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Research Article

Resolution of Max-Product Fuzzy Relation Equation with Interval-Valued Parameter

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Considering the application background on P2P network system, we investigate the max-product fuzzy relation equation with interval-valued parameter in this paper. Order relation on the set of all interval-valued numbers plays key role in the construction and resolution of the interval-valued-parameter fuzzy relation equation (IPFRE). The basic operations supremum ($a \vee b$) and infimum ($a \wedge b$) in the IPFRE should be defined depending on the order relation. A novel total order is introduced for establishing the IPFRE. We also discuss some properties of the IPFRE system, including the consistency and structure of the complete solution set. Concepts of close index set and open index set are defined, helping us to construct the resolution method of the IPFRE system. We further provide a detailed algorithm for obtaining the complete solution set. Besides, the solution set is compared to that of the classical max-T fuzzy relation equations system.

1. Introduction

A crisp relation represents the presence or absence of association, interaction, or interconnectedness between the elements of two or more sets [1]. Its importance is almost self-evident [2]. Relation exists everywhere widely and trying to discovery the relation is one of the most important targets in science research. In classical two-valued logic relation, one object is either relevant or irrelevant to another one. The degree or strength of the relation is not able to be described. To overcome such shortage, the typical crisp relation was naturally extended to fuzzy relation [3]. As inverse problem of fuzzy linear system, fuzzy relation equations were first proposed by E. Sanchez [4], motivated by its application in medical diagnosis. The potential maximal solution could be easily obtained, used for checking the consistency of a max-min fuzzy relation equations system. However, searching all the minimal solutions is much more difficult and important for the complete solution set. Since the traditional linear algebra methods are no longer effective for the fuzzy relation equations, various resolution methods were proposed and illustrated [5–23].

The composition plays important role in fuzzy relation equations. It was extended from max-min to max-product [24–27] and other ones [21]. It has been demonstrated the max-product composition is superior to the max-min one in some cases [25]. J. Loetamonphong and S.-C. Fang first investigated the optimization problem with linear-objective function and max-product fuzzy relation equations constraint [28]. The resolution idea was picked from [29], in which the main problem was divided into two subproblems and solved by the branch-and-bound method. In fact the two-subproblem approach was widely adopted to deal with linear-objective optimization problem subject to fuzzy relation equations or inequalities with various composition such as max-min, max-product and max-average [30, 31]. Nonlinear-objective optimization problem subject to fuzzy relation system was also studied. There exist three kinds of solution methods for such nonlinear optimization problem: (i) genetic algorithm for optimization problem with general nonlinear-objective function and fuzzy relation constraint. By this kind of method, only approximate solution could be found [32, 33]; (ii) method to separate the main problem into a finite number of subproblems according to the minimal solutions of the constraint [34, 35]; (iii) specific method for the fuzzy relation problem with special objective functions [36, 37].

In recent years some new types of fuzzy relation inequalities or equations appeared. Fuzzy relation inequalities with addition-min composition were employed to describe the P2P file sharing system. Some properties of its solution set [38] and corresponding optimization models were studied [39–41]. Besides, bipolar fuzzy relation equations was another interesting research object. The scholars were interested in the relevant linear optimization problem [42–44].

As a special form of fuzzy number, interval was a powerful tool in both theoretical and practical application aspects [45–47]. Considering the fuzziness in the parameters, interval-valued fuzzy relation equations attracted some researchers' attentions [48, 49]. The authors investigated three types of solutions, i.e., tolerable solution set, united solution set, and controllable solution set, respectively. Before definition of these solutions, a group of fuzzy relations was converted into two groups of fuzzy relation inequalities. In the existing works on interval-valued fuzzy relation equations, the compositions in the system contained *max-t-norm* [50], *min-s-norm* [51], and *max-plus* [52]. In these existing works, an interval-valued number was considered as a set of real numbers, but not a fuzzy number. Analogously, the interval-valued vector and matrix were both considered as some sets. However as we know, an interval-valued number is indeed a fuzzy number. There exists its own operations, such as addition and multiplication. Based on such consideration, we introduce the max-product fuzzy relation equations with interval-valued parameters in this paper. The rest part is organized as follows. In Section 2 we define some basic operations and an order relation on the set of all interval- valued numbers. Section 3 is the formulation of max-product fuzzy relation equation with interval-valued parameter. In Section 4 we provides its solution method based on concepts of close index set and open index set. Moreover, structure and resolution algorithm of the complete solution set is investigated. A numerical example is given in Section 5 to illustrated the algorithm. Simple discussion and conclusion are set in Sections 6 and 7 respectively.

2. Basic Operation and Order Relation on the Set of All Interval-Valued Numbers

Interval-valued number is an important extension of the precise real number for describing the quantities in the real world. An interval-valued number is usually with the form of $[\underline{u},\overline{u}]$, where $\underline{u},\overline{u} \in R$ and $\underline{u} \le \overline{u}$, R represents the real number set.

Definition 1 ([53] (fuzzy number)). A fuzzy set $\widetilde{A}$ in R is called a fuzzy number, if it satisfies the following conditions:

(i) $\widetilde{A}$ is normal; i.e., the cut set A_1 is nonempty,

(ii) the α-level cut set A_α is a closed interval for any $\alpha \in (0,1]$,

(iii) the support of $\widetilde{A}$ is bounded.

An interval-valued number could be viewed as a special type of fuzzy number [54]. For interval-valued number $[\underline{u},\overline{u}]$, the membership function is

$$\mu(x) = \begin{cases} 1, & \text{if } \underline{u} \le x \le \overline{u}, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

It is easy to check that $[\underline{u},\overline{u}]$ satisfies the conditions in Definition 1. For more detail, the readers could refer to [55–57].

Denote the set of all interval-valued numbers by $IV = \{[\underline{u},\overline{u}] \mid \underline{u} \le \overline{u},\ \underline{u},\overline{u} \in R\}$. We first present two basic operations on the set IV, which are commonly used in the existing works [58]. Let $[\underline{u},\overline{u}], [\underline{v},\overline{v}] \in IV$ and $k \in R$. Then

(a) (addition) $[\underline{u},\overline{u}] + [\underline{v},\overline{v}] = [\underline{u}+\underline{v}, \overline{u}+\overline{v}]$;

(b) (scalar-multiplication) $k[\underline{u},\overline{u}] = [\underline{u},\overline{u}]k = \{[k\underline{u},k\overline{u}],\ \text{if } k \ge 0;\ [k\overline{u},k\underline{u}],\ \text{if } k < 0\}$.

Besides, supremum ($\vee$) and infimum ($\wedge$) are basic operations for establishing the fuzzy relation equations. These two operations depend on the order relation on the set IV. Now we investigate the order relation on the set IV first.

In general, an order relation is defined for comparing two elements in a set. In the set IV, the most common order relation is the classical "product" order [59], denoted by "$\lesssim$" in this paper. Let $[\underline{u},\overline{u}], [\underline{v},\overline{v}] \in IV$. We have $[\underline{u},\overline{u}] \lesssim [\underline{v},\overline{v}]$ if and only if both $\underline{u} \le \underline{v}$ and $\overline{u} \le \overline{u}$ hold, while $[\underline{u},\overline{u}] = [\underline{v},\overline{v}]$ if and only if both $\underline{u} = \underline{v}$ and $\overline{u} = \overline{u}$ hold. It is easy to check that "$\lesssim$" is a partial order on IV, but not a total order. A *partial order* on a set is a binary relation satisfying *reflexivity*, *antisymmetry*, and *transitivity*. A *total order* is a partial order which makes any two elements in the set *comparable*. Under the above-mentioned order "$\lesssim$", the supremum ($\tilde{\vee}$) and infimum ($\tilde{\wedge}$) operations on the set IV are as follows:

$$\begin{aligned} [\underline{u},\overline{u}] \tilde{\vee} [\underline{v},\overline{v}] &= [\underline{u} \vee \underline{v}, \overline{u} \vee \overline{v}], \\ [\underline{u},\overline{u}] \tilde{\wedge} [\underline{v},\overline{v}] &= [\underline{u} \wedge \underline{v}, \overline{u} \wedge \overline{v}]. \end{aligned} \tag{2}$$

The advantage of the product order "$\lesssim$" lies in its convenience in computation. However, it is not a total order. Some pairs of interval-valued numbers could not be compared by the product order. For example, under such order relation, we are not able to compare $[0.1, 0.6]$ and $[0.2, 0.3]$. Both $[0.1, 0.6] \lesssim [0.2, 0.3]$ and $[0.2, 0.3] \lesssim [0.1, 0.6]$ are incorrect.

However, as pointed out in [60], a total order is indispensable in some situations. It enables us to compare any pair of interval-valued numbers. Specific total order was introduced for comparing any two intuitionistic fuzzy sets [61]. Moreover, admissible order, which refined both the total order and the classical product order, was defined on the interval-valued number set [62–64]. We recall the definition of admissible order on IV below.

Definition 2 ([62] (admissible order)). Let $(IV, \preceq)$ be a poset, with partial order "$\preceq$". The order $\preceq$ is said to be admissible order, if it satisfies the following conditions:

(i) $\preceq$ is a total (linear) order on IV;

(ii) for any $[\underline{u},\overline{u}], [\underline{v},\overline{v}] \in IV$, $[\underline{u},\overline{u}] \lesssim [\underline{v},\overline{v}]$ implies that $[\underline{u},\overline{u}] \preceq [\underline{v},\overline{v}]$, where $\lesssim$ represents the classical product order.

For more detailed examples on admissible order, the readers could refer to [59, 62–64].

In this paper, motivated by the order defined in [61] for the intuitionistic fuzzy set and the order introduced in [65–69] for comparison of two fuzzy numbers in solving fuzzy linear programming, we define the following order relation "$<$" on the set IV.

Definition 3. For any $[\underline{u},\overline{u}],[\underline{v},\overline{v}]\in IV$, we say $[\underline{u},\overline{u}]$ is less than $[\underline{v},\overline{v}]$, denoted by $[\underline{u},\overline{u}]<[\underline{v},\overline{v}]$, if it holds that

(i) $(1/2)(\underline{u}+\overline{u})<(1/2)(\underline{v}+\overline{v})$; or

(ii) $(1/2)(\underline{u}+\overline{u})=(1/2)(\underline{v}+\overline{v})$ and $(1/2)(\overline{u}-\underline{u})>(1/2)(\overline{v}-\underline{v})$;

Besides, $[\underline{u},\overline{u}]$ is equal to $[\underline{v},\overline{v}]$, i.e., $[\underline{u},\overline{u}]=[\underline{v},\overline{v}]$, if both $(1/2)(\underline{u}+\overline{u})=(1/2)(\underline{v}+\overline{v})$ and $(1/2)(\overline{u}-\underline{u})=(1/2)(\overline{v}-\underline{v})$ hold. Inequality $[\underline{u},\overline{u}]\leq[\underline{v},\overline{v}]$ represents either $[\underline{u},\overline{u}]<[\underline{v},\overline{v}]$ or $[\underline{u},\overline{u}]=[\underline{v},\overline{v}]$ holds. The dual symbols of $<$ and $\leq$ are $>$ and $\geq$, respectively.

