Research Article

Maximum Independent Sets Partition of \((n,k)\)-Star Graphs

Fu-Tao Hu

School of Mathematical Sciences, Anhui University, Hefei, 230601, China

Correspondence should be addressed to Fu-Tao Hu; hufu@ahu.edu.cn

Received 19 April 2019; Accepted 24 June 2019; Published 4 July 2019

Academic Editor: Alejandro F. Villaverde

Copyright © 2019 Fu-Tao Hu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

For graph-theoretical notation and terminology not defined here, we follow [1]. In particular, let \(G = (V,E)\) be a simple undirected graph without loops and multiedges, where \(V = V(G)\) is the vertex set and \(E = E(G)\) is the edge set. If \(xy \in E(G)\), we call that two vertices \(x\) and \(y\) are adjacent. For a vertex \(x\), all the vertices adjacent to it are the neighbors of \(x\).

A subset \(S\) of \(V\) is said to be an independent set if no two of vertices are adjacent in \(S\) of a graph \(G\). The cardinality of a maximum independent set in a graph \(G\) is called the independent number of \(G\) and is denoted by \(\alpha(G)\). Let \(C\) be a set of \(k\) colours.\(A \ k\)-vertex-colouring (simply a \(k\)-colouring) is a mapping \(c : V \rightarrow C\) such that any two adjacent vertices are assigned the different colours of graph \(G\). A graph \(G\) is \(k\)-colourable if it has a \(k\)-colouring. The chromatic number, which is denoted by \(\chi(G)\), is the minimum \(k\), for which graph \(G\) is \(k\)-colourable.

As we know, the interconnection networks take an important part in the parallel computing/communication systems. An interconnection network can be modeled by a graph, where the processors are the vertices and the edges are the communication links.

In 1989, Akers and Krishnamurthy [2] introduced the \(n\)-dimensional star graph \(S_n\), which has superior degree and diameter compared to the hypercube and it is highly hierarchical and symmetrical [3]. However, the vertex cardinality of the \(n\)-dimensional star is \(n!\). The gap between \(n!\) and \((n+1)!\) is very large when \(S_n\) is extended to \(S_{n+1}\). Chiang and Chen [4] in 1995 generalized the star graph \(S_n\) to the \((n,k)\)-star graph, which preserves many good properties of the star graph and has smaller scale. Since the \((n,k)\)-star graph was introduced, it has received great attention in the literature [4–21].

The independent number and chromatic number of a graph are two important parameters in graph theory. However, we did not know the values of these two parameters of the \((n,k)\)-star graph since it was proposed. In this paper, we show a maximum independent sets partition of \((n,k)\)-star graph. From that, we can immediately deduce the exact value of the independent number and chromatic number of \((n,k)\)-star graph.

2. Preliminary Results

We use \([n]\) to denote the set \([1,2,...,n]\), where \(n\) is a positive integer. A permutation of \([n]\) is a sequence of \(n\) distinct symbols of \(u_i \in [n], u_1u_2...u_n\). The \(n\)-dimensional star network, denoted by \(S_n\), is a graph with the vertex set

\[ V(S_n) = \{ u_1u_2...u_n : u_i \in [n], u_i \neq u_j \text{ for } i \neq j \} \]

The edges are specified as follows:

\[ E(S_n): \text{ } u_1u_2...u_n \text{ is adjacent to } v_1v_2...v_n \text{ if there exists } i \text{ with } 2 \leq i \leq n \text{ such that } v_j = u_j \text{ for } j \neq i, v_1 = u_1, \text{ and } v_i = u_i. \]

The star graphs are vertex-transitive \((n-1)\)-regular of order \(n!\).

Let \(n\) and \(k\) be two positive integers with \(k \in [n-1]\), and let \(\Gamma_{n,k}\) be the set of all \(k\)-permutations on \([n]\); that is,
Γ_{n,k} = \{p_1, p_2, \ldots, p_k : p_i \in [n] \text{ and } p_i \neq p_j \text{ for } i \neq j\}. In 1995, Chiang and Chen [4] generalized the star graph to (n, k)-star graph denoted by $S_{n,k}$ with vertex set $V(S_{n,k}) = \Gamma_{n,k}$. The adjacency is defined as follows: $p_1p_2 \ldots p_k$ is adjacent to

1. $1 \leq i \leq k$;
2. $x (p_2 \ldots p_k)$, where $x \in [n] \setminus \{p_i : i \in [k]\}$.

