \Theta_{0,22} \end{bmatrix} < 0,
$$

(10c)

where

$$
\Theta_{N,11} = -M_{ij_{j-1}}^{f_{j-1}-f_{j}} + A_{j_{j-1}}^T P_{ij_{j-1}}^{f_{j-1}-f_{j}} A_{j_{j-1}} + Q,
$$

$$
\Theta_{N,12} = A_{j_{j-1}}^T P_{ij_{j-1}}^{f_{j-1}-f_{j}} A_{j_{j-1}}^T,
$$

$$
\Theta_{N,22} = -Q + A_{d_{j_{j-1}}}^T \tilde{P}_{ij_{j-1}}^{f_{j-1}-f_{j}} A_{d_{j_{j-1}}},
$$

$$
\Theta_{Z,[n-1],11} = -M_{ij_{j-1}}^{f_{j-2}-f_{j-1}} + A_{j_{j-1}}^T M_{ij_{j-1}}^{f_{j-2}-f_{j-1}} + M_{ij_{j-1}}^{f_{j-2}-f_{j-1}} + Q,
$$

$$
\Theta_{Z,[n-1],12} = A_{j_{j-1}}^T M_{ij_{j-1}}^{f_{j-2}-f_{j-1}} - A_{d_{j_{j-1}}}^T,
$$

$$
\Theta_{Z,[n-1],22} = -Q + A_{d_{j_{j-1}}}^T M_{ij_{j-1}}^{f_{j-2}-f_{j-1}} A_{d_{j_{j-1}}},
$$

$$
\Theta_{0,11} = -P_{ij_{0}} + A_{j_{0}}^T M_{ij_{0}}^{f_{j-2}-f_{j}} A_{j_{0}} + Q,
$$

$$
\Theta_{0,12} = A_{j_{0}}^T M_{ij_{0}}^{f_{j-2}-f_{j}} A_{d_{j_{0}}},
$$

$$
\Theta_{0,22} = -Q + A_{d_{j_{0}}}^T M_{ij_{0}}^{f_{j-2}-f_{j}} A_{d_{j_{0}}},
$$

$$
\overline{M}_{ij_{j-1}}^{f_{j-2}-f_{j-1}} = \sum_{l_{j-1}=1}^{\gamma} \tau_{ij_{j-1}}^{f_{j-2}-f_{j-1}} M_{ij_{j-1}}^{f_{j-1}},
$$

$$
\overline{P}_{ij_{j-1}}^{f_{j-2}-f_{j-1}} = \sum_{l_{j-1}=1}^{\gamma} \tau_{ij_{j-1}}^{f_{j-2}-f_{j-1}} P_{ij_{j-1}}^{f_{j-1}},
$$

$$
\overline{M}_{ij_{j-1}}^{f_{j-2}-f_{j-1}} = \sum_{l_{j-1}=1}^{\gamma} \tau_{ij_{j-1}}^{f_{j-2}-f_{j-1}} M_{ij_{j-1}}^{f_{j-1}},
$$

$$
\overline{M}_{ij_{j-1}}^{f_{j-2}-f_{j-1}} = \sum_{l_{j-1}=1}^{\gamma} \tau_{ij_{j-1}}^{f_{j-2}-f_{j-1}} M_{ij_{j-1}}^{f_{j-1}},
$$

Proof. Consider a LKF at time $k$ as follows

$$
V (\varphi_k) \triangleq V (\mu (k), r_k, \xi_k)
$$

$$
= \xi^T_k \left( \sum_{j=1}^{\gamma} \mu_{f_j} (k) P_{ij}^{f_j} \right) \xi_k + \sum_{m=k+1-d}^{k} \xi^T_m Q \xi_m,
$$

(12)

and denote

$$
\Omega_{ij} (k) \triangleq \sum_{j=1}^{\gamma} \mu_{f_j} (k) \sum_{j=1}^{\gamma} \mu_{f_j} (k + 1) \cdots \sum_{j=1}^{\gamma} \mu_{f_j} (k + l),
$$

(13)

$$
\eta_k \triangleq \left[ \xi_k^T T \right]^T,
$$

(14)

$$
V' (\varphi_k) \triangleq \xi_k^T \left( \sum_{j=1}^{\gamma} \mu_{f_j} (k) M_{ij}^{f_j} \right) \xi_k + \sum_{m=k+1-d}^{k} \xi^T_m Q \xi_m.
$$

(15)

By virtue of Definition 1, we develop the following steps to prove that

$$
\Delta N V (\varphi_k) = \mathcal{E} \left( V (\varphi_{k-N}) | \mathcal{F}_k \right) - V (\varphi_k) < 0
$$

(16)

establishes the mean-square stability of disturbance-free NMJDS (6).

Subtracting and adding a group of conditional mathematical expectations, \( \mathcal{E} [V' (\varphi_{k-N-1}) | \mathcal{F}_k] \), \( \mathcal{E} [V' (\varphi_{k-N-2}) | \mathcal{F}_k] \), \( \mathcal{E} [V' (\varphi_{k-1}) | \mathcal{F}_k] \) from to (16), one has

$$
\Delta N V (\varphi_k) = \mathcal{E} \left[ V (\varphi_{k-N}) | \mathcal{F}_k \right] - V (\varphi_k)
$$

$$
= \mathcal{E} \left[ V (\varphi_{k-N}) | \mathcal{F}_k \right] - \mathcal{E} \left[ V' (\varphi_{k-N-1}) | \mathcal{F}_k \right] + \mathcal{E} \left[ V' (\varphi_{k-N-2}) | \mathcal{F}_k \right] + \cdots + \mathcal{E} \left[ V' (\varphi_{k+1}) | \mathcal{F}_k \right] - V (\varphi_k).
$$

Step I. Bearing in mind that

$$
\xi_{k+N} = \tilde{A}_{j_{k+N-1}} \xi_{k+N-1} + \tilde{A}_{d_{j_{k+N-1}}} \xi_{k-d+N-1}
$$