Under the above-defined order relation, it is easy to check that $[\underline{u},\overline{u}]=[\underline{v},\overline{v}]$ if and only if $\underline{u}=\underline{v}$ and $\overline{u}=\overline{v}$. This is coincident with the classical product order. In Definition 3, $(1/2)(\underline{u}+\overline{u})$ is the midpoint of $[\underline{u},\overline{u}]$, while $(1/2)(\overline{u}-\underline{u})$ means the radius of $[\underline{u},\overline{u}]$ [70].

Proposition 4. *For arbitrary $[\underline{u},\overline{u}],[\underline{v},\overline{v}],[\underline{w},\overline{w}]\in IV$, the following statements are true:*

(i) *(reflexivity) $[\underline{u},\overline{u}]\leq[\underline{u},\overline{u}]$;*

(ii) *(antisymmetry) $[\underline{u},\overline{u}]\leq[\underline{v},\overline{v}]$ and $[\underline{v},\overline{v}]\leq[\underline{u},\overline{u}]$ indicate $[\underline{u},\overline{u}]=[\underline{v},\overline{v}]$;*

(iii) *(transitivity) $[\underline{u},\overline{u}]\leq[\underline{v},\overline{v}]$ and $[\underline{v},\overline{v}]\leq[\underline{w},\overline{w}]$ indicate $[\underline{u},\overline{u}]\leq[\underline{w},\overline{w}]$;*

(iv) *(comparability) either $[\underline{u},\overline{u}]\leq[\underline{v},\overline{v}]$ or $[\underline{v},\overline{v}]\leq[\underline{u},\overline{u}]$ holds;*

(v) *(refinement of the classical product order) if $[\underline{u},\overline{u}]\lesssim[\underline{v},\overline{v}]$, then $[\underline{u},\overline{u}]\leq[\underline{v},\overline{v}]$.*

Proof. Omitted. □

It is shown in Proposition 4 that the order relation "$\leq$" in Definition 3 is an admissible order. Moreover, $(IV,\leq)$ forms a total order set. Under such order relation, any pair of interval-valued numbers could be compared. Next we provide the supremum ($\vee$) and infimum ($\wedge$) operations on IV, based on the above-defined order "$\leq$". Let $[\underline{u},\overline{u}],[\underline{v},\overline{v}]\in IV$. Then we have

(c) (supremum) $[\underline{u},\overline{u}]\vee[\underline{v},\overline{v}]=\{[\underline{v},\overline{v}],\ \text{if } [\underline{u},\overline{u}]\leq[\underline{v},\overline{v}];\ [\underline{u},\overline{u}],\ \text{otherwise}\}$;

(d) (infimum) $[\underline{u},\overline{u}]\wedge[\underline{v},\overline{v}]=\{[\underline{u},\overline{u}],\ \text{if } [\underline{u},\overline{u}]\leq[\underline{v},\overline{v}];\ [\underline{v},\overline{v}],\ \text{otherwise}\}$.

3. Form of Max-Product Fuzzy Relation Equation with Interval-Valued Parameter

Motivated by the application background of P2P network system, we introduce the form of max-product fuzzy relation equations with interval-valued parameters in this section.

3.1. Unit Interval-Valued Number. An interval-valued number $[\underline{u},\overline{u}]$ is said to be unit interval-valued number, if $0\leq\underline{u}\leq\overline{u}\leq 1$. We denote the set of all unit interval-valued numbers by

$$IV([0,1])=\{[\underline{u},\overline{u}]\mid 0\leq\underline{u}\leq\overline{u}\leq 1,\ \underline{u},\overline{u}\in R\},\tag{3}$$

and the set of all nonzero (or positive) unit interval-valued numbers by

$$IV((0,1])=\{[\underline{u},\overline{u}]\mid\overline{u}>0,\ [\underline{u},\overline{u}]\in IV([0,1])\}.\tag{4}$$

Remark 5. According to the order given in Definition 3, the set $IV((0,1])$ can also be written as

$$\begin{aligned}&IV((0,1])\\&\quad=\{[\underline{u},\overline{u}]\mid[\underline{u},\overline{u}]\neq[0,0],\ [\underline{u},\overline{u}]\in IV([0,1])\},\end{aligned}\tag{5}$$

or

$$\begin{aligned}&IV((0,1])\\&\quad=\{[\underline{u},\overline{u}]\mid[\underline{u},\overline{u}]>[0,0],\ [\underline{u},\overline{u}]\in IV([0,1])\}.\end{aligned}\tag{6}$$

Obviously both $IV([0,1])$ and $IV((0,1])$ are subsets of IV. Hence the operations and order relation defined on IV in Section 2 keep valid for $IV([0,1])$ and $IV((0,1])$.

3.2. Classical Max-Min and Max-Product Fuzzy Relation Equations. For the convenience of expression, we denote two index sets I and J as follows:

$$\begin{aligned}I&=\{1,2,\ldots,m\},\\J&=\{1,2,\ldots,n\}.\end{aligned}\tag{7}$$

System of fuzzy relation equations, with the most commonly used *max-min* composition [4], could be written as [19–21, 27, 29, 31]

$$\begin{gathered}(a_{11}\wedge x_1)\vee(a_{12}\wedge x_2)\vee\cdots\vee(a_{1n}\wedge x_n)=b_1,\\(a_{21}\wedge x_1)\vee(a_{22}\wedge x_2)\vee\cdots\vee(a_{2n}\wedge x_n)=b_2,\\\vdots\\(a_{m1}\wedge x_1)\vee(a_{m2}\wedge x_2)\vee\cdots\vee(a_{mn}\wedge x_n)=b_m,\end{gathered}\tag{8}$$

where $a_{ij},x_j,b_i\in[0,1]$, $i\in I$, $j\in J$. But soon after, the *max-min* composition was extended to the *max-product* one.

Analogously, system of max-product fuzzy relation equations was expressed by

$$\begin{aligned} a_{11}x_1 \vee a_{12}x_2 \vee \cdots \vee a_{1n}x_n &= b_1, \\ a_{21}x_1 \vee a_{22}x_2 \vee \cdots \vee a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 \vee a_{m2}x_2 \vee \cdots \vee a_{mn}x_n &= b_m, \end{aligned} \tag{9}$$

with the same parameters and variables appear in system (8).

In [71], the max-product fuzzy relation equation was applied to describe the quantitative relation in a P2P (peer-to-peer) network system. Suppose there exist n terminals in such system, denoted by T_j, $j \in J$. Each pair of terminals was wireless-connected. They transformed their local resources through electromagnetic wave. The terminal T_j transmits the electromagnetic wave with intensity x_j. Due to the decreasing of the intensity, associated with the distance, the intensity of the electromagnetic wave is no more than x_j, when it is received by T_i, $i \in J$ and $i \neq j$. Hence the intensity of electromagnetic wave at T_i, which is send out by T_j, could be represented by

$$a_{ij}x_j, \tag{10}$$

where $0 \leq a_{ij} \leq 1$. We call a_{ij} the decreasing coefficient. In general case, T_i selects the terminal with highest intensity, to download the target resource. The intensity of electromagnetic wave reflects the download quality level. Hence if the requirement of download quality level of T_i is $b_i > 0$, then the intensities of electromagnetic wave should satisfy

$$a_{i1}x_1 \vee a_{i2}x_2 \vee \cdots \vee a_{in}x_n = b_i. \tag{11}$$

Here, without loss of generality, we assume the first m terminals require the download quality level. Then we have $i \in I = \{1, 2, \ldots, m\}$. After normalization of the parameters and variables, the requirements of download quality level of the terminals could be described the max-product fuzzy relation equations as system (9).

3.3. Max-Product Fuzzy Relation Equation with Interval-Valued Parameter. When applying system (8) to describe the P2P network system as shown in last subsection, the decreasing coefficient a_{ij} is determined by the distance between the terminals T_i and T_j. The value of a_{ij} will become smaller, if the distance of T_i and T_j becomes farther. However, in real world application, the distance might not be the unique factor influencing the decreasing coefficient a_{ij}. For example, weather condition and artificially disturbing of the electromagnetic wave are also potential influence factors. Notice that these influence factors are usually uncertain or random. Hence the decreasing coefficient a_{ij} is not a precise number. It could take any possible value within limits. Here we assume a_{ij} as an interval-valued number, i.e., $a_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]$. Correspondingly, the requirement of the terminal T_i is also assumed to be a value range and denoted by $[\underline{b}_i, \overline{b}_i]$, $i \in I$.

As a consequence, considering the uncertain potential influence factors of the decreasing coefficient, the corresponding system for describing the P2P network system could be written as

$$\begin{aligned} &[\underline{a}_{11}, \overline{a}_{11}]\, x_1 \vee [\underline{a}_{12}, \overline{a}_{12}]\, x_2 \vee \cdots \\ &\quad \vee [\underline{a}_{1n}, \overline{a}_{1n}]\, x_n = [\underline{b}_1, \overline{b}_1], \\ &[\underline{a}_{21}, \overline{a}_{21}]\, x_1 \vee [\underline{a}_{22}, \overline{a}_{22}]\, x_2 \vee \cdots \\ &\quad \vee [\underline{a}_{2n}, \overline{a}_{2n}]\, x_n = [\underline{b}_2, \overline{b}_2], \\ &\qquad \vdots \\ &[\underline{a}_{m1}, \overline{a}_{m1}]\, x_1 \vee [\underline{a}_{m2}, \overline{a}_{m2}]\, x_2 \vee \cdots \\ &\quad \vee [\underline{a}_{mn}, \overline{a}_{mn}]\, x_n = [\underline{b}_m, \overline{b}_m], \end{aligned} \tag{12}$$

where the parameters $[\underline{a}_{ij}, \overline{a}_{ij}], [\underline{b}_i, \overline{b}_i] \in IV((0,1])$ are unit interval-valued numbers, $i \in I$, $j \in J$. System (12) consists of m max-product fuzzy relation equations with interval-valued parameters. System (12) could also be written as the following matrix form,

$$A \odot x^T = b^T, \tag{13}$$

where $A = (a_{ij})_{m\times n} = ([\underline{a}_{ij}, \overline{a}_{ij}])_{m\times n}$, $b = (b_i)_{1\times m} = ([\underline{b}_i, \overline{b}_i])_{1\times m}$, $x = (x_j)_{1\times n}$. In system (12), the order relation and operations are as defined in Section 2.