By definition, $S_{n,k}$ is an $(n-1)$-regular vertex-transitive with $n/(n-k)!$ vertices. Moreover, $S_{n,n-1}$ and $S_{n,1}$ is isomorphic to $K_n$.

Let $S_{i-1,k-1}$ denote a subgraph of $S_{n,k}$ induced by all the vertices with the same last symbol $i$, for each $i \in [n]$. See Figure 1 for instance.

**Lemma 1** (Chiang and Chen [4], 1995). $S_{n,k}$ can be decomposed into $n$ subgraphs $S_{i-1,k-1}$, $i \in [n]$, and each subgraph $S_{i-1,k-1}$ is isomorphic to $S_{n-1,k-1}$.

**Lemma 2** (Li and Xu [14], 2014). For any $\alpha = p_2p_3 \ldots p_k \in \Gamma_{n,k-1}$ ($k \geq 2$), let $V_{\alpha} = \{p_2 \alpha : p_1 \in [n], p_1 \neq p_i, 2 \leq i \leq k\}$. Then the subgraph of $S_{n,k}$ induced by $V_{\alpha}$ is a complete graph of order $n-k+1$, denoted by $K_{n-k+1}^\alpha$.

### 3. Maximum Independent Sets Partition of $S_{n,k}$

**Proposition 3.** The independent number of $S_{n,k}$ is $\alpha(S_{n,k}) \leq n!/(n-k+1)!$.

**Proof.** This conclusion is true for $k = 1$ since $S_{n,1} \cong K_n$. Next, assume that $k \geq 2$. Let $I$ be any maximum independent set of $S_{n,k}$. For any $\alpha = p_2p_3 \ldots p_k \in \Gamma_{n,k-1}$ ($k \geq 2$), let $V_{\alpha} = \{p_2 \alpha : p_1 \in [n], p_1 \neq p_i, 2 \leq i \leq k\}$. Then the subgraph of $S_{n,k}$ induced by $V_{\alpha}$ is a complete graph of order $n-k+1$, denoted by $K_{n-k+1}^\alpha$. By Lemma 2. Thus, $I$ contains at most one vertex in $K_{n-k+1}^\alpha$. By definition, there are exactly $n!/(n-k+1)!$ such $K_{n-k+1}^\alpha$. Therefore, $\alpha(S_{n,k}) = |I| \leq n!/(n-k+1)!$. □

**Proposition 4.** Let $I_1 = \{1\}$, $I_2 = \{2\}$, $\ldots$, $I_n = \{n\}$. Then $\{I_1, I_2, \ldots, I_n\}$ is a maximum independent sets partition of $S_{n,1}$.

**Proposition 5.** Let

\begin{align*}
I_1^2 &= \{21, 32, 43, \ldots, n(n-1), 1n\}, \\
I_2^2 &= \{31, 42, 53, \ldots, 1(n-1), 2n\}, \\
&\vdots \\
I_{n-1}^2 &= \{n1, 12, 23, \ldots, (n-2)(n-1), (n-1)n\}.
\end{align*}

Then $\{I_1^2, I_2^2, \ldots, I_{n-1}^2\}$ is a maximum independent sets partition of $S_{n,2}$.

For each $j \in [n-k+2]$, we use $I_j^2$ of $S_{n-k+2,2}$ to generate a maximum independent set $I_j^3$ of $S_{n-k+3,3}$. Step by step, we generate a maximum independent set $I_j^3$ of $S_{n,k}$ in the following. For each $i \geq \lceil k \rceil \setminus \{1, 2\}$, $\pi \in I_{i-1}^3$, and $x \in [n-k+i-1]$, denote by $\pi(x, n-k+i)$ a permutation that replaces $x$ by $n-k+i$ if $x \in \pi$ (x is in $\pi$ means $x$ is equal to some symbol in $\pi$); otherwise $\pi(x, n-k+i) = \pi$. Let $\pi = p_1p_2 \ldots p_k$ be any vertex in $S_{n-k+i}$, Denote by $\pi' = p_1p_2 \ldots p_k$, the vertex by exchanging the first two symbols in $\pi$.

**Step 1.** By Proposition 5, denoted by $I_j^2 = \{(j+1)1, (j+2)2, \ldots, (n-k+2)\}$ and $I_{n-k+2}^3(x) = \{\pi(x, n-k+3) : \pi \in I_j^3\}$ for each $x \in [n-k+2]$. Let $I_j^3 = \bigcup_{x \in [n-k+3]} I_j^3(x)$.