(18)

one has

$$
\mathcal{E} \left[ V (\varphi_{k-N}) | \mathcal{F}_k \right] - \mathcal{E} \left[ V' (\varphi_{k-N-1}) | \mathcal{F}_k \right] \bigg( \tilde{A}_{j_{k+N-1}} \xi_{k+N-1} + \tilde{A}_{d_{j_{k+N-1}}} \xi_{k-d+N-1} \bigg)^T \Omega_{0,N-1} (k)
$$

$$
= \left( \tilde{A}_{j_{k+N-1}} \xi_{k+N-1} + \tilde{A}_{d_{j_{k+N-1}}} \xi_{k-d+N-1} \right)^T \Omega_{0,N-1} (k)
$$

$$
\cdot \left( \tilde{A}_{j_{k+N-1}} \xi_{k+N-1} + \tilde{A}_{d_{j_{k+N-1}}} \xi_{k-d+N-1} \right)
$$

$$
= \xi_{k+N-1}^T \Omega_{0,N-1} (k) \tilde{A}_{j_{k+N-1}} \xi_{k+N-1} + \Omega_{0,N-2} (k) \sum_{l_{j+1}=1}^{\gamma} \tau_{ij_{j+1}}^{f_{j+1}} \tilde{A}_{j_{k+N-1}} \xi_{k+N-1}
$$

$$
+ \Omega_{0,N-2} (k) \sum_{l_{j+1}=1}^{\gamma} \tau_{ij_{j+1}}^{f_{j+1}} \tilde{A}_{d_{j_{k+N-1}}} \xi_{k+N-1}
$$

$$
= \eta_k^T \Omega_{0,N-1} (k) \tilde{A}_{j_{k+N-1}} \xi_{k+N-1}.
$$

(19)
It can be inferred from (10a) that
\[
\mathbb{E} \{ V'(\phi_k + N) \mid \mathcal{F}_k \} - \mathbb{E} \{ V'(\phi_{k+1}) \mid \mathcal{F}_k \} < 0. 
\] (20)

Therefore, we have
\[
\Delta_N V(\phi_k) < \mathbb{E} \{ V'(\phi_k + N) \mid \mathcal{F}_k \} - \mathbb{E} \{ V'(\phi_{k+1}) \mid \mathcal{F}_k \} + \cdots + \mathbb{E} \{ V'(\phi_k) \mid \mathcal{F}_k \} - V(\phi_k). 
\] (21)

**Step 2.** Bearing in mind that
\[
\xi_{k+1} = \tilde{A}_{d2} \xi_k + \tilde{A}_{d0} \xi_{d-k} 
\] (22)
one obtains
\[
\mathbb{E} \{ V'(\phi_k + N) \mid \mathcal{F}_k \} - \mathbb{E} \{ V'(\phi_{k+1}) \mid \mathcal{F}_k \} = (\tilde{A}_{d2} \xi_k + \tilde{A}_{d0} \xi_{d-k})^T \Omega_0 N_j \xi_k 
\] (23)
\[
- \xi_{k+1}^T \Omega_0 N_j N_{j+1} \xi_{k+1}, \quad \xi_{k+1}^T \Omega_0 N_j \xi_{k+1}. 
\]

If (10b) holds for \( q = 1 \), then it results in
\[
\mathbb{E} \{ V'(\phi_k + N) \mid \mathcal{F}_k \} - \mathbb{E} \{ V'(\phi_{k+1}) \mid \mathcal{F}_k \} < 0, 
\] (24)
which further implies
\[
\Delta_N V(\phi_k) < \mathbb{E} \{ V'(\phi_k + N) \mid \mathcal{F}_k \} - \mathbb{E} \{ V'(\phi_{k+1}) \mid \mathcal{F}_k \} + \cdots + \mathbb{E} \{ V'(\phi_k) \mid \mathcal{F}_k \} - V(\phi_k). 
\] (25)

Repeating the above procedure \( N - 2 \) times yields
\[
\Delta_N V(\phi_k) < \mathbb{E} \{ V'(\phi_k + N) \mid \mathcal{F}_k \} - \mathbb{E} \{ V'(\phi_{k+1}) \mid \mathcal{F}_k \} - V(\phi_k). 
\] (26)

**Step 3.** Bearing in mind that
\[
\xi_{k+1} = \tilde{A}_{d2} \xi_k + \tilde{A}_{d0} \xi_{d-k} 
\] (27)
on one has
\[
\mathbb{E} \{ V'(\phi_k + N) \mid \mathcal{F}_k \} - \mathbb{E} \{ V'(\phi_{k+1}) \mid \mathcal{F}_k \} - \mathbb{E} \{ V'(\phi_k) \mid \mathcal{F}_k \} < 0, 
\] (28)

It can be inferred from (10c) that
\[
\mathbb{E} \{ V'(\phi_k + N) \mid \mathcal{F}_k \} - \mathbb{E} \{ V'(\phi_{k+1}) \mid \mathcal{F}_k \} - \mathbb{E} \{ V'(\phi_k) \mid \mathcal{F}_k \} < 0, 
\] (29)
which implies that
\[
\eta_k^T \Omega_0 \eta_k < 0. 
\] (30)

**Step 4.** By observing (30), we obtain
\[
\Delta_N V(\phi_k) < \mathbb{E} \{ V'(\phi_k + N) \mid \mathcal{F}_k \} - \mathbb{E} \{ V'(\phi_{k+1}) \mid \mathcal{F}_k \} - \mathbb{E} \{ V'(\phi_k) \mid \mathcal{F}_k \} < 0 \iff L(\phi(k)) - L(\phi(k + 1)) < 0 \iff L(\phi(k + N)) - L(\phi(k)) < 0. 
\] (31)

Therefore, it can be yielded from (31) and (32) that
\[
\mathbb{E} \{ L(\phi_{k+1}) \mid \mathcal{F}_k \} - \mathbb{E} \{ L(\phi_{k}) \mid \mathcal{F}_k \} = \alpha \left( \sum_{k=0}^{S} \| \eta_k \|_2^2 \right) \Rightarrow \mathbb{E} \{ \Delta N V(\phi_k) \mid \mathcal{F}_k \} \leq \alpha^{-1} [L(\phi_0) - \mathbb{E} \{ L(\phi_{N+1}) \mid \mathcal{F}_k \}] \leq \alpha^{-1} L(\phi_0) \Rightarrow \lim_{S \to \infty} \mathbb{E} \left\{ \sum_{k=0}^{S} \| \eta_k \|_2^2 \mid \mathcal{F}_k \right\} \leq \alpha^{-1} L(\phi_0) < \infty. 
\] (32)

This implies, from **Definition 1**, that the disturbance-free NMJDS (6) is mean-square stable. The proof is completed.