4. Resolution of Max-Product Fuzzy Relation Equation with Interval-Valued Parameter

4.1. Resolution of $[\underline{a}_0, \overline{a}_0]x_0 = [\underline{b}_0, \overline{b}_0]$ and $[\underline{a}_0, \overline{a}_0]x_0 \leq [\underline{b}_0, \overline{b}_0]$

Proposition 6. *Let $[\underline{a}_0, \overline{a}_0], [\underline{b}_0, \overline{b}_0] \in IV((0,1])$. The equation*

$$[\underline{a}_0, \overline{a}_0]\, x_0 = [\underline{b}_0, \overline{b}_0] \tag{14}$$

is solvable (has at least one solution) if and only if $\underline{a}_0\overline{b}_0 = \overline{a}_0\underline{b}_0$.

Proof. ($\Longrightarrow$) Notice that the equation $[\underline{a}_0, \overline{a}_0]x_0 = [\underline{b}_0, \overline{b}_0]$ is equivalent to

$$\begin{aligned} \underline{a}_0 x_0 &= \underline{b}_0, \\ \overline{a}_0 x_0 &= \overline{b}_0. \end{aligned} \tag{15}$$

If $\underline{a}_0 = 0$, then $\underline{a}_0 x_0 = \underline{b}_0$ indicates $\underline{b}_0 = 0$. So we get $\underline{a}_0\overline{b}_0 = 0 = \overline{a}_0\underline{b}_0$.

If $\underline{a}_0 \neq 0$, then $\underline{a}_0 x_0 = \underline{b}_0$ indicates $x_0 = \underline{b}_0/\underline{a}_0$, while $\overline{a}_0 x_0 = \overline{b}_0$ indicates $x_0 = \overline{b}_0/\overline{a}_0$. Combinatorially, we have $\underline{b}_0/\underline{a}_0 = x_0 = \overline{b}_0/\overline{a}_0$, i.e., $\underline{a}_0\overline{b}_0 = \overline{a}_0\underline{b}_0$.

($\Longleftarrow$) Since $[\underline{a}_0, \overline{a}_0] \in IV((0,1])$, it is clear that $\underline{a}_0 + \overline{a}_0 > 0$. Set

$$x_0 = \frac{\underline{b}_0 + \overline{b}_0}{\underline{a}_0 + \overline{a}_0}. \tag{16}$$

Then it follows from $\underline{a}_0\overline{b}_0 = \overline{a}_0\underline{b}_0$ that

$$\begin{aligned}
[\underline{a}_0,\overline{a}_0]\,x_0 &= [\underline{a}_0,\overline{a}_0]\,\frac{\underline{b}_0+\overline{b}_0}{\underline{a}_0+\overline{a}_0} \\
&= \left[\frac{\underline{a}_0\left(\underline{b}_0+\overline{b}_0\right)}{\underline{a}_0+\overline{a}_0}, \frac{\overline{a}_0\left(\underline{b}_0+\overline{b}_0\right)}{\underline{a}_0+\overline{a}_0}\right] \\
&= \left[\frac{\underline{a}_0\underline{b}_0+\underline{a}_0\overline{b}_0}{\underline{a}_0+\overline{a}_0}, \frac{\overline{a}_0\underline{b}_0+\overline{a}_0\overline{b}_0}{\underline{a}_0+\overline{a}_0}\right] \\
&= \left[\frac{\underline{a}_0\underline{b}_0+\overline{a}_0\underline{b}_0}{\underline{a}_0+\overline{a}_0}, \frac{\underline{a}_0\overline{b}_0+\overline{a}_0\overline{b}_0}{\underline{a}_0+\overline{a}_0}\right] \\
&= \left[\frac{\left(\underline{a}_0+\overline{a}_0\right)\underline{b}_0}{\underline{a}_0+\overline{a}_0}, \frac{\left(\underline{a}_0+\overline{a}_0\right)\overline{b}_0}{\underline{a}_0+\overline{a}_0}\right] = \left[\underline{b}_0,\overline{b}_0\right].
\end{aligned} \tag{17}$$

Hence x_0 is a solution of (14). □

Lemma 7. *Let $[\underline{a}_0,\overline{a}_0] \in IV((0,1])$. We have*

(i) *if $x_0 \le y_0$, then $[\underline{a}_0,\overline{a}_0]x_0 \le [\underline{a}_0,\overline{a}_0]y_0$;*

(ii) *if $x_0 < y_0$, then $[\underline{a}_0,\overline{a}_0]x_0 < [\underline{a}_0,\overline{a}_0]y_0$.*

Proof. The proof is trivial. □

According Lemma 7, we get the following Corollary 8 immediately.

Corollary 8. *Let $[\underline{a}_0,\overline{a}_0] \in IV((0,1])$. If y_1 is a solution of the inequality $[\underline{a}_0,\overline{a}_0]x_0 \le [\underline{b}_0,\overline{b}_0]$, then any y_2 satisfying $y_2 \le y_1$ is also a solution of $[\underline{a}_0,\overline{a}_0]x_0 \le [\underline{b}_0,\overline{b}_0]$.*

Furthermore, combining Proposition 6 and Lemma 7, we get Corollary 9 below.

Corollary 9. *Let $[\underline{a}_0,\overline{a}_0], [\underline{b}_0,\overline{b}_0] \in IV((0,1])$. If $\underline{a}_0\overline{b}_0 = \overline{a}_0\underline{b}_0$, then the equation $[\underline{a}_0,\overline{a}_0]x_0 = [\underline{b}_0,\overline{b}_0]$ has a unique solution $x_0 = (\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0)$.*

Proposition 10. *Let $[\underline{a}_0,\overline{a}_0], [\underline{b}_0,\overline{b}_0] \in IV((0,1])$ and $x_0 \in [0,1]$. The inequality*

$$[\underline{a}_0,\overline{a}_0]\,x_0 \le \left[\underline{b}_0,\overline{b}_0\right] \tag{18}$$

is solvable and its solution set is either in form of

$$[0, y_0), \tag{19}$$

or in form of

$$[0, y_0], \tag{20}$$

where $y_0 = (\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0)\wedge 1$. In particular, the solution set of (18) is $[0, y_0)$ if and only if $\underline{a}_0\overline{b}_0 > \overline{a}_0\underline{b}_0$, and $\underline{a}_0+\overline{a}_0 \ge \underline{b}_0+\overline{b}_0$.

Proof. We consider the solution set of (18) in three cases according to its parameters.

Case 1. If $\underline{a}_0((\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0)) \le \underline{b}_0$, i.e., $\underline{a}_0\overline{b}_0 \le \overline{a}_0\underline{b}_0$, then

$$\begin{aligned}
[\underline{a}_0,\overline{a}_0]\,y_0 &\le [\underline{a}_0,\overline{a}_0]\,\frac{\underline{b}_0+\overline{b}_0}{\underline{a}_0+\overline{a}_0} \\
&= \left[\frac{\underline{a}_0\left(\underline{b}_0+\overline{b}_0\right)}{\underline{a}_0+\overline{a}_0}, \frac{\overline{a}_0\left(\underline{b}_0+\overline{b}_0\right)}{\underline{a}_0+\overline{a}_0}\right]
\end{aligned} \tag{21}$$

Denote $[\underline{a}_0(\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0), \overline{a}_0(\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0)] = [\underline{u},\overline{u}]$. Now we check that

$$[\underline{u},\overline{u}] \le \left[\underline{b}_0,\overline{b}_0\right] \tag{22}$$

by the following points (i) and (ii), according to Definition 3.

(i) $(1/2)(\underline{u}+\overline{u}) = (1/2)(\underline{a}_0(\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0) + \overline{a}_0(\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0)) = (1/2)(\underline{b}_0+\overline{b}_0)$.

(ii) $(1/2)(\overline{u}-\underline{u}) = (1/2)(\overline{a}_0(\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0) - \underline{a}_0(\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0)) = (1/2)((\overline{a}_0\underline{b}_0+\overline{a}_0\overline{b}_0-\underline{a}_0\underline{b}_0-\underline{a}_0\overline{b}_0)/(\underline{a}_0+\overline{a}_0))$. Since $\underline{a}_0\overline{b}_0 \le \overline{a}_0\underline{b}_0$, we have

$$\begin{aligned}
\frac{1}{2}(\overline{u}-\underline{u}) &= \frac{1}{2}\left(\frac{\overline{a}_0\underline{b}_0+\overline{a}_0\overline{b}_0-\underline{a}_0\underline{b}_0-\underline{a}_0\overline{b}_0}{\underline{a}_0+\overline{a}_0}\right) \\
&\ge \frac{1}{2}\left(\frac{\underline{a}_0\overline{b}_0+\overline{a}_0\overline{b}_0-\underline{a}_0\underline{b}_0-\overline{a}_0\underline{b}_0}{\underline{a}_0+\overline{a}_0}\right) \\
&= \frac{1}{2}\left(\overline{b}_0-\underline{b}_0\right).
\end{aligned} \tag{23}$$

Combining (21) and (22) we get $[\underline{a}_0,\overline{a}_0]y_0 \le [\underline{u},\overline{u}] \le [\underline{b}_0,\overline{b}_0]$; i.e., y_0 is a solution of (18). According to Corollary 8, $[0, y_0]$ is a subset of the solution set of (18). Next we verify that any $y > y_0$ is not a solution of (18) in two situations.

When $(\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0) \le 1$, we have $y_0 = (\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0)$. Since $y > y_0 \ge 0$ and $\underline{a}_0+\overline{a}_0 > 0$, it is clear that $(\underline{a}_0+\overline{a}_0)y > (\underline{a}_0+\overline{a}_0)y_0$. Hence the midpoint of $[\underline{a}_0,\overline{a}_0]y = [\underline{a}_0 y, \overline{a}_0 y]$ is

$$\begin{aligned}
\frac{\underline{a}_0 y+\overline{a}_0 y}{2} &= \frac{\underline{a}_0+\overline{a}_0}{2}y > \frac{\underline{a}_0+\overline{a}_0}{2}y_0 \\
&= \frac{\underline{a}_0+\overline{a}_0}{2}\,\frac{\underline{b}_0+\overline{b}_0}{\underline{a}_0+\overline{a}_0} = \frac{\underline{b}_0+\overline{b}_0}{2}.
\end{aligned} \tag{24}$$

According to Definition 2, $[\underline{a}_0,\overline{a}_0]y > [\underline{b}_0,\overline{b}_0]$; i.e., y is not a solution of (18).