**Step 2.** $I_j^3(n-k+3) = \{\pi'(n-k+3) : \pi \in I_j^3\}$ and $I_j^3(x) = \{\pi(x, n-k+3) : \pi \in I_j^3\}$ for each $x \in [n-k+2]$. Let $I_j^3 = \bigcup_{x \in [n-k+2]} I_j^3(x)$.

**Step i-1.** Let $I_j^3(n-k+i) = \{\pi'(n-k+i) : \pi \in I_j^3\}$ and $I_j^3(x) = \{\pi(x, n-k+i) : \pi \in I_j^3\}$ for each $x \in [n-k+i-1]$. Let $I_j^3 = \bigcup_{x \in [n-k+i]} I_j^3(x)$.

**Step k-1.** Let $I_j^3(n) = \{\pi'(n) : \pi \in I_j^3\}$ and $I_j^3(x) = \{\pi(x, n) : \pi \in I_j^3\}$ for each $x \in [n-l]$. Let $I_j^3 = \bigcup_{x \in [n-l]} I_j^3(x)$.

**Proposition 6.** For each $j \in [n-k+1]$ and $i \in [k]\setminus\{1\}$, $|I_j^i| = (n-k+i)!/(n-k+1)!$ and $I_j^i \cap I_j^j = \emptyset$ for any $j' \in [n-k+1]\setminus\{j\}$.

Therefore $\{I_1^1, I_2^2, \ldots, I_{n-k+i}^i\}$ is a vertex sets partition of $S_{n-k+i}$.

In the following, we show that $I_j^i$ is an independent set of $S_{n-k+i}$ for each $i \in \{3, 4, \ldots, k\}, k \geq 3$, and $j \in [n-k+1]$. $I_j^i$ is an independent set of $S_{n-k+i,j}$, then $I_j^1$ is an independent set of $S_{n-k+i-1,j-1}$.

**Lemma 7.** Let $i \in \{3, 4, \ldots, k\}, k \geq 3$, and $j \in [n-k+1]$. If $I_j^i$ is an independent set of $S_{n-k+i,j}$, then $I_j^1$ is an independent set of $S_{n-k+i-1,j-1}$.
Proof. By Proposition 5, \( I_3^i \) is an independent set of \( S_{n-k+i-1,j-1} \) for each \( j \in [n-k+1] \). Next assume that \( i \geq 4 \). Suppose to the contrary that \( I_3^i \) is not an independent set of \( S_{n-k+i-1,j-1} \). Firstly, assume that \( p_1 p_2 \cdots p_i \) and \( p_1 p_2 \cdots p_i p_{i+1} \) are two adjacent vertices in \( I_3^i \) of \( S_{n-k+i-1,j-1} \). If \( s = 2 \), then \( p_1 (n-k+i) \cdots p_{i-1} p_2 \) and \( (n-k+i) p_1 \cdots p_{i-1} p_2 \) belong to \( I_3^i (p_2) \subseteq I_3^i \) by the construction of \( I_3^i \); the one to one correspondence is

\[
\begin{align*}
p_1 p_2 \cdots p_{i-1} & \leftrightarrow p_1 (n-k+i) \cdots p_{i-1} p_2, \\
p_2 p_1 \cdots p_{i-1} & \leftrightarrow (n-k+i) p_1 \cdots p_{i-1} p_2.
\end{align*}
\]

However, \( p_1 (n-k+i) \cdots p_{i-1} p_2 \) and \( (n-k+i) p_1 \cdots p_{i-1} p_2 \) are adjacent in \( S_{n-k+i,j} \), a contradiction. If \( s > 2 \), then \( p_1 (n-k+i) \cdots p_{i-1} p_2 \) and \( p_1 (n-k+i) \cdots p_{i-1} p_2 \) belong to \( I_3^i (p_2) \subseteq I_3^i \) by the construction of \( I_3^i \); the one to one correspondence is

\[
\begin{align*}
p_1 p_2 \cdots p_{i-1} & \leftrightarrow p_1 (n-k+i) \cdots p_{i-1} p_2, \\
p_2 p_1 \cdots p_{i-1} & \leftrightarrow p_1 (n-k+i) \cdots p_{i-1} p_2.
\end{align*}
\]