**Remark 4.** Compared with the multistep LF approach developed in [34], the proposed NALKF approach used the \( N \)-step-ahead TPs \( \sum \eta_{i+1} \) to replace the multiple multiplication \( \sum \eta_{i+1} \). This will greatly reduce the number of decision variables, especially for the time-delayed scenario.

The following lemma provides sufficient conditions under which the filtering error system, i.e., NMJDS (6), is mean-square stable and the filtering error \( e_k \) satisfies the dissipative requirement (9).

**Lemma 5.** The NMJDS (6) possesses the disturbance attenuation level in (9) for all \( w_k \in \mathbb{L}_{2[0,0]}^2 \), if there exist a set of symmetric matrices \( \Pi_{i,j}^k \) such that
\[
\Pi_{i,j}^k = (\Pi_{i,j}^k)^T > 0, \quad \text{for all } i, j \leq N. 
\]
the $i_l \in \mathbb{Z}_{[1,p]}$, $f_l \in \mathbb{Z}_{[1,p]}$, $l \in \mathbb{Z}_{[0,N]}$, $q \in \mathbb{Z}_{[1,N-1]}$, and $Q = Q^T > 0$ such that

$$\Lambda_N = \begin{bmatrix} \Lambda_{N,11} & \Lambda_{N,12} & \Lambda_{N,13} \\ \Lambda_{N,22} & \Lambda_{N,23} & \Lambda_{N,33} \\ \Lambda_{N,32} & \Lambda_{N,33} & \Lambda_{N,33} \end{bmatrix} < 0, \quad (34a)$$

$$\Lambda_{N,N-1} = \begin{bmatrix} \Lambda_{N,N-1,11} & \Lambda_{N,N-1,12} & \Lambda_{N,N-1,13} \\ \Lambda_{N,N-1,22} & \Lambda_{N,N-1,23} & \Lambda_{N,N-1,23} \\ \Lambda_{N,N-1,32} & \Lambda_{N,N-1,33} & \Lambda_{N,N-1,33} \end{bmatrix} < 0, \quad (34b)$$

$$\Lambda_0 = \begin{bmatrix} \Lambda_{0,11} & \Lambda_{0,12} & \Lambda_{0,13} \\ \Lambda_{0,22} & \Lambda_{0,23} & \Lambda_{0,23} \\ \Lambda_{0,33} & \Lambda_{0,33} & \Lambda_{0,33} \end{bmatrix} < 0, \quad (34c)$$

where

$$\Lambda_{N,11} = -\bar{M}_{N,p,N-1}^{f_N^{N-2}f_N^{N-1}} + \bar{A}_{N,p,N-1}^{f_N^{N-2}f_N^{N-1}} + Q + \bar{C}_{N-1}^T \bar{C}_{N-1},$$

$$\Lambda_{N,12} = -\bar{A}_{N-1}^{f_N^{N-1}f_N^{N-2}} \bar{P}_{N,p,N-1}^{f_N^{N-2}f_N^{N-1}} \bar{A}_{N-1},$$

$$\Lambda_{N,13} = \bar{A}_{N-1}^{f_N^{N-1}f_N^{N-2}} \bar{P}_{N,p,N-1}^{f_N^{N-1}f_N^{N-2}} \bar{B}_{N-1},$$

$$\Lambda_{N,22} = -Q + \bar{A}_{d,N-1}^T \bar{P}_{N,p,N-1}^{f_N^{N-1}f_N^{N-2}} \bar{A}_{d,N-1},$$

$$\Lambda_{N,23} = \bar{A}_{d,N-1}^T \bar{P}_{N,p,N-1}^{f_N^{N-1}f_N^{N-2}} \bar{B}_{N-1},$$

$$\Lambda_{N,33} = -\gamma^2 \mathbf{I} + \bar{B}_{N-1}^T \bar{P}_{N,p,N-1}^{f_N^{N-2}f_N^{N-1}} \bar{B}_{N-1} + \bar{D}_{N-1}^T \bar{D}_{N-1},$$

$$\Lambda_{z_{N,N-1},11} = -\bar{M}_{N,p,N-1}^{f_N^{N-2}f_N^{N-1}} + \bar{A}_{N,p,N-1}^{f_N^{N-2}f_N^{N-1}} + Q + \bar{C}_{N-1}^T \bar{C}_{N-1},$$

$$\Lambda_{z_{N,N-1},12} = \bar{A}_{N-1}^{f_N^{N-1}f_N^{N-2}} \bar{P}_{N,p,N-1}^{f_N^{N-2}f_N^{N-1}} \bar{A}_{N-1},$$

$$\Lambda_{z_{N,N-1},13} = \bar{A}_{N-1}^{f_N^{N-1}f_N^{N-2}} \bar{P}_{N,p,N-1}^{f_N^{N-1}f_N^{N-2}} \bar{B}_{N-1},$$

$$\Lambda_{z_{N,N-1},22} = -Q + \bar{A}_{d,N-1}^T \bar{P}_{N,p,N-1}^{f_N^{N-1}f_N^{N-2}} \bar{A}_{d,N-1},$$

$$\Lambda_{z_{N,N-1},23} = \bar{A}_{d,N-1}^T \bar{P}_{N,p,N-1}^{f_N^{N-1}f_N^{N-2}} \bar{B}_{N-1},$$

$$\Lambda_{z_{N,N-1},33} = -\gamma^2 \mathbf{I} + \bar{B}_{N-1}^T \bar{P}_{N,p,N-1}^{f_N^{N-2}f_N^{N-1}} \bar{B}_{N-1} + \bar{D}_{N-1}^T \bar{D}_{N-1},$$

and $\bar{M}_{N,p,N-1}^{f_N^{N-2}f_N^{N-1}}, \bar{P}_{N,p,N-1}^{f_N^{N-2}f_N^{N-1}}, \bar{M}_{d,N-1}^{f_N^{N-2}f_N^{N-1}}, \bar{M}_{d,N-1}^{f_N^{N-2}f_N^{N-1}}, \bar{M}_{d,N-1}^{f_N^{N-2}f_N^{N-1}}, \bar{M}_{d,N-1}^{f_N^{N-2}f_N^{N-1}}$ are the same as in Lemma 3.

**Proof.** Consider a LKF $V(\varphi_k)$ and an auxiliary functional $V'(\varphi_k)$ noted in (12) and (15), respectively. It is clear that inequalities (34a)–(34c) incorporate sufficient conditions presented in Lemma 3 and therefore Lemma 5 implies the mean-square stability of the NMJDS (6). Next, we direct our attention to the $H_\infty$ performance analysis via the following steps.