When $(\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0) > 1$, we have $y_0 = 1$, then it is obvious that any $y > y_0$ is not a solution of (18), due to the assumption of the variable that $x_0 \in [0,1]$.

Consequently, in this case, the solution set of (18) is $[0, y_0]$.

Case 2. If $\underline{a}_0((\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0)) > \underline{b}_0$, i.e., $\underline{a}_0\overline{b}_0 > \overline{a}_0\underline{b}_0$, and $(\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0) > 1$, then $y_0 = (\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0)\wedge 1 = 1$ and $(1/2)(\underline{a}_0+\overline{a}_0) < (1/2)(\underline{b}_0+\overline{b}_0)$. It follows from Definition 3 that $[\underline{a}_0,\overline{a}_0] < [\underline{b}_0,\overline{b}_0]$, i.e., $[\underline{a}_0,\overline{a}_0]1 < [\underline{b}_0,\overline{b}_0]$. Considering Lemma 7 and the assumption of the variable that $x_0 \in [0,1]$, the solution set of (18) is $[0,1] = [0, y_0]$.

Case 3. If $\underline{a}_0((\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0)) > \underline{b}_0$, i.e., $\underline{a}_0\overline{b}_0 > \overline{a}_0\underline{b}_0$, and $(\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0) \le 1$, then $y_0 = (\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0) \wedge 1 = (\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0)$.

Now we compare the interval-value number $[\underline{a}_0,\overline{a}_0]x_0$ with $[\underline{b}_0,\overline{b}_0]$ in three cases as below.

(i) When $x_0 = (\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0) = y_0$, in the similar way of (i) and (ii) in case 1, the midpoint of $[\underline{a}_0,\overline{a}_0]x_0$ is exactly $(1/2)(\underline{u}+\overline{u}) = (1/2)(\underline{b}_0+\overline{b}_0)$, which is equal to that of $[\underline{b}_0,\overline{b}_0]$. On the other hand, since $\underline{a}_0\overline{b}_0 > \overline{a}_0\underline{b}_0$, the radius of $[\underline{a}_0,\overline{a}_0]x_0$ is

$$\begin{aligned}\frac{1}{2}(\overline{u}-\underline{u}) &= \frac{1}{2}\left(\frac{\overline{a}_0\underline{b}_0+\overline{a}_0\overline{b}_0-\underline{a}_0\underline{b}_0-\underline{a}_0\overline{b}_0}{\underline{a}_0+\overline{a}_0}\right)\\ &< \frac{1}{2}\left(\frac{\underline{a}_0\overline{b}_0+\overline{a}_0\overline{b}_0-\underline{a}_0\underline{b}_0-\overline{a}_0\underline{b}_0}{\underline{a}_0+\overline{a}_0}\right)\\ &= \frac{1}{2}\left(\overline{b}-\underline{b}\right),\end{aligned} \tag{25}$$

which is less than that of $[\underline{b}_0,\overline{b}_0]$. Hence it holds that $[\underline{a}_0,\overline{a}_0]x_0 = [\underline{a}_0,\overline{a}_0]y_0 > [\underline{b}_0,\overline{b}_0]$.

(ii) When $x_0 > (\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0) = y_0$, according to Lemma 7 and the result in the above point (i), it is clear that $[\underline{a}_0,\overline{a}_0]x_0 > [\underline{a}_0,\overline{a}_0]y_0 > [\underline{b}_0,\overline{b}_0]$.

(iii) When $x_0 < (\underline{b}_0+\overline{b}_0)/(\underline{a}_0+\overline{a}_0) = y_0$, we have

$$\frac{1}{2}(\underline{a}_0+\overline{a}_0)x_0 < \frac{1}{2}(\underline{a}_0+\overline{a}_0)\frac{\underline{b}_0+\overline{b}_0}{\underline{a}_0+\overline{a}_0} = \frac{1}{2}\left(\underline{b}_0+\overline{b}_0\right), \tag{26}$$

i.e., the midpoint of $[\underline{a}_0,\overline{a}_0]x_0$ is less than that of $[\underline{b}_0,\overline{b}_0]$. Hence $[\underline{a}_0,\overline{a}_0]x_0 < [\underline{b}_0,\overline{b}_0]$.

Combining the above points (i)-(iii), it is obvious that the solution set of (18) is $[0, y_0)$.

The rest of the proof is evident based on cases 1, 2, and 3. □

Proposition 11. *For the inequality $[\underline{a}_0,\overline{a}_0]x_0 \le [\underline{b}_0,\overline{b}_0]$, we have the following:*

(i) *if its solution set is $[0, y_0)$, then*

$$[\underline{a}_0,\overline{a}_0]x_0 \begin{cases} < [\underline{b}_0,\overline{b}_0], & \text{if } 0 \le x_0 < y_0,\\ > [\underline{b}_0,\overline{b}_0], & \text{if } y_0 \le x_0 \le 1;\end{cases} \tag{27}$$

(ii) *if its solution set is $[0, y_0]$, then*

$$[\underline{a}_0,\overline{a}_0]x_0 \begin{cases} \le [\underline{b}_0,\overline{b}_0], & \text{if } 0 \le x_0 \le y_0,\\ > [\underline{b}_0,\overline{b}_0], & \text{if } y_0 < x_0 \le 1.\end{cases} \tag{28}$$

In particular, if $[\underline{a}_0,\overline{a}_0]x_0 = [\underline{b}_0,\overline{b}_0]$, then the solution set should be $[0, y_0]$ and $x_0 = y_0$.

Proof. The proof lies in the process of the proof of Proposition 10. □

4.2. Basic Concepts and Properties of Max-Product IPFRE. This subsection presents some basic concept and properties of the max-product fuzzy relation equations with interval-valued numbers, i.e., system (12).

We adopt the following order relation for comparing two n-dimension vectors. For any $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in [0,1]^n$, $x \le y$ $(x < y, x = y)$ means $x_j \le y_j$ $(x_j < y_j, x_j = y_j)$ holds for all $j \in J$, while $x \lneq y$ means both $x \le y$ and $x \ne y$ hold.

Denote the complete solution set of system (12) by

$$X(A,b) = \left\{x \in [0,1]^n \mid A \odot x^T = b^T\right\}. \tag{29}$$

Definition 12. If $X(A,b) \ne \emptyset$ $(X(A,b) = \emptyset)$, system (12) is said to be consistent (inconsistent),.

Definition 13. A minimal solution $\check{x} \in X(A,b)$ is a solution satisfying $x \in X(A,b)$ and $x \le \check{x}$ indicate $x = \check{x}$; a maximum solution $\hat{x} \in X(A,b)$ is a solution satisfying $x \le \hat{x}$ for any $x \in X(A,b)$.

Theorem 14. *Let $x \in [0,1]^n$ be an n-dimension vector. Then x is a solution of system (12) if and only if for any $i \in I$, $[\underline{a}_{ij},\overline{a}_{ij}]x_j \le [\underline{b}_i,\overline{b}_i]$ holds for all $j \in J$ and there exists $j_i \in J$ such that $[\underline{a}_{ij_i},\overline{a}_{ij_i}]x_{j_i} = [\underline{b}_i,\overline{b}_i]$.*

Proof. ($\Longrightarrow$) Since x is a solution of (12), it satisfies the equation s

$$\begin{aligned}&\left[\underline{a}_{i1},\overline{a}_{i1}\right]x_1 \vee \left[\underline{a}_{i2},\overline{a}_{i2}\right]x_2 \vee \cdots \vee \left[\underline{a}_{in},\overline{a}_{in}\right]x_n\\ &\quad = \left[\underline{b}_i,\overline{b}_i\right], \quad i \in I.\end{aligned} \tag{30}$$

According to the scalar-multiplication and supremum operations (see (b) and (c) in Section 2),

$$\begin{aligned}&\left[\underline{a}_{i1},\overline{a}_{i1}\right]x_1 \vee \left[\underline{a}_{i2},\overline{a}_{i2}\right]x_2 \vee \cdots \vee \left[\underline{a}_{in},\overline{a}_{in}\right]x_n\\ &\quad = \bigvee_{j\in J}\left[\underline{a}_{ij},\overline{a}_{ij}\right]x_j = \bigvee_{j\in J}\left[\underline{a}_{ij}x_j,\overline{a}_{ij}x_j\right],\end{aligned} \tag{31}$$

Hence

$$\left[\underline{a}_{ij}x_j,\overline{a}_{ij}x_j\right] \le \bigvee_{j\in J}\left[\underline{a}_{ij}x_j,\overline{a}_{ij}x_j\right] = \left[\underline{b}_i,\overline{b}_i\right] \tag{32}$$

for any $j \in J$ and there exists some $j_i \in J$ such that

$$\left[\underline{a}_{ij_i}x_{j_i},\overline{a}_{ij_i}x_{j_i}\right] = \bigvee_{j\in J}\left[\underline{a}_{ij}x_j,\overline{a}_{ij}x_j\right] = \left[\underline{b}_i,\overline{b}_i\right], \tag{33}$$

i.e., $[\underline{a}_{ij},\overline{a}_{ij}]x_j \le [\underline{b}_i,\overline{b}_i]$ and $[\underline{a}_{ij_i},\overline{a}_{ij_i}]x_{j_i} = [\underline{b}_i,\overline{b}_i]$.

($\Longleftarrow$) Analogously, it is easy to check the feasibility of x as a solution of (12), according to the scalar-multiplication and supremum operations on IV. □

According to Proposition 10, the single-variable inequality

$$[\underline{a}_{ij}, \overline{a}_{ij}]\, x_j \le [\underline{b}_i, \overline{b}_i] \tag{34}$$

could be solved, $i \in I$, $j \in J$. Assume that the solution set of (34) is

$$[0, y_{ij}] \text{ or } [0, y_{ij}), \tag{35}$$

$i \in I$, $j \in J$. We define the vector $\hat{x}$ based on the above solution sets (35) as follows:

$$\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) = \left(\bigwedge_{i\in I} y_{i1}, \bigwedge_{i\in I} y_{i2}, \ldots, \bigwedge_{i\in I} y_{in}\right). \tag{36}$$

Theorem 15. *Suppose $X(A, b) \neq \emptyset$. Then for any $z \in X(A, b)$, it holds that $z \le \hat{x}$.*

Proof. Since $z = (z_1, z_2, \ldots, z_n,) \in X(A, b)$, according to Theorem 14 we have

$$[\underline{a}_{ij}, \overline{a}_{ij}]\, z_j \le [\underline{b}_i, \overline{b}_i], \quad \forall i \in I,\ j \in J. \tag{37}$$

That is to say, z_j is a solution of the equation $[\underline{a}_{ij}, \overline{a}_{ij}]x_j \le [\underline{b}_i, \overline{b}_i]$. Notice that the solution set of $[\underline{a}_{ij}, \overline{a}_{ij}]x_j \le [\underline{b}_i, \overline{b}_i]$ is either $[0, y_{ij}]$ or $[0, y_{ij})$. Hence $z_j \le y_{ij}$ for all $i \in I$, $j \in J$. So we further get

$$z_j \le \bigwedge_{i\in I} y_{ij} = \hat{x}_j, \quad j \in J, \tag{38}$$

i.e., $z \le \hat{x}$. □

As shown in Theorem 15, there does not exist any solution bigger that $\hat{x}$. We call $\hat{x}$ the *potential maximum solution* of system (12). Moreover, $\hat{x}$ is called *maximum solution* if $\hat{x} \in X(A, b)$, while called *pseudomaximum solution* if $\hat{x} \notin X(A, b)$.