However, \( p_1 (n-k+i) \cdots p_{i-1} p_2 \) and \( p_1 (n-k+i) \cdots p_{i-1} p_2 \) are adjacent in \( S_{n-k+i,j} \), a contradiction. Secondly, assume that \( p_1 p_2 \cdots p_{i-1} \) and \( p_1 p_2 \cdots p_{i-1} \) are two adjacent vertices in \( I_3^i \) of \( S_{n-k+i,j} \). By the construction of \( I_3^i \), \( p_1 (n-k+i) \cdots p_{i-1} p_2 \) and \( p_1 (n-k+i) \cdots p_{i-1} p_2 \) belong to \( I_3^i (p_2) \subseteq I_3^i \); the one to one correspondence is

\[
\begin{align*}
p_1 p_2 \cdots p_{i-1} & \leftrightarrow p_1 (n-k+i) \cdots p_{i-1} p_2, \\
p_2 p_1 \cdots p_{i-1} & \leftrightarrow p_1 (n-k+i) \cdots p_{i-1} p_2.
\end{align*}
\]

However, \( p_1 (n-k+i) \cdots p_{i-1} p_2 \) and \( p_1 (n-k+i) \cdots p_{i-1} p_2 \) are adjacent in \( S_{n-k+i,j} \), a contradiction. □

**Lemma 8.** Let \( i \in \{3, \ldots, k\}, j \geq 3, \) and \( j \in [n-k+1]. \) If \( I_j^i \) is an independent set of \( S_{n-k+i-1,j+1} \), then the two vertices \( p \sigma_1 \) and \( p \sigma_2 \) cannot both belong to \( I_j \) of \( S_{n-k+i,j} \) where \( \sigma_1 \) and \( \sigma_2 \) are adjacent in \( S_{n-k+i,j+1} \).

**Proof.** By Proposition 5, \( I_j^i = \{(j+1), (j+2), \ldots, (n-k+2)\} \) is an independent set of \( S_{n-k+i,j} \). Suppose to the contrary that there exist two vertices \( p \sigma_1 \) and \( p \sigma_2 \) in \( I_j^i \) but \( \sigma_1 \) and \( \sigma_2 \) are adjacent in \( S_{n-k+i,j+1} \).

We consider the case for \( i = 3 \). Suppose to the contrary that there exist two vertices \( p \sigma_1 \) and \( p \sigma_2 \) in \( I_j^i \) but \( \sigma_1 \) and \( \sigma_2 \) are adjacent in \( S_{n-k+i,j+1} \).

Assume that \( \sigma_1 = p_1 p_2 \) and \( \sigma_2 = p_2 p_1 \). Then \( pp_1 p_2 \in I_j^i (p_2) \) and \( pp_1 p_2 \in I_j^i (p_1) \). If \( p = n-k+i \), then \( p \sigma_3 p_1 \) and \( p \sigma_2 p_1 \) are in \( I_j^i \) by the construction of \( I_j^i \); the one to one correspondence is

\[
\begin{align*}
(n-k+i) p_1 p_2 & \leftrightarrow p_2 p_1, \\
(n-k+i) p_2 p_1 & \leftrightarrow p_1 p_2.
\end{align*}
\]

However, \( p_3 p_1 \) and \( p_1 p_2 \) are adjacent in \( S_{n-k+i+2} \), a contradiction with \( I_j^i \) being an independent set of \( S_{n-k+i+2} \). If \( p < n-k+3 \), then \( pp_1 p_2 \) and \( pp_2 p_1 \) are two vertices in \( I_j^i \) by the construction of \( I_j^i \); the one to one correspondence is

\[
\begin{align*}
pp_1 p_2 & \leftrightarrow PP_1, \\
pp_2 p_1 & \leftrightarrow PP_2,
\end{align*}
\]
a contradiction with the construction of \( I_j^i \).

Now, assume that \( \sigma_1 = p_1 p_2 \) and \( \sigma_2 = p_2 p_3 \). Then \( pp_1 p_2 \in I_j^i (p_2) \) and \( pp_2 p_3 \in I_j^i (p_2) \). If \( p = n-k+3 \), then \( p \sigma_3 p_1 \) and \( p \sigma_2 p_3 \) are in \( I_j^i \) by the construction of \( I_j^i \); the one to one correspondence is

\[
\begin{align*}
(n-k+3) p_1 p_2 & \leftrightarrow p_2 p_1, \\
(n-k+3) p_2 p_3 & \leftrightarrow p_2 p_3,
\end{align*}
\]
a contradiction with the construction of \( I_j^i \). If \( p < n-k+3 \), then \( pp_1 p_2 \) and \( pp_3 p_2 \) are two vertices in \( I_j^i \) by the construction of \( I_j^i \); the one to one correspondence is

\[
\begin{align*}
pp_1 p_2 & \leftrightarrow PP_1, \\
pp_2 p_1 & \leftrightarrow PP_2,
\end{align*}
\]
a contradiction with the construction of \( I_j^i \).