Define new vectors $\xi_k \triangleq [\xi_k^T \xi_{k-1}^T w_k^T]^T$ and $s(k) \triangleq e_k^T e_k - \gamma^2 w_k^T w_k$. We try to prove that

$$\Delta_N V(\varphi_k) + \sum_{i=0}^{N-1} \theta_{k+i} < 0 \quad (36)$$

guarantees $H_\infty$ performance of the NMJDS (6).

Adding and subtracting a group of mathematical items

$$\begin{align*}
&- \mathbb{E}[V'(\varphi_{k-1}) | \mathcal{F}_k] + \theta_{k+N-1} \\
&- \mathbb{E}[V'(\varphi_{k-2}) | \mathcal{F}_k] + \theta_{k+N-2} - \cdots \\
&- \mathbb{E}[V'(\varphi_{k+1}) | \mathcal{F}_k] + \theta_{k+1}
\end{align*} \quad (37)$$

to and from $\Delta_N V(\varphi_k)$, one has

$$\Delta_N V(\varphi_k) = \mathbb{E}[V(\varphi_{k+1}) | \mathcal{F}_k]$$

$$\begin{align*}
&- \mathbb{E}[V'(\varphi_{k-1}) | \mathcal{F}_k] + \theta_{k+N-1} \\
&+ \mathbb{E}[V'(\varphi_{k-1}) | \mathcal{F}_k] + \theta_{k+N-2} - \cdots \\
&+ \mathbb{E}[V'(\varphi_{k+1}) | \mathcal{F}_k] - V(\varphi_k) + \theta_k \\
&- \sum_{i=0}^{N-1} \theta_{k+i}
\end{align*} \quad (38)$$

First, following the same lines of the proof of Lemma 3, we know that (34a) implies that

$$\begin{align*}
&\mathbb{E}[V(\varphi_{k+N}) | \mathcal{F}_k] - \mathbb{E}[V'(\varphi_{k+N}) | \mathcal{F}_k] + \theta_{k+N-1} \\
= \sum_{i=0}^{N-1} \theta_{k+i} \Lambda_N \xi_{k+N-1} < 0,
\end{align*} \quad (39)$$
which further yields
\[
\Delta_N V (\phi_k) < \mathcal{E} \left\{ V' (\phi_{k+N-1}) \mid \mathcal{F}_k \right\} \\
- \mathcal{E} \left\{ V' (\phi_{k+N-2}) \mid \mathcal{F}_k \right\} + \theta_{k+N-2} + \cdots \\\n+ \mathcal{E} \left\{ V' (\phi_{k+1}) \mid \mathcal{F}_k \right\} - V (\phi_k) + \theta_k \\
- \sum_{l=0}^{N-1} \theta_{k+l}.
\]
(40)

Then, it can be concluded form (34b) that
\[
\mathcal{E} \left\{ V (\phi_{k+N-q}) \mid \mathcal{F}_k \right\} - \mathcal{E} \left\{ V (\phi_{k+N-q-1}) \mid \mathcal{F}_k \right\} \\
+ \theta_{k+N-q-1} \\
= \mathcal{E} \left\{ V' (\phi_{k+1}) \mid \mathcal{F}_k \right\} - V (\phi_k) + \theta_k \\
= \mathcal{E} \left\{ V' (\phi_{k+1}) \mid \mathcal{F}_k \right\} - V (\phi_k) + \theta_k \\
\leq r^T k_{N-q-1} \Omega_{q,N-q-1} (k) \Lambda z_{[0,N-1]} k_{N-q-1} < 0,
\]
(41)

which implies that
\[
\Delta_N V (\phi_k) < \mathcal{E} \left\{ V' (\phi_{k+1}) \mid \mathcal{F}_k \right\} - V (\phi_k) + \theta_k \\
- \sum_{l=0}^{N-1} \theta_{k+l}.
\]
(42)

Finally, form (34c), we obtain
\[
\mathcal{E} \left\{ V' (\phi_{k+1}) \mid \mathcal{F}_k \right\} - V (\phi_k) + \theta_k = r^T k_{N-q-1} \Omega_{q,N-q-1} (k) \Lambda z_{[0,N-1]} k_{N-q-1} < 0,
\]
(43)

which directly yields that (36) holds by considering (38), (40), (42), and (43) together.

In the following, we assume zero-initial conditions, i.e.,
\[ x_{-N} = x_{-N+1} = \cdots = x_0 = 0. \]

Define
\[
J_k \triangleq \mathcal{E} \left\{ \sum_{k=0}^{S} (z^T k z_k - \gamma^2 w^T k w_k) \right\}.
\]
(44)

Inequality (36) results in
\[
\sum_{k=-N}^{k=N} \left\{ \Delta_N V (\phi_k) + \sum_{l=0}^{N-1} \theta_{k+l} \right\} < 0 \implies \mathcal{E} \left\{ V (\phi_{k+N}) \mid \mathcal{F}_k \right\} \\
+ \mathcal{E} \left\{ V (\phi_{k+N-1}) \mid \mathcal{F}_{k-1} \right\} + \cdots + \mathcal{E} \left\{ V (\phi_{k+1}) \mid \mathcal{F}_{k+N-1} \right\} \\
- \mathcal{E} \left\{ V (\phi_k) \mid \mathcal{F}_{k-N} \right\} \\
- \mathcal{E} \left\{ V (\phi_{k-N}) \mid \mathcal{F}_{k-N+1} \right\} - \cdots - \mathcal{E} \left\{ V (\phi_1) \mid \mathcal{F}_{k-N+S} \right\} \\
+ \sum_{k=-N}^{k=N} (\theta_k + \theta_{k+1} + \cdots + \theta_{k+N-1}) < 0.
\]
(45)

As \( S \to \infty \), all the negative items tend to zero and we obtain
\[
NV (\phi_{\infty}) + NJ_{\infty} < 0
\]
(46)

which yields \( J_{\infty} < 0 \). Therefore, the dissipative inequality (9) holds for \( S > 0 \). This completes the proof.
\[ \Box \]

**Remark 6.** The \( N \)-step-ahead LKF approach is an \( N \)-step ahead of the current time \( k \) and thus it has a potential to allow the LKF to increase during a finite-time interval \([k, k + N]\). Hence, a more relaxed stability condition and a better \( H_\infty \) attenuation level could be obtained for the filtering error system (6). Therefore, a better filtering performance could be further expected.