Theorem 16. *Let $z^1, z^2 \in [0, 1]$ and $z^1 \le z \le z^2$. If $z^1, z^2 \in X(A, b)$, then $z \in X(A, b)$.*

Proof. According to Theorem 14, $z^2 \in X(A, b)$ indicates

$$[\underline{a}_{ij}, \overline{a}_{ij}]\, z_j^2 \le [\underline{b}_i, \overline{b}_i], \quad \forall i \in I. \tag{39}$$

Besides, by Lemma 7, $z \le z^2$ implies that

$$[\underline{a}_{ij}, \overline{a}_{ij}]\, z_j \le [\underline{a}_{ij}, \overline{a}_{ij}]\, z_j^2 \le [\underline{b}_i, \overline{b}_i], \quad \forall i \in I. \tag{40}$$

On the other hand, again by Theorem 14, $z^1 \in X(A, b)$ indicates, for any $i \in I$, there exists $j_i \in J$ such that $[\underline{a}_{ij_i}, \overline{a}_{ij_i}]z_{j_i}^1 = [\underline{b}_i, \overline{b}_i]$. It follows from Lemma 7 and the inequality $z^1 \le z$ that

$$[\underline{a}_{ij_i}, \overline{a}_{ij_i}]\, z_{j_i} \ge [\underline{a}_{ij_i}, \overline{a}_{ij_i}]\, z_{j_i}^1 = [\underline{b}_i, \overline{b}_i]. \tag{41}$$

Considering inequality (40) we have $[\underline{a}_{ij_i}, \overline{a}_{ij_i}]z_{j_i} \le [\underline{b}_i, \overline{b}_i]$. Thus it holds that $[\underline{a}_{ij_i}, \overline{a}_{ij_i}]z_{j_i} = [\underline{b}_i, \overline{b}_i]$. Combining (40), it follows from Theorem 14 that $z \in X(A, b)$. □

Next we define some new concepts and consider the consistency of system (12).

Define the *close index* set

$$J^{close} = \left\{ j \in J \mid [\underline{a}_{ij}, \overline{a}_{ij}]\, \hat{x}_j \le [\underline{b}_i, \overline{b}_i] \text{ for all } i \in I \right\}, \tag{42}$$

and the *open index* set

$$J^{open} = \left\{ j \in J \mid [\underline{a}_{ij}, \overline{a}_{ij}]\, \hat{x}_j > [\underline{b}_i, \overline{b}_i] \text{ for some } i \in I \right\}. \tag{43}$$

It is obvious that

$$\begin{aligned} J^{close} \cap J^{open} &= \emptyset, \\ J^{close} \cup J^{open} &= J. \end{aligned} \tag{44}$$

Proposition 17. *In system (12), if $J^{open} \neq \emptyset$, then $\hat{x} \notin X(A, b)$.*

Proof. If $J^{open} \neq \emptyset$, then there exists $j \in J$ and $i \in I$ such that $[\underline{a}_{ij}, \overline{a}_{ij}]\hat{x}_j > [\underline{b}_i, \overline{b}_i]$ by (43). According to Theorem 14, $\hat{x}$ is not a solution of system (12). □

The following Corollary 18 could be easily obtained by Theorem 15 and Proposition 17.

Corollary 18. *The vector $\hat{x}$ is the unique maximum solution of system (12) if and only if $J^{open} = \emptyset$ and $\hat{x} \in X(A, b)$.*

Based on the potential maximum solution $\hat{x}$ and the open index set, we define the vector

$$\underline{\hat{x}} = (\underline{\hat{x}}_1, \underline{\hat{x}}_2, \ldots, \underline{\hat{x}}_n), \tag{45}$$

where

$$\underline{\hat{x}}_j = \begin{cases} 0, & \text{if } j \in J^{open}; \\ \hat{x}_j, & \text{if } j \notin J^{open}; \end{cases} \tag{46}$$

The vector $\underline{\hat{x}}$ could be used to check the consistency of system (12) by the following Theorem 19. We call $\underline{\hat{x}}$ the maximum close solution, if it is a solution of system (12).

Theorem 19. *System (12) is consistent, i.e., $X(A, b) \neq \emptyset$, if and only if $\underline{\hat{x}} \in X(A, b)$.*

Proof. ($\Longleftarrow$) It is obvious.

($\Longrightarrow$) According to Theorem 14, we complete the proof from two aspects as follows:

(i) Take arbitrary $i \in I$, $j \in J$. If $j \in J^{open}$, then $[\underline{a}_{ij}, \overline{a}_{ij}]\underline{\hat{x}}_j = [\underline{a}_{ij}, \overline{a}_{ij}]0 = [0, 0] \le [\underline{b}_i, \overline{b}_i]$. While if $j \notin J^{open}$, then $j \in J^{close}$, and $[\underline{a}_{ij}, \overline{a}_{ij}]\underline{\hat{x}}_j = [\underline{a}_{ij}, \overline{a}_{ij}]\hat{x}_j \le [\underline{b}_i, \overline{b}_i]$.

(ii) Notice that system (12) is consistent. There exists a solution $z \in X(A, b)$ to system (12). By Theorem 14, for arbitrary $i \in I$, there exists some $j_i \in J$, such that $[\underline{a}_{ij_i}, \overline{a}_{ij_i}]z_{j_i} = [\underline{b}_i, \overline{b}_i]$.

Firstly, we verify that $z_{j_i} = \hat{x}_{j_i}$. Since $[\underline{a}_{ij_i}, \overline{a}_{ij_i}]z_{j_i} = [\underline{b}_i, \overline{b}_i]$, it follows from Proposition 11 that the solution of the inequality $[\underline{a}_{ij_i}, \overline{a}_{ij_i}]x_{j_i} = [\underline{b}_i, \overline{b}_i]$ is $[0, y_{ij_i}]$ and $z_{j_i} = y_{ij_i}$. So we get

$$z_{j_i} = y_{ij_i} \geq \bigwedge_{k=1}^{m} y_{kj_i} = \widehat{x}_{j_i}. \tag{47}$$

On the other hand, it follows from Theorem 15 that $z \leq \widehat{x}$, which indicates $z_{j_i} \leq \widehat{x}_{j_i}$. Thus it holds that $z_{j_i} = \widehat{x}_{j_i}$.

Secondly, we prove $j_i \notin J^{open}$ by contradiction. Assume that $j_i \in J^{open}$. Then there exists some $k \in I$ such that $[\underline{a}_{kj_i}, \overline{a}_{kj_i}]\widehat{x}_{j_i} > [\underline{b}_k, \overline{b}_k]$. Considering $z_{j_i} = \widehat{x}_{j_i}$, we have

$$\begin{aligned} \bigvee_{j=1}^{n} \left[\underline{a}_{kj}, \overline{a}_{kj}\right] z_j &\geq \left[\underline{a}_{kj_i}, \overline{a}_{kj_i}\right] z_{j_i} = \left[\underline{a}_{kj_i}, \overline{a}_{kj_i}\right] \widehat{x}_{j_i} \\ &> \left[\underline{b}_k, \overline{b}_k\right]. \end{aligned} \tag{48}$$

Inequality (48) shows that the vector z does not satisfy the kth equation in system (12). Hence z is not a solution of (12), which is conflict with the assumption that $z \in X(A, b)$.

At last, since $j_i \notin J^{open}$, it follows from (46) that $\underline{\widehat{x}}_{j_i} = \widehat{x}_{j_i}$. So we get

$$\left[\underline{a}_{ij_i}, \overline{a}_{ij_i}\right] \underline{\widehat{x}}_{j_i} = \left[\underline{a}_{ij_i}, \overline{a}_{ij_i}\right] \widehat{x}_{j_i} = \left[\underline{a}_{ij_i}, \overline{a}_{ij_i}\right] z_{j_i} = \left[\underline{b}_i, \overline{b}_i\right] \tag{49}$$

The above-proved points (i) and (ii) contribute to $\underline{\widehat{x}} \in X(A, b)$ by Theorem 14. □

Note. In general, $\underline{\widehat{x}}$ is not the maximum solution even if system (12) is consistent. Moreover, it is possible that system (12) has no maximum solution.

4.3. Structure of the Solution Set of System (12). Let $P = J_1 \times J_2 \times \cdots \times J_m$, where

$$J_i = \left\{j \in J \mid j \in J^{close} \text{ and } \left[\underline{a}_{ij}, \overline{a}_{ij}\right] \widehat{x}_j = \left[\underline{b}_i, \overline{b}_i\right]\right\}, \quad i \in I. \tag{50}$$

Each $p \in P$ is called a path of the characteristic matrix C (or system (12)). Thus P is the set of all paths.

Let $p \in P$ be a path. Denote the index sets

$$I_j^p = \{i \in I \mid j_i = j\}, \quad j \in J. \tag{51}$$

Based on the path P and its corresponding index sets $I_1^p, I_2^p, \ldots, I_n^p$, we can define vector

$$\check{x}^p = \left(\check{x}_1^p, \check{x}_2^p, \ldots, \check{x}_n^p\right), \tag{52}$$

where

$$\check{x}_j^p = \begin{cases} 0, & \text{if } I_j^p = \emptyset; \\ \widehat{x}_j, & \text{otherwise}. \end{cases} \tag{53}$$

Theorem 20. *Let $p \in P$ be a path and $\check{x}^p$ be the vector defined by (52) and (53). Then $\check{x}^p$ is a solution of system (12).*

Proof. (i) Take arbitrary $i \in I$ and $j \in J$.