Therefore, the conclusion is true for \( i = 3 \).

We prove this Lemma by induction on \( i \). Assume that the induction hypothesis is true for \( i-1 \) with \( i \geq 4 \). We prove the case for \( i \geq 4 \). Assume that \( I_j^{i-1} \) is an independent set of \( S_{n-k+i-1,j-1} \). Then \( I_j^{i-2} \) is an independent set of \( S_{n-k+i+2,j-2} \) by Lemma 7. Suppose to the contrary that there exist two vertices \( p \sigma_1 \) and \( p \sigma_2 \) in \( I_j^i \) but \( \sigma_1 \) and \( \sigma_2 \) are adjacent in \( S_{n-k+i+2,j-2} \).

Firstly, assume that \( \sigma_1 = p_3 p_5 \cdots p_{i-1} p_i \) and \( \sigma_2 = p_3 p_5 \cdots p_{i-1} p_i \). Suppose that \( s = i \). If \( p = n-k+i \), then \( p \sigma_3 p_1 \) and \( p \sigma_2 p_3 \) are in \( I_j^{i-1} \) by the construction of \( I_j^i \); the one to one correspondence is

\[
\begin{align*}
(n-k+i) p_3 p_5 \cdots p_{i-1} p_i & \leftrightarrow p_3 p_5 \cdots p_{i-1} p_i, \\
(n-k+i) p_3 p_5 \cdots p_{i-1} p_i & \leftrightarrow p_2 p_3 \cdots p_{i-1} p_i,
\end{align*}
\]

However, \( p_3 p_5 \cdots p_{i-1} p_i \) and \( p_3 p_5 \cdots p_{i-1} p_i \) are adjacent in \( S_{n-k+i+2,j-2} \), a contradiction. Next, suppose that \( s \neq i \). If \( p = n-k+i \), then \( p \sigma_3 p_1 \) and \( p \sigma_2 p_3 \) are two vertices in \( I_j^i \) by the construction of \( I_j^i \). Now assume that
Proof. Assume that \( p < n - k + i \). Then \( pp_p \ldots p_i \) and \( pp_p \ldots p_i \) are two vertices in \( I_j \) by the construction of \( I_j \). However, \( pp_p \ldots p_i \) and \( p_p \ldots p_i \) are two adjacent vertices in \( S_{n-k+i-j-2} \), a contradiction with the induction hypothesis.

Secondly, assume that \( \pi_1 = p_p \ldots p_i \) and \( \pi_2 = p_p \ldots p_i \). If \( p = n - k + i \) then \( p_p \ldots p_i \) and \( p_p \ldots p_i \) are two vertices in \( I_1 \) by the construction of \( I_1 \). Suppose that \( p < n - k + i \). Then \( pp_p \ldots p_i \) and \( pp_p \ldots p_i \) are two vertices in \( I_1 \) by the construction of \( I_1 \). However, \( pp_p \ldots p_i \) and \( p_p \ldots p_i \) are two adjacent vertices in \( S_{n-k+i+j-2} \), a contradiction with the induction hypothesis.

By the principle of induction, this Lemma completes.

Lemma 9. Let \( i \in \{2, 3, \ldots, k\} \), \( k \geq 3 \), and \( j \in [n-k+1] \). If \( I_j \) is an independent set of \( S_{n-k+i} \), then \( I_j \) is an independent set of \( S_{n-k+i} \) for each \( x \in [n-k+i] \).

Proof. Assume that \( i = 3 \). Suppose to the contrary that \( I_j \) is not an independent set of \( S_{n-k+i+j} \). Assume that \( p_1, p_2 \) and \( p_3 \) are two adjacent vertices in \( I_j \). By the construction of \( I_j \), \( p_1, p_2, p_3 \in I_j \) if \( n - k + 3 \notin \{p_1, p_2\} \) and \( x \in I_j \) if \( p_1 = n - k + 3 \) (the case for \( p_2 = n - k + 3 \) is similar), a contradiction with the construction of \( I_j \) in Proposition 5. Assume that \( p_1, p_2 \) and \( p_3 \) are two adjacent vertices in \( I_j \). By the construction of \( I_j \), \( p_1, p_2 \), \( p_3 \) and \( p_3 \) are two vertices in \( I_j \) if \( p_1 = n - k + 3 \), \( x \in [n-k+i] \), otherwise.