**Remark 7.** Although mathematical difficulties lie in the derivation of conditions for the \( H_\infty \) analysis (as compared with [32]), when the system goes to future steps, we develop the main results from stability condition step by step.

### 4. Robust \( H_\infty \) Filter Design

According to \( H_\infty \) performance analysis for the NMJDS (6), we are now in the position to provide numerical testable conditions to the robust \( H_\infty \) filter design. Before giving the main results, the following well known lemma must be recalled.

**Lemma 8** (see [41]). Let matrix \( \Psi = \Psi^T \), \( E \) and \( H \) be real matrices of appropriate dimensions, with \( \mathcal{L}(k) \) satisfying \( \mathcal{L}(k)^T \mathcal{L}(k) \leq I \); then
\[
\Psi + E \mathcal{L}(k) H + H^T \mathcal{L}(k)^T E^T < 0
\]
(47)

if and only if there exists a positive scalar \( \epsilon > 0 \) such that
\[
\Psi + \epsilon^{-1} EE^T + \epsilon H^T H < 0,
\]
(48)
or equivalently
\[
\begin{bmatrix}
\Psi & E \\
E^T & \epsilon H^T
\end{bmatrix} < 0.
\]
(49)

In the following theorem, the addressed filter design problem is solvable for all possible time-varying transition probabilities and all admissible parameter uncertainties and time-delays if a set of LMIs are feasible.

**Theorem 9.** The NMJDS (6) is mean-square stable and the dissipative constraint (9) with an optimized attenuation level
\[
\gamma \text{ is achieved for all nonzero } u_k, \text{ if there exist a set of symmetric matrices } M_{l,j}^{f_i}, M_{l,j}^{f_i+1}, M_{l,j}^{f_i+2}, X_{l,j}, Z_{l,j}, W_{l,j}, V_{l,j}, P_{l,j} = (P_{l,j}^*)^T > 0, \ P_{l,j}^{f_i} = (P_{l,j}^{f_i})^T > 0, \ P_{l,j}^{f_i+1} = (P_{l,j}^{f_i+1})^T > 0, \ Q_1 = Q_1^* > 0, \ Q_2 = Q_2^* > 0, Q_3 = Q_3^* > 0 \text{ for all the } i \in \mathbb{Z}_{[1,\beta]}, f_1 \in \mathbb{Z}_{[1,\rho]}, l \in \mathbb{Z}_{[0,N]}, q \in \mathbb{Z}_{[1,N-1]}, \text{ and matrices } C_{F_{l,j}}, D_{F_{l,j}}, \text{ scalar } e \ \text{ such that}
\]

\[
\Xi_{N} = \begin{bmatrix}
\Xi_{N,11} & 0 & \Xi_{N,13} & \Xi_{N,14} & \Xi_{N,15} \\
* & \Xi_{N,22} & 0 & \Xi_{N,34} & 0 \\
* & * & \Xi_{N,44} & 0 & \Xi_{N,45} \\
* & * & * & \Xi_{N,54} & \Xi_{N,55} \\
\end{bmatrix} < 0,
\]

\[
\Xi_{N,11} = \begin{bmatrix}
-M_{f_{l,j}+f_{l,j-1}}^{f_{l,j}+f_{l,j-1}} & +Q_1 & -M_{f_{l,j}+f_{l,j+1}}^{f_{l,j}+f_{l,j-1}} + Q_2 \\
* & -M_{f_{l,j}+f_{l,j-1}}^{f_{l,j}+f_{l,j-1}} + Q_3 \\
\end{bmatrix},
\]

\[
\Xi_{N,13} = \begin{bmatrix}
0 & \mathcal{G}_{l,j}^{f_{l,j}+1} - \mathcal{C}_{l,j}^{f_{l,j}+1} D_{F_{l,j+1}}^T \\
0 & -\mathcal{C}_{l,j+1}^{f_{l,j}+1} \\
\end{bmatrix},
\]

\[
\Xi_{N,14} = \begin{bmatrix}
(X_{l,j} A_{l,j})^T + (W_{l,j} C_{l,j})^T & (Z_{l,j} A_{l,j})^T + (W_{l,j} C_{l,j})^T & V_{l,j}^T \\
\end{bmatrix},
\]

\[
\Xi_{N,15} = \begin{bmatrix}
0 & 0 & 0 & \left( H_{l,j}^a \right)^T & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
\Xi_{N,22} = \begin{bmatrix}
-Q_1 & -Q_2 \\
* & -Q_3 \\
\end{bmatrix},
\]

\[
\Xi_{N,24} = \begin{bmatrix}
(X_{l,j} A_{l,j})^T & (Z_{l,j} A_{l,j})^T \\
0 & 0 \\
\end{bmatrix},
\]

\[
\Xi_{N,25} = \begin{bmatrix}
0 & 0 & 0 & 0 & \left( H_{l,j}^d \right)^T & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
\Xi_{N,33} = \begin{bmatrix}
-\gamma^2 I & F_{l,j}^T & -D_{l,j}^T & D_{F_{l,j+1}}^T & -I \\
* & 0 \\
\end{bmatrix},
\]

\[
\Xi_{N,34} = \begin{bmatrix}
(X_{l,j} B_{l,j})^T & (W_{l,j} D_{l,j})^T & (Z_{l,j} B_{l,j})^T & (W_{l,j} D_{l,j})^T \\
0 & 0 \\
\end{bmatrix},
\]

\[
\Xi_{N,44} = \begin{bmatrix}
-X_{l,j-1}^T & -X_{l,j-1}^T + P_{l,j-1}^{f_{l,j-1}+f_{l,j}} & -Y_{l,j-1}^T & -Y_{l,j-1}^T + P_{l,j-1}^{f_{l,j-1}+f_{l,j}} & -Y_{l,j-1}^T + P_{l,j-1}^{f_{l,j-1}+f_{l,j}} \\
* & 0 \\
\end{bmatrix},
\]

\[
\Xi_{N,45} = \begin{bmatrix}
X_{l,j-1} E_{l,j-1} & 0 & X_{l,j-1} E_{l,j-1} & 0 & 0 & 0 \\
Z_{l,j-1} E_{l,j-1} & 0 & Z_{l,j-1} E_{l,j-1} & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
\Xi_{N,55} = \text{diag} \left( -\epsilon I \right),
\]