If $I_j^p = \emptyset$, then $\check{x}_j^p = 0$. Hence $[\underline{a}_{ij}, \overline{a}_{ij}]\check{x}_j^p = [\underline{a}_{ij}, \overline{a}_{ij}]0 = [0, 0] \leq [\underline{b}_i, \overline{b}_i]$.

If $I_j^p \neq \emptyset$, then $\check{x}_j^p = \widehat{x}_j$. By (51), there exists $i \in I$ such that $j_i = j$. Remind that $p = (j_1, j_2, \ldots, j_m) \in J_1 \times J_2 \times \cdots \times J_m$. It is clear $j_i \in J_i$, which indicates $j_i \in J^{close}$, i.e., $j \in J^{close}$. According to the definition of J^{close} (see (42)), we have $[\underline{a}_{ij}, \overline{a}_{ij}]\check{x}_j^p = [\underline{a}_{ij}, \overline{a}_{ij}]\widehat{x}_j \leq [\underline{b}_i, \overline{b}_i]$.

(ii) Take arbitrary $i \in I$. Notice $p = (j_1, j_2, \ldots, j_m)$. We denote $j_i = j'$. It is clear that $i \in I_{j'}^p$, which indicates $I_{j'}^p \neq \emptyset$. Hence

$$\begin{aligned} \check{x}_{j'}^p &= \widehat{x}_{j'}, \\ \text{i.e., } \check{x}_{j_i}^p &= \widehat{x}_{j_i}. \end{aligned} \tag{54}$$

On the other hand, $j_i \in J_i$ implies

$$\left[\underline{a}_{ij_i}, \overline{a}_{ij_i}\right] \widehat{x}_{j_i} = \left[\underline{b}_i, \overline{b}_i\right]. \tag{55}$$

Equations (54) and (55) contribute to $[\underline{a}_{ij_i}, \overline{a}_{ij_i}]\check{x}_{j_i}^p = [\underline{b}_i, \overline{b}_i]$.

Based on (i) and (ii), it follows from Theorem 14 that $\check{x}^p \in X(A, b)$. □

Theorem 20 shows that, for any path $p \in P$ of system (12), $\check{x}^p$ is a solution corresponding to p. Next we further construct the solution interval corresponding to p. Based on the path $p \in P$, define the vector $X^p = (X_1^p, X_2^p, \ldots, X_n^p)$, where

$$X_j^p = \begin{cases} \left[\check{x}_j^p, \widehat{x}_j\right], & \text{if } j \in J^{close}; \\ \left[\check{x}_j^p, \widehat{x}_j\right), & \text{if } j \in J^{open}. \end{cases} \tag{56}$$

We call X^p the *solution interval* corresponding to the path p.

Remark 21. It could be easily found from (56) that

(i) X^p is a close set (or close interval) if and only if $J^{open} = \emptyset$;

(ii) X^p is an open set (or open interval) if and only if $J^{close} = \emptyset$.

Theorem 22. *Let $p \in P$ be a path and X^p be defined by (56) based on p. Then it holds that $X^p \subseteq X(A, b)$.*

Proof. Take arbitrary $y = (y_1, y_2, \ldots, y_n) \in X^p$. Then it follows from (56) that

$$\begin{aligned} j \in J^{close} &\Longrightarrow \\ \check{x}_j^p &\leq y_j \leq \widehat{x}_j, \end{aligned} \tag{57}$$

$$\begin{aligned} j \in J^{open} &\Longrightarrow \\ \check{x}_j^p &\leq y_j < \widehat{x}_j, \end{aligned} \tag{58}$$

(i) We first prove it holds for all $i \in I$ and $j \in J$ that $[\underline{a}_{ij}, \overline{a}_{ij}]y_j \leq [\underline{b}_i, \overline{b}_i]$ in tow cases.

Case 1. If $j \in J^{close}$, then $[\underline{a}_{ij}, \overline{a}_{ij}]y_j \leq [\underline{a}_{ij}, \overline{a}_{ij}]\hat{x}_j \leq [\underline{b}_i, \overline{b}_i]$, $\forall i \in I$.

Case 2. If $j \in J^{open}$, then the index set $I_j^{>} = \{i \in I | [\underline{a}_{ij}, \overline{a}_{ij}]\hat{x}_j > [\underline{b}_i, \overline{b}_i]\} \neq \emptyset$. Obviously, when $i \neq I_j^{>}$, it holds that $[\underline{a}_{ij}, \overline{a}_{ij}]\hat{x}_j \leq [\underline{b}_i, \overline{b}_i]$. Combining (58) and Lemma 7, we have

$$\left[\underline{a}_{ij}, \overline{a}_{ij}\right] y_j < \left[\underline{a}_{ij}, \overline{a}_{ij}\right] \hat{x}_j \leq \left[\underline{b}_i, \overline{b}_i\right]. \tag{59}$$

On the other hand, when $i \in I_j^{>}$, we have

$$\left[\underline{a}_{ij}, \overline{a}_{ij}\right] \hat{x}_j > \left[\underline{b}_i, \overline{b}_i\right]. \tag{60}$$

Suppose the solution set of the inequality $[\underline{a}_{ij}, \overline{a}_{ij}]x_j > [\underline{b}_i, \overline{b}_i]$ is $[0, y_{ij})$ or $[0, y_{ij}]$. Then

$$\hat{x}_j = \bigwedge_{k \in I} y_{kj} \leq y_{ij}. \tag{61}$$

Thus

$$\left[\underline{a}_{ij}, \overline{a}_{ij}\right] y_{ij} \geq \left[\underline{a}_{ij}, \overline{a}_{ij}\right] \hat{x}_j > \left[\underline{b}_i, \overline{b}_i\right]. \tag{62}$$

According to Proposition 11, the solution set of $[\underline{a}_{ij}, \overline{a}_{ij}]x_j > [\underline{b}_i, \overline{b}_i]$ should be $[0, y_{ij})$. It follows from (58) and (61) that $y_j < y_{ij}$. Again by Proposition 11 we get $[\underline{a}_{ij}, \overline{a}_{ij}]y_j < [\underline{b}_i, \overline{b}_i]$.

(ii) For any $i \in I$, there exists $j_i \in J_i \subseteq J$ (due to the existence of $p = (j_1, j_2, \ldots, j_m) \in J_1 \times J_2 \times \cdots \times J_m$), such that $[\underline{a}_{ij_i}, \overline{a}_{ij_i}]\check{x}_{j_i}^p = [\underline{b}_i, \overline{b}_i]$. By (57) and (58),

$$\left[\underline{a}_{ij_i}, \overline{a}_{ij_i}\right] y_{j_i} \geq \check{x}_{j_i}^p = \left[\underline{b}_i, \overline{b}_i\right]. \tag{63}$$

On the other hand, as a result of (i), $[\underline{a}_{ij_i}, \overline{a}_{ij_i}]y_{j_i} \leq [\underline{b}_i, \overline{b}_i]$ holds. These two aspects lead to $[\underline{a}_{ij_i}, \overline{a}_{ij_i}]y_{j_i} = [\underline{b}_i, \overline{b}_i]$.

Combining (i) and (ii), it follows from Theorem 14 that $y \in X(A, b)$. Due to the arbitrariness of y, the inclusion relation $X^p \subseteq X(A, b)$ holds. □

Theorem 23. *For any $y \in X(A, b)$, there exists a path $p \in P$ such that $y \in X^p$.*

Proof. Since $y \in X(A, b)$, it follows from Theorem 14 that

$$\left[\underline{a}_{ij}, \overline{a}_{ij}\right] y_j \leq \left[\underline{b}_i, \overline{b}_i\right], \tag{64}$$

and for any $i \in I$, there exists $j_i \in J$ such that

$$\left[\underline{a}_{ij_i}, \overline{a}_{ij_i}\right] y_{j_i} = \left[\underline{b}_i, \overline{b}_i\right]. \tag{65}$$

Let $p = (j_1, j_2, \ldots, j_m)$.

(i) We first verify that $p \in P$, i.e., $j_i \in J_i$, $\forall i \in I$.

Notice that $[\underline{a}_{ij_i}, \overline{a}_{ij_i}]y_{j_i} = [\underline{b}_i, \overline{b}_i]$. Considering the inequality $[\underline{a}_{ij_i}, \overline{a}_{ij_i}]x_{j_i} = [\underline{b}_i, \overline{b}_i]$, it follows from Proposition 11 that its solution set should be $[0, y_{ij_i}]$, and moreover,

$$y_{j_i} = y_{ij_i}. \tag{66}$$

Thus

$$\hat{x}_{j_i} = \bigwedge_{k \in I} y_{kj_i} \leq y_{ij_i} = y_{j_i}. \tag{67}$$

On the other hand, Theorem 15 indicates $y_{j_i} \leq \hat{x}_{j_i}$. So we get

$$\hat{x}_{j_i} = y_{j_i}. \tag{68}$$

As a result of (65) and (68),

$$\left[\underline{a}_{ij_i}, \overline{a}_{ij_i}\right] \hat{x}_{j_i} = \left[\underline{a}_{ij_i}, \overline{a}_{ij_i}\right] y_{j_i} = \left[\underline{b}_i, \overline{b}_i\right]. \tag{69}$$

Combining (64) and (68), we get

$$\left[\underline{a}_{kj_i}, \overline{a}_{kj_i}\right] \hat{x}_{j_i} = \left[\underline{a}_{kj_i}, \overline{a}_{kj_i}\right] y_{j_i} \leq \left[\underline{b}_k, \overline{b}_k\right], \quad \forall k \in I, \tag{70}$$

which indicates

$$j \in J^{close}. \tag{71}$$

(69) and (71) contribute to $j_i \in J_i$.

(ii) We now check $\check{x}^p \leq y$.

Take arbitrary $j \in J$. If $I_j^p \neq \emptyset$, then $\check{x}_j^p = \hat{x}_j$. Besides, since $I_j^p \neq \emptyset$, there exists $i \in I_j^p \subseteq I$ such that $j_i = j$. By (68) we have $\hat{x}_j = \hat{x}_{j_i} = y_{j_i} = y_j$. Hence

$$\check{x}_j^p = \hat{x}_j = y_j. \tag{72}$$

If $I_j^p = \emptyset$, then $\check{x}_j^p = 0 \leq y_j$. Consequently, it holds that $\check{x}_j^p \leq y_j$ for all $j \in J$, i.e., $\check{x}^p \leq y$.