Any case above makes a contradiction with the construction of \( I_j \) in Proposition 5.

We proceed by induction on \( i \geq 3 \). Assume that \( i \) is true for \( i - 1 \) with \( i \geq 4 \). Next we prove that \( I_j \) is an independent set of \( S_{n-k+i+j} \) for each \( x \in [n-k+i] \). Suppose to the contrary that \( I_j \) is not an independent set of \( S_{n-k+i+j} \).

Firstly, assume that \( p_1, p_2 \) and \( p_3 \) are two adjacent vertices in \( I_j \). On one hand, suppose that \( x = n - k + i \). By the construction of \( I_j \), \( p_1, p_2 \) should be in \( I_j \) by the construction of \( I_j \), but they are two adjacent vertices in \( I_j \), a contradiction. Now assume that \( p_1 \) is not \( n - k + i \) for each \( s \in [i] \). Then \( p_2 \) and \( p_3 \) should be in \( I_j \) by the construction of \( I_j \), but they are two adjacent vertices in \( I_j \), a contradiction. This Theorem completes.

Theorem 10. The vertex set \( I_j \) is an independent set of \( S_{n-k+i+j} \) for each \( j \in [n-k+1] \) and \( i \in [2, 3, \ldots, k] \).

Proof. We proceed by induction on \( i \geq 2 \). By Proposition 5, \( I_2 \) is an independent set of \( S_{n-k+i+j} \). Assume that the induction hypothesis is true for \( i - 1 \) with \( i \geq 3 \). Assume that \( I_j \) is not independent. Assume that \( p_1, p_2 \) and \( p_3 \) are two adjacent vertices in \( I_j \) (this is the only possible case since \( I_j \) is an independent set of \( S_{n-k+i+j} \) for each \( x \in [n-k+i+1] \)). If \( p_1 = n - k + i \), then \( p_2 \) and \( p_3 \) should be in \( I_2 \) by the construction of \( I_2 \), but they are two adjacent vertices in \( I_2 \), a contradiction. This Theorem completes.
Proof. By Proposition 3, $\alpha(S_{n,k}) \leq n!/(n-k+1)!$. By Proposition 6, $I^k_j$ is an independent set of $S_{n,k}$. Therefore, $I^k_j$ for each $j \in [n-k+1]$ is a maximum independent set of $S_{n,k}$ and $\alpha(S_{n,k}) = n!/(n-k+1)!$. By Proposition 6, $\{I^k_1, I^k_2, \ldots, I^k_{n-k+1}\}$ is a vertex sets partition of $S_{n,k}$, so $\{I^k_1, I^k_2, \ldots, I^k_{n-k+1}\}$ is a maximum independent sets partition of $S_{n,k}$.

Since $\{I^k_1, I^k_2, \ldots, I^k_{n-k+1}\}$ is a maximum independent sets partition of $S_{n,k}$, we immediately obtain the chromatic number of $S_{n,k}$.

**Corollary 12.** The chromatic number of $S_{n,k}$ is $\chi(S_{n,k}) = n-k+1$.

Next, we show the maximum independent sets partition of $S_{4,3}$ by our construction.

**Example 13** (see Figure 2). By Proposition 5, $I^2_1 = \{21, 32, 13\}$ and $I^2_2 = \{31, 12, 23\}$ are two maximum independent sets of $S_{3,2}$. The constructed two maximum independent sets of $S_{4,3}$ are

$$I^3_1 = \{124, 234, 314\} \cup \{241, 321, 431\}$$
$$\cup \{412, 342, 132\} \cup \{213, 423, 143\}$$

$$= I^3_1(4) \cup I^3_1(1) \cup I^3_1(2) \cup I^3_1(3),$$

$$I^3_2 = \{134, 214, 324\} \cup \{341, 421, 231\}$$
$$\cup \{312, 142, 432\} \cup \{413, 123, 243\}$$

$$= I^3_2(4) \cup I^3_2(1) \cup I^3_2(2) \cup I^3_2(3).$$

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.

**Acknowledgments**

The work was supported by NSFC (no. 11401004) and Anhui Provincial Natural Science Foundation (no. 1708085MA01).

**References**