\[
\Xi_{Z_{l,j-1},11} = \begin{bmatrix}
-M_{f_{l,j-1}+f_{l,j-2}}^{f_{l,j-1}+f_{l,j-2}} & +Q_1 & -M_{f_{l,j-1}+f_{l,j-2}}^{f_{l,j-1}+f_{l,j-2}} + Q_2 \\
* & -M_{f_{l,j-1}+f_{l,j-2}}^{f_{l,j-1}+f_{l,j-2}} + Q_3 \\
\end{bmatrix},
\]

\[
\Xi_{Z_{l,j-1},13} = \begin{bmatrix}
0 & \mathcal{G}_{l,j}^{f_{l,j-1}} - \mathcal{C}_{l,j}^{f_{l,j-1}} D_{F_{l,j-1}}^T \\
0 & -\mathcal{C}_{F_{l,j-1}}^{f_{l,j-1}} \\
\end{bmatrix},
\]

\[
\Xi_{Z_{l,j-1},14} = \begin{bmatrix}
(X_{l,j-1} A_{l,j-1} + W_{l,j-1} C_{l,j-1})^T & (Z_{l,j-1} A_{l,j-1} + W_{l,j-1} C_{l,j-1})^T & V_{l,j-1}^T \\
\end{bmatrix},
\]
\[
\mathbb{E}_{Z_{1,N-1,42}} = \begin{bmatrix}
0 & 0 & 0 & \left( H_{N,q}^A \right)^T & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]
\[
\mathbb{E}_{Z_{1,N-1,22}} = \begin{bmatrix}
-Q_1 & -Q_2 \\
* & -Q_3
\end{bmatrix},
\]
\[
\mathbb{E}_{Z_{1,N-1,24}} = \begin{bmatrix}
X_{i,N,q+1}A_{i,j,N,q+1}^T & \left( Z_{i,N,q+1}A_{i,j,N,q+1} \right)^T
\end{bmatrix},
\]
\[
\mathbb{E}_{Z_{1,N-1,25}} = \begin{bmatrix}
0 & 0 & 0 & 0 & \left( H_{N,q}^A \right)^T \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]
\[
\mathbb{E}_{Z_{1,N-1,33}} = \begin{bmatrix}
-Y^2 I - D_{i,N,q-1}^T D_{i,j,N,q-1}^T - I
\end{bmatrix},
\]
\[
\mathbb{E}_{Z_{1,N-1,34}} = \begin{bmatrix}
X_{i,N,q+1}B_{i,j,N,q+1} + W_{i,N,q+1}D_{i,j,N,q+1}^T & \left( Z_{i,N,q+1}B_{i,j,N,q+1} + W_{i,N,q+1}D_{i,j,N,q+1} \right)^T
\end{bmatrix},
\]
\[
\mathbb{E}_{Z_{1,N-1,44}} = \begin{bmatrix}
\begin{bmatrix}
X_{i,N,q+1} - X_{i,j,q+1}^T + \bar{M}_{i,j,N,q+1} \\
- Y_{i,N,q+1} - Z_{i,N,q+1}^T + \bar{M}_{i,j,N,q+1}^2
\end{bmatrix} & \begin{bmatrix}
- Y_{i,N,q+1} - Y_{i,j,q+1}^T + \bar{M}_{i,j,N,q+1}^3
\end{bmatrix}
\end{bmatrix},
\]
\[
\mathbb{E}_{Z_{1,N-1,45}} = \begin{bmatrix}
X_{i,N,q+1}E_{i,j,N,q+1} & 0 & X_{i,N,q+1}E_{i,j,N,q+1} & 0 & 0 & 0 \\
Z_{i,N,q+1}E_{i,j,N,q+1} & 0 & Z_{i,N,q+1}E_{i,j,N,q+1} & 0 & 0 & 0
\end{bmatrix},
\]
\[
\mathbb{E}_{Z_{1,N-1,55}} = \text{diag}_e(-eI),
\]
\[
\mathbb{E}_{0,11} = \begin{bmatrix}
-P_{f,i,j+1}^1 + Q_1 & -P_{f,i,j+1}^2 + Q_2 \\
* & -P_{f,i,j+1}^3 + Q_3
\end{bmatrix},
\]
\[
\mathbb{E}_{0,13} = \begin{bmatrix}
0 & G_{i,j}^T - C_{i,j}^T D_{F,j}^T \\
0 & -C_{F,j}^T
\end{bmatrix},
\]
\[
\mathbb{E}_{0,14} = \begin{bmatrix}
\begin{bmatrix}
X_{i,j}A_{i,j}^T + \left( V_i C_i \right)^T \\
V_i^T
\end{bmatrix} \\
\begin{bmatrix}
Z_{i,j}A_{i,j}^T + \left( W_i C_i \right)^T \\
V_i^T
\end{bmatrix}
\end{bmatrix},
\]
\[
\mathbb{E}_{0,15} = \begin{bmatrix}
0 & 0 & 0 & \left( H_{i,j}^A \right)^T \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]
\[
\mathbb{E}_{0,22} = \begin{bmatrix}
-Q_1 & -Q_2 \\
* & -Q_3
\end{bmatrix},
\]
\[
\mathbb{E}_{0,25} = \begin{bmatrix}
0 & 0 & 0 & 0 & \left( H_{i,j}^A \right)^T \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]
\[
\mathbb{E}_{0,33} = \begin{bmatrix}
-Y^2 I - D_{i,j}^T D_{F,j}^T - I
\end{bmatrix},
\]
\[
\mathbb{E}_{N,34} = \begin{bmatrix}
\begin{bmatrix}
X_{i,j}B_{i,j}^T + \left( V_i D_i \right)^T \\
0
\end{bmatrix} \\
\begin{bmatrix}
Z_{i,j}B_{i,j}^T + \left( W_i D_i \right)^T \\
0
\end{bmatrix}
\end{bmatrix},
\]
\[
\mathbb{E}_{N,44} = \begin{bmatrix}
\begin{bmatrix}
X_{i,j} - X_{i,j}^T + \bar{M}_{i,j,N,q+1} \\
- Y_{i,j} - Z_{i,j}^T + \bar{M}_{i,j,N,q+1}^2
\end{bmatrix} & \begin{bmatrix}
- Y_{i,j} - Y_{i,j}^T + \bar{M}_{i,j,N,q+1}^3
\end{bmatrix}
\end{bmatrix},
\]
\[
\mathbb{E}_{N,45} = \begin{bmatrix}
X_{i,j}E_{i,j} & 0 & X_{i,j}E_{i,j} & 0 & 0 & 0 \\
Z_{i,j}E_{i,j} & 0 & Z_{i,j}E_{i,j} & 0 & 0 & 0
\end{bmatrix},
\]
\[
\mathbb{E}_{N,55} = \text{diag}_e(-eI),
\]
and $\overline{M}_{h_{d}j_{N},1}$, $\overline{P}_{h_{d}j_{N}}$, $\overline{M}_{h_{d}j_{N},1}f_{N-1}$, $\overline{M}_{h_{d}j_{N},2}$, $\overline{M}_{h_{d}j_{N},3}$ are the same as in Lemma 3. Moreover, if (50a)–(50c) are true, the desired filtering parameters of the form (5) can be obtained as