(iii) According to Theorem 15,

$$y_j \leq \hat{x}_j, \quad \forall j \in J. \tag{73}$$

Obviously $y_j \leq \hat{x}_j$ holds for any $j \in J^{close}$. While for $j \in J^{open}$, we have to verify that $y_j \leq \hat{x}_j$ (by contradiction). Otherwise, due to (73), it turns out to be $y_j = \hat{x}_j$. By (43), $j \in J^{open}$ indicates there exists $i \in I$ such that $[\underline{a}_{ij}, \overline{a}_{ij}]\hat{x}_j > [\underline{b}_i, \overline{b}_i]$. So we get

$$\begin{aligned} &\left[\underline{a}_{i1}, \overline{a}_{i1}\right] y_1 \vee \left[\underline{a}_{i2}, \overline{a}_{i2}\right] y_2 \vee \cdots \vee \left[\underline{a}_{in}, \overline{a}_{in}\right] y_n \\ &\quad \geq \left[\underline{a}_{ij}, \overline{a}_{ij}\right] y_j = \left[\underline{a}_{ij}, \overline{a}_{ij}\right] \hat{x}_j > \left[\underline{b}_i, \overline{b}_i\right]. \end{aligned} \tag{74}$$

Thus y does not satisfy the jth equation in system (12). This is conflict with the assumption that $y \in X(A, b)$.

The above points (i)-(iii) contribute to $y \in X^p$ and the proof is complete. □

Theorem 24 (structure of the solution set). *Suppose system (12) is consistent. Then its complete solution set is*

$$X(A, b) = \bigcup_{p \in P} X^p, \tag{75}$$

where $P = J_1 \times J_2 \times \cdots \times J_m$ is the set of all paths and X^p is the solution interval corresponding to the path p.

Proof. It is simple corollary of Theorems 22 and 23. □

4.4. Algorithm for Solving System (12) Based on the Index Sets. Based on the above-defined index sets J^{close}, J^{open}, and J_i, $i \in I$, we propose the following algorithm for obtaining the solution set of system (12).

Algorithm for Solving System (12)

Step 1. Solve the inequality $[\underline{a}_{ij}, \overline{a}_{ij}]x_j \le [\underline{b}_i, \overline{b}_i]$, for any $i \in I$ and $j \in J$, by Proposition 10. Suppose the solution set is $S_{ij} = [0, y_{ij})$ or $S_{ij} = [0, y_{ij}]$, where $i \in I$ and $j \in J$.

Step 2. Compute the potential maximum solution $\widehat{x}$ by (36).

Step 3. Obtain the close index set J^{close} and open index set J^{open} by (42) and (43), respectively.

Step 4. Compute the vector $\underline{\widehat{x}}$ by (45) and (46).

Step 5. Check the consistency of system (12) by the vector $\underline{\widehat{x}}$ according to Theorems 14 and 19. If $\underline{\widehat{x}} \in X(A, b)$, then system (12) is consistent and continue to Step 6. Otherwise, system (12) is inconsistent and stop.

Step 6. Compute the index sets $J_1, J_2, \dots, J_m$ by (50). Let $P = J_1 \times J_2 \times \cdots \times J_m$.

Step 7. For any $p \in P$, compute the index sets $I_1^p, I_2^p, \dots, I_n^p$ by (51).

Step 8. Compute the corresponding solution $\check{x}^p$ by (52) and (53).

Step 9. Construct the solution interval X^p by (56).

Step 10. According to Theorem 24, generate the complete solution set of system (12) by

$$X(A, b) = \bigcup_{p \in P} X^p. \tag{76}$$

5. Numerical Example

Example 1. Consider the following interval-valued-parameter max-product fuzzy relation equations:

$$A \odot x^T = b^T, \tag{77}$$

where

$$\begin{gathered} A = \begin{bmatrix} [0.2, 0.3] & [0.5, 0.7] & [0, 0.5] & [0.3, 0.45] & [0.1, 0.3] & [0.3, 0.5] & [0.5, 1.0] & [0.25, 0.5] \\ [0.6, 0.8] & [0.3, 0.5] & [0.2, 0.4] & [0.4, 0.5] & [0.2, 0.5] & [0.3, 0.6] & [0.75, 1.0] & [0.4, 0.6] \\ [0.6, 0.7] & [0.5, 0.6] & [0.3, 0.5] & [0.4, 0.6] & [0.3, 0.4] & [0.6, 0.8] & [0.5, 0.7] & [0.6, 0.8] \\ [0.7, 0.98] & [0.7, 0.98] & [0.28, 0.45] & [0.5, 0.7] & [0.36, 0.42] & [0.48, 0.72] & [0.5, 0.9] & [0.45, 0.75] \\ [0.4, 0.6] & [0.7, 0.8] & [0.3, 0.4] & [0.5, 0.7] & [0.4, 0.5] & [0.6, 0.7] & [0.85, 0.95] & [0.5, 0.7] \end{bmatrix}, \\ b = ([0.2, 0.4], [0.3, 0.4], [0.36, 0.48], [0.35, 0.49], [0.4, 0.5]), \\ x = (x_1, x_2, \dots, x_8) \in [0, 1]^8. \end{gathered} \tag{78}$$

Solution

Step 1. Solve the inequality $[\underline{a}_{ij}, \overline{a}_{ij}]x_j \le [\underline{b}_i, \overline{b}_i]$ for $i = 1, 2, \dots, 5$, $j = 1, 2, \dots, 8$ by Proposition 10. Denote the corresponding solution set by S_{ij}. After calculation we get all the solution sets as follows:

$$(S_{ij}) = \begin{bmatrix} [0,1] & \left[0, \frac{1}{2}\right) & [0,1] & \left[0, \frac{4}{5}\right) & [0,1] & \left[0, \frac{3}{4}\right) & \left[0, \frac{2}{5}\right] & \left[0, \frac{4}{5}\right] \\ \left[0, \frac{1}{2}\right] & \left[0, \frac{7}{8}\right] & [0,1] & \left[0, \frac{7}{9}\right) & [0,1] & \left[0, \frac{7}{9}\right] & \left[0, \frac{2}{5}\right] & \left[0, \frac{7}{10}\right] \\ \left[0, \frac{42}{65}\right) & \left[0, \frac{42}{55}\right) & [0,1] & \left[0, \frac{21}{25}\right] & [0,1] & \left[0, \frac{3}{5}\right] & \left[0, \frac{7}{10}\right] & \left[0, \frac{3}{5}\right] \\ \left[0, \frac{1}{2}\right] & \left[0, \frac{1}{2}\right] & [0,1] & \left[0, \frac{7}{10}\right] & [0,1] & \left[0, \frac{7}{10}\right] & \left[0, \frac{3}{5}\right] & \left[0, \frac{7}{10}\right] \\ \left[0, \frac{9}{10}\right] & \left[0, \frac{3}{5}\right) & [0,1] & \left[0, \frac{3}{4}\right] & [0,1] & \left[0, \frac{9}{13}\right) & \left[0, \frac{1}{2}\right) & \left[0, \frac{3}{4}\right] \end{bmatrix}. \tag{79}$$

Step 2. After calculation by (36), the potential maximum solution of system (77) is

$$\hat{x} = (0.5, 0.5, 1, 0.7, 1, 0.6, 0.4, 0.6)\,. \tag{80}$$

Step 3. According to (42) and (43), we find the close index set and open index set as

$$\begin{aligned} J^{close} &= \{1, 3, 4, 5, 6, 7, 8\}\,, \\ J^{open} &= \{2\}\,, \end{aligned} \tag{81}$$

respectively.

Step 4. Computing the vector $\underline{\hat{x}}$ by (45) and (46), we get

$$\underline{\hat{x}} = (0.5, 0, 1, 0.7, 1, 0.6, 0.4, 0.6)\,. \tag{82}$$

Step 5. Check the consistency of system (77) by the vector $\underline{\hat{x}}$ according to Theorems 14 and 19. After detailed calculation, it is not difficult to check that $A \odot \underline{\hat{x}}^T = b^T$, i.e., $\underline{\hat{x}}$ is a solution of (77). Hence system (77) is consistent and we could continue to Step 6.

Step 6. Computing the index sets $J_1, J_2, \ldots, J_5$ by (50), we get

$$\begin{aligned} J_1 &= \{7\}\,; \\ J_2 &= \{1, 7\}\,; \\ J_3 &= \{6, 8\}\,; \\ J_4 &= \{1, 4\}\,; \\ J_5 &= \{5\}\,. \end{aligned} \tag{83}$$

Let $P = J_1 \times J_2 \times \cdots \times J_5$. Obviously the set P contains 8 elements, i.e., there exist 8 paths of system (77).

Step 7. For any $p \in P$, compute the index sets $I_1^p, I_2^p, \ldots, I_8^p$ by (51).

For $p_1 = (7, 1, 6, 1, 5)$, the corresponding index sets are

$$\begin{aligned} I_1^{p_1} &= \{2, 4\}\,; \\ I_2^{p_1} &= \emptyset; \\ I_3^{p_1} &= \emptyset; \\ I_4^{p_1} &= \emptyset; \\ I_5^{p_1} &= \{5\}\,; \\ I_6^{p_1} &= \{3\}\,; \\ I_7^{p_1} &= \{1\}\,; \\ I_8^{p_1} &= \emptyset. \end{aligned} \tag{84}$$

For $p_2 = (7, 1, 6, 4, 5)$, the corresponding index sets are

$$\begin{aligned} I_1^{p_2} &= \{2\}\,; \\ I_2^{p_2} &= \emptyset; \\ I_3^{p_2} &= \emptyset; \\ I_4^{p_2} &= \{4\}\,; \\ I_5^{p_2} &= \{5\}\,; \\ I_6^{p_2} &= \{3\}\,; \\ I_7^{p_2} &= \{1\}\,; \\ I_8^{p_2} &= \emptyset. \end{aligned} \tag{85}$$

For $p_3 = (7, 1, 8, 1, 5)$, the corresponding index sets are

$$\begin{aligned} I_1^{p_3} &= \{2, 4\}\,; \\ I_2^{p_3} &= \emptyset; \\ I_3^{p_3} &= \emptyset; \\ I_4^{p_3} &= \emptyset; \\ I_5^{p_3} &= \{5\}\,; \\ I_6^{p_3} &= \emptyset; \\ I_7^{p_3} &= \{1\}\,; \\ I_8^{p_3} &= \{3\}\,. \end{aligned} \tag{86}$$

For $p_4 = (7, 1, 8, 4, 5)$, the corresponding index sets are

$$\begin{aligned} I_1^{p_4} &= \{2\}\,; \\ I_2^{p_4} &= \emptyset; \\ I_3^{p_4} &= \emptyset; \\ I_4^{p_4} &= \{4\}\,; \\ I_5^{p_4} &= \{5\}\,; \end{aligned}$$

$$\begin{aligned} I_6^{p_4} &= \emptyset; \\ I_7^{p_4} &= \{1\}; \\ I_8^{p_4} &= \{3\}. \end{aligned} \tag{87}$$