$$
A_{F_{j_0}} = Y_{i_{0}}^{-1} V_{i_{0}}, \\
B_{F_{j_0}} = Y_{i_{0}}^{-1} W_{i_{0}}, \\
C_{F_{j_0}}, \\
D_{F_{j_0}}.
$$

Proof. Noting that $-R_{j_{N-1}}(\overline{P}_{h_{d}j_{N}})^{-1} R_{j_{N-1}}^T \leq -R_{j_{N-1}} - R_{j_{N-1}}^T + \overline{P}_{h_{d}j_{N}}$, applying the well-known Schur complement [41] to (34a), and performing a congruence transformation $\text{diag}(I, I, I, I, R_{j_{N-1}})$, $\text{diag}(I, I, I, 1, R_{j_{N-1}})$ to (34a), we have the following.

\[
\begin{bmatrix}
-\overline{M}_{h_{d}j_{N},1}f_{N-1}^{-1} & 0 & 0 & \overline{C}_{j_{N-1}}^T & (R_{j_{N-1}} - \overline{A}_{j_{N-1}})^T \\
* & -Q & 0 & 0 & (R_{j_{N-1}} - \overline{A}_{d,j_{N-1}})^T \\
* & * & -\gamma^2 I & \overline{D}_{j_{N-1}}^T & (R_{j_{N-1}} - \overline{B}_{j_{N-1}})^T \\
* & * & * & -I & 0 \\
* & * & * & * & -R_{j_{N-1}} - R_{j_{N-1}}^T + \overline{P}_{h_{d}j_{N}}^{-1} \\
\end{bmatrix} < 0
\]  \tag{53}

Letting

\[
V_{i_{N-1}} = Y_{i_{N-1}} A_{F_{j_{N-1}}}, \\
W_{i_{N-1}} = Y_{i_{N-1}} B_{F_{j_{N-1}}}, \\
\overline{M}_{h_{d}j_{N},1}f_{N-1}^{-1} = \begin{bmatrix} \overline{M}_{h_{d}j_{N},1}f_{N-1}^{-1} & \overline{M}_{h_{d}j_{N},2}f_{N-1}^{-1} \\
* & \overline{M}_{h_{d}j_{N},3}f_{N-1}^{-1} \end{bmatrix}, \\
\overline{P}_{h_{d}j_{N}} = \begin{bmatrix} \overline{P}_{h_{d}j_{N},1} & \overline{P}_{h_{d}j_{N},2} \\
* & \overline{P}_{h_{d}j_{N},3} \end{bmatrix},
\]

and substituting (54) into (53), one has

\[
Y_N = \begin{bmatrix} \Xi_{N,11} & 0 & \Xi_{N,13} & Y_{N,14} \\
* & \Xi_{N,22} & 0 & Y_{N,24} \\
* & * & \Xi_{N,33} & Y_{N,34} \\
* & * & * & \Xi_{N,44} \end{bmatrix} < 0,  \tag{55}
\]

where

\[
Y_{N,14} = \left( \frac{X_{i_{N-1}} \overline{A}_{j_{N-1}}}{V_{i_{N-1}}} \right)^T + \left( \frac{W_{i_{N-1}} C_{i_{N-1}}}{V_{i_{N-1}}} \right)^T, \\
Y_{N,24} = \left( \frac{X_{i_{N-1}} \overline{A}_{d,j_{N-1}}}{0} \right)^T + \left( \frac{W_{i_{N-1}} C_{i_{N-1}}}{0} \right)^T.
\]  \tag{56}

By taking (2) into account, one further has

\[
Y_N = \Psi_N + \mathcal{L}_{j_{N-1}}(k)
\]  \tag{57}

where

\[
\begin{bmatrix}
H_{d_{N-1}}^d & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]
By employing Schur complement and Lemma 8 to (57), we know that $\Upsilon_N < 0$ implies $\Xi_N < 0$. Therefore, it can be concluded that condition (34a) can be transformed into an LMI formulation (50a) with designed filter gains. Following the same lines of the proof of (50a), we can prove that the filtering gain matrices obtained by Theorem 9 can achieve a bounded filtering error (9). This completes the proof. \hfill \Box

Remark 10. The NALKF approach also has potentials to be extended to solve other filtering or control problems of time-delayed systems, such as sampled-data synchronization control [42, 43], time-varying delay analysis [44, 45], and $H_{\infty}$ repetitive-control [46]. However, it should be noted that the NALKF approach can only be developed for discrete-time systems. For the continuous-time case, the higher derivative of the Lyapunov functional should be considered, such as $\min(\dot{V}(t), \ddot{V}(t)) < 0$ or $\dot{V}(t) + \ddot{V}(t) < 0$. Moreover, for time-delayed systems, a trade-off must be considered between the conservatism reduction and computational burden.

5. Illustrative Example

In this section, we provide an illustrative example to verify the effectiveness and applicability of the proposed NALKF method.

Consider NMJDS (1) with parameters as follows

$$A_1 = \begin{bmatrix} 0.5 & -0.6 \\ -0.4 & -0.2 \end{bmatrix},$$
$$A_{d,1} = \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & 0.15 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.15 & 0.3 \end{bmatrix},$$
$$A_{d,2} = \begin{bmatrix} 0.38 & 0 \\ 0.01 & 0.16 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}.$$
The indicator functions are chosen as
\[
\mu_1 (k) = \frac{1 - x_{k,1}}{2}, \\
\mu_2 (k) = \frac{1 + x_{k,1}}{2}.
\]

Here \( x_{k,1} \) represents the first state of \( x_k \). The initial state, initial estimated state, initial \( \hat{z}_k \), and initial \( \bar{z}_k \) are taken as \( x_0 = \bar{x}_0 = [0 \ 0]^T \), \( z_0 = \hat{z}_0 = [0 \ 0]^T \). The initial operating mode and time delay are assumed as \( \tau_0 = 1 \) and \( d = 4 \). The exogenous disturbance \( w_k \) is taken as \( 0.2e^{-0.5k} \).

In this case, we would like to provide an optimal \( H_\infty \) performance for designing the filter; that is, we are interested in the optimization problem

\[
\min \quad \gamma \\
\text{subject to} \quad (50a)-(50c)
\]

under zero-initial condition. Solving (50a)-(50c) by using the LMI Toolbox, the robust \( H_\infty \) filter gains obtained via two-step-ahead approach are listed as

\[
A_{F,1} = \begin{bmatrix}
-0.0214 & 0.2915 \\
-0.0636 & -0.6562
\end{bmatrix}, \\
A_{F,2} = \begin{bmatrix}
1.2150 & 0.3716 \\
-0.1274 & 0.1425
\end{bmatrix}, \\
B_{F,1} = \begin{bmatrix}
3.2391 \\
-0.9403
\end{bmatrix}, \\
B_{F,2} = \begin{bmatrix}
0.1913 \\
1.5089
\end{bmatrix}, \\
C_{F,1} = \begin{bmatrix}
-0.6039 & -0.4670
\end{bmatrix}, \\
C_{F,2} = \begin{bmatrix}
-0.1541 & -0.4506
\end{bmatrix}, \\
D_{F,1} = -0.5960, \\
D_{F,2} = -0.1448.
\]

Similarly, the robust \( H_\infty \) filter gains obtained via three-step-ahead approach are listed as

\[
A_{F,1} = \begin{bmatrix}
-0.3931 & 0.3412 \\
0.7966 & -0.9720
\end{bmatrix}, \\
A_{F,2} = \begin{bmatrix}
0.4723 & 0.3298 \\
0.4882 & 0.3683
\end{bmatrix}, \\
B_{F,1} = \begin{bmatrix}
-7.3667 \\
6.3261
\end{bmatrix}, \\
B_{F,2} = \begin{bmatrix}
1.2150 & 0.3716 \\
-0.1274 & 0.1425
\end{bmatrix}, \\
C_{F,1} = \begin{bmatrix}
-0.6039 & -0.4670
\end{bmatrix}, \\
C_{F,2} = \begin{bmatrix}
-0.1541 & -0.4506
\end{bmatrix}, \\
D_{F,1} = -0.5960, \\
D_{F,2} = -0.1448.
\]

It must be pointed out that the conventional filtering approach, such as in [18], can be viewed as a special case of the NALKF approach, i.e., the one-step-ahead approach. The filtering gains can be obtained as

\[
A_{F,1} = \begin{bmatrix}
0.0531 & -0.0076 \\
-0.0727 & -0.5311
\end{bmatrix}, \\
A_{F,2} = \begin{bmatrix}
0.1398 & 1.7352 \\
0.0533 & 0.7602
\end{bmatrix}, \\
B_{F,1} = \begin{bmatrix}
-9.8681 \\
4.2420
\end{bmatrix}, \\
B_{F,2} = \begin{bmatrix}
-1.3763 \\
-0.5301
\end{bmatrix}, \\
C_{F,1} = \begin{bmatrix}
0.0761 & -1.0295
\end{bmatrix}, \\
C_{F,2} = \begin{bmatrix}
-0.0433 & -0.6545
\end{bmatrix}, \\
D_{F,1} = -6.9897, \\
D_{F,2} = -0.5292.
\]

We also get the minimum value of \( \gamma \) and the number of decision variables of NALKF approach, which are shown in Table 1. Obviously, with the increase of the predictive step, we obtain better \( H_\infty \) performance. However, it must be pointed out that the number of decision variables of the developed NALKF approach is not increased with the predictive step because the knowledge of the multistep TPs \( \pi_{i,j,N-q} \) is included.

The jumping mode path shown in Figure 1 from time step 0 to time step 50 is generated randomly by employing TP matrices \( \Pi_1, \Pi_2 \) and indicator functions \( \mu_1 (k), \mu_2 (k) \). The number of iterations is chosen as 50 and each iteration unit length is taken as 1.

The comparison of the estimated error under jumping mode path in Figure 1 is demonstrated in Figure 2, where the red line, green line, and blue line represent the error under conventional filtering approach, two-step-ahead approach, and three-step-ahead approach, respectively. It also can be concluded that with the increase of the predictive step, the estimated error becomes smaller.

Moreover, the estimate of the output \( \hat{z}_k \) under jumping mode path in Figure 1 is also drawn in Figure 3, where the black line is the output \( z_k \). It can be observed that the
Table 1: Minimum value of $\gamma$ and number of the decision variables.

<table>
<thead>
<tr>
<th>Approach</th>
<th>Minimum Value of $\gamma$</th>
<th>Number of Decision Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional</td>
<td>8.7815</td>
<td>81</td>
</tr>
<tr>
<td>Two-step</td>
<td>1.8374</td>
<td>117</td>
</tr>
<tr>
<td>Three-step</td>
<td>1.6383</td>
<td>117</td>
</tr>
</tbody>
</table>

three-step-ahead approach (blue line) has the best tracking performance, which also justifies the results of Figure 2.

Finally, a Monte Carlo simulation of the estimated error, which is generated under 20 different jumping mode paths, is given in Figure 4. Such a simulation uses the two-step-ahead approach. It can be seen that the overall estimation quality is good. The extreme case is less than 25%. Therefore, it justifies the effectiveness of the developed approach from another side.

6. Conclusion

The robust $H_{\infty}$ filtering problem has been considered in this paper for nonhomogeneous Markovian jump delay systems via $N$-step-ahead Lyapunov-Krasovskii functional approach. The robust $H_{\infty}$ filter has been designed in terms of a feasible optimization problem subject to LMI constraints, which guarantees that the filtering error system is mean-square stable and the filtering error satisfies smaller $H_{\infty}$ dissipative level for all possible time-varying transition probabilities and all admissible parameter uncertainties and time-delays. The
$N$-step-ahead approach can be extended to deal with the robust $H_{\infty}$ dynamic output feedback control problem.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**