For $p_5 = (7, 7, 6, 1, 5)$, the corresponding index sets are

$$\begin{aligned} I_1^{p_5} &= \{4\}; \\ I_2^{p_5} &= \emptyset; \\ I_3^{p_5} &= \emptyset; \\ I_4^{p_5} &= \emptyset; \\ I_5^{p_5} &= \{5\}; \\ I_6^{p_5} &= \{3\}; \\ I_7^{p_5} &= \{1, 2\}; \\ I_8^{p_5} &= \emptyset. \end{aligned} \tag{88}$$

For $p_6 = (7, 7, 6, 4, 5)$, the corresponding index sets are

$$\begin{aligned} I_1^{p_6} &= \emptyset; \\ I_2^{p_6} &= \emptyset; \\ I_3^{p_6} &= \emptyset; \\ I_4^{p_6} &= \{4\}; \\ I_5^{p_6} &= \{5\}; \\ I_6^{p_6} &= \{3\}; \\ I_7^{p_6} &= \{1, 2\}; \\ I_8^{p_6} &= \emptyset. \end{aligned} \tag{89}$$

For $p_7 = (7, 7, 8, 1, 5)$, the corresponding index sets are

$$\begin{aligned} I_1^{p_7} &= \{4\}; \\ I_2^{p_7} &= \emptyset; \\ I_3^{p_7} &= \emptyset; \\ I_4^{p_7} &= \emptyset; \\ I_5^{p_7} &= \{5\}; \\ I_6^{p_7} &= \emptyset; \\ I_7^{p_7} &= \{1, 2\}; \\ I_8^{p_7} &= \{3\}. \end{aligned} \tag{90}$$

For $p_8 = (7, 7, 8, 4, 5)$, the corresponding index sets are

$$\begin{aligned} I_1^{p_8} &= \emptyset; \\ I_2^{p_8} &= \emptyset; \\ I_3^{p_8} &= \emptyset; \\ I_4^{p_8} &= \{4\}; \\ I_5^{p_8} &= \{5\}; \\ I_6^{p_8} &= \emptyset; \\ I_7^{p_8} &= \{1, 2\}; \\ I_8^{p_8} &= \{3\}. \end{aligned} \tag{91}$$

Step 8. Compute the corresponding solution $\check{x}^p$ by (52) and (53).

$$\begin{aligned} \check{x}^{p_1} &= (0.5, 0, 0, 0, 1, 0.6, 0.4, 0), \\ \check{x}^{p_2} &= (0.5, 0, 0, 0.7, 1, 0.6, 0.4, 0), \\ \check{x}^{p_3} &= (0.5, 0, 0, 0, 1, 0, 0.4, 0.6), \\ \check{x}^{p_4} &= (0.5, 0, 0, 0.7, 1, 0, 0.4, 0.6), \\ \check{x}^{p_5} &= (0.5, 0, 0, 0, 1, 0.6, 0.4, 0), \\ \check{x}^{p_6} &= (0, 0, 0, 0.7, 1, 0.6, 0.4, 0), \\ \check{x}^{p_7} &= (0.5, 0, 0, 0, 1, 0, 0.4, 0.6), \\ \check{x}^{p_8} &= (0, 0, 0, 0.7, 1, 0, 0.4, 0.6). \end{aligned} \tag{92}$$

Step 9. Construct the solution interval X^p by (56).

$$\begin{aligned} &X^{p_1} \\ &\quad = (0.5, [0, 0.5), [0, 1], [0, 0.7], 1, 0.6, 0.4, [0, 0.6]), \\ &X^{p_2} = (0.5, [0, 0.5), [0, 1], 0.7, 1, 0.6, 0.4, [0, 0.6]), \\ &X^{p_3} \\ &\quad = (0.5, [0, 0.5), [0, 1], [0, 0.7], 1, [0, 0.6], 0.4, 0.6), \\ &X^{p_4} = (0.5, [0, 0.5), [0, 1], 0.7, 1, [0, 0.6], 0.4, 0.6), \\ &X^{p_5} \\ &\quad = (0.5, [0, 0.5), [0, 1], [0, 0.7], 1, 0.6, 0.4, [0, 0.6]), \end{aligned}$$

$$\begin{aligned}&X^{p_6}\\&\quad=([0,0.5],[0,0.5),[0,1],0.7,1,0.6,0.4,[0,0.6]),\\&X^{p_7}\\&\quad=(0.5,[0,0.5),[0,1],[0,0.7],1,[0,0.6],0.4,0.6),\\&X^{p_8}\\&\quad=([0,0.5],[0,0.5),[0,1],0.7,1,[0,0.6],0.4,0.6).\end{aligned} \tag{93}$$

Step 10. Generate the complete solution set of system (77) according to Theorem 24. Since $X^{p_1} = X^{p_5}$, $X^{p_3} = X^{p_7}$, $X^{p_2} \subseteq X^{p_1}$, $X^{p_4} \subseteq X^{p_3}$, we have

$$\begin{aligned}X(A,b)&=\bigcup_{p\in P}X^p=\bigcup_{t=1}^{8}X^{p_t}\\&=X^{p_1}\cup X^{p_3}\cup X^{p_6}\cup X^{p_8},\end{aligned} \tag{94}$$

where $X^{p_1}, X^{p_3}, X^{p_6}, X^{p_8}$ are as shown in Step 9.

6. Discussion

6.1. Some Further Properties on System (12).

Proposition 1. *If $\check{x} = (\check{x}_1, \check{x}_2, \ldots, \check{x}_n)$ is a minimal solution of system (12), then it holds for any $j \in J$ that $\check{x}_j \in \{0, \widehat{x}_j\}$, where $\widehat{x}$ is the potential maximum solution of (12) as defined by (36). In particular, when $j \in J^{open}$, it holds that $\check{x}_j = 0$.*

Proof. Based on Theorems 23 and 24, any minimal solution could be selected from the set $\{\check{x}^p \mid p \in P\}$, i.e.,

$$\check{x} \in \{\check{x}^p \mid p \in P\}. \tag{95}$$

It follows from (53) that $\check{x}_j \in \{0, \widehat{x}_j\}$ for any $j \in J$. Moreover, when $j \in J^{open}$, it should hold that $\check{x}_j = 0$. Otherwise, it turns out to be $\check{x}_j = \widehat{x}_j$. According to (43), there exists some $i \in I$ such that $[\underline{a}_{ij}, \overline{a}_{ij}]\widehat{x}_j > [\underline{b}_i, \overline{b}_i]$. So we get

$$\begin{aligned}&[\underline{a}_{i1},\overline{a}_{i1}]\,\check{x}_1\vee[\underline{a}_{i2},\overline{a}_{i2}]\,\check{x}_2\vee\cdots\vee[\underline{a}_{in},\overline{a}_{in}]\,\check{x}_n\\&\quad\geq[\underline{a}_{ij},\overline{a}_{ij}]\,\check{x}_j=[\underline{a}_{ij},\overline{a}_{ij}]\,\widehat{x}_j>[\underline{b}_i,\overline{b}_i].\end{aligned} \tag{96}$$

This is conflict with $\check{x} \in X(A, b)$. □

Proposition 2. *Let system (12) be consistent and J^{open} is its open index set. There exists a unique maximum solution if and only if $J^{open} = \emptyset$. In particular, when the maximum solution exists, it should be $\widehat{x}$.*

Proof. ($\Longrightarrow$) If system (12) has a maximum solution, according to the structure of its solution set, i.e.,

$$X(A,b)=\bigcup_{p\in P}X^p, \tag{97}$$

and the expression of X^p (see (56)), the maximum solution should be $\widehat{x}$. Then it follows from Corollary 18 that $J^{open} = \emptyset$.

($\Longleftarrow$) If $J^{open} = \emptyset$, it holds that $\underline{\widehat{x}} = \widehat{x}$ by (46). Since system (12) is consistent, it follows from Theorem 19 that $\widehat{x} = \underline{\widehat{x}} \in X(A, b)$. According to Corollary 18, $\widehat{x}$ is the unique maximum solution of system (12).

Consequently, when the maximum solution exists, it should be $\widehat{x}$. □

Theorem 3. *Let system (12) be consistent with solution set $X(A, b)$. Then the following statements are equivalent:*

(i) *$X(A, b)$ is a close set;*

(ii) *$J^{open} = \emptyset$;*

(iii) *$\widehat{x} \in X(A, b)$;*

(iv) *there exists a unique maximum solution of system (12).*

Proof. The proof of (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) lies in Proposition 2.

(i) $\Longleftrightarrow$ (ii) Considering $X(A, b) = \bigcup_{p\in P} X^p$ and the expression of X^p (see (56)), $X(A, b)$ is a close set if and only if X^p is close set for any $p \in P$. This is equivalent to $J^{open} = \emptyset$. □

6.2. Comparing the Solution Set of System (12) to That of the Classical Max-T Fuzzy Relation Equations. As well known to everyone, the solution set of a system of classical max-T fuzzy relation equations, when nonempty, is determined by a unique maximum solution and a finite number of minimal solutions. The solution set could also be considered as a union of finite close intervals in form of

$$X(A,b)=\bigcup_{\check{x}\in\check{X}(A,b)}[\check{x},\widehat{x}]. \tag{98}$$

Differences between the solution set of the interval-valued-parameter max-product fuzzy relation equation (i.d. system (12)) and that of the classical max-T fuzzy relation equation are shown as below.

(i) The classical max-T fuzzy relation equation always has a maximum solution, when it is consistent. But in the consistent system (12), this property no longer holds. According to Proposition 2, the maximum solution exists if and only if its open index set J^{open} is empty.

(ii) As a union of finite close intervals, the solution set of classical max-T fuzzy relation equation is always a close set. However, the solution set of system (12) might not be a close set. It turns out to be close set unless $J^{open} = \emptyset$; i.e., the maximum solution exists, according to Theorem 3.

7. Conclusion

This paper pay attention to interval-valued fuzzy relation equations with max-product composition. Basic operations and order relation of the interval-valued numbers are introduced before description of the fuzzy relation equations system. In order to deal with such kind of system, we define concepts of close index set and open index set. We characterize the structure of the complete solution set to the interval-valued fuzzy relation equation, which is different from that to the classical one. Detailed algorithm is developed for obtaining the complete solution set with illustrative example. In the future research we will focus on the optimization problem subject to interval-valued fuzzy relation equations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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