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Research Article

Maximum Independent Sets Partition of (n, k)-Star Graphs

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The (n, k)-star graph is a very important computer modelling. The independent number and chromatic number of a graph are two important parameters in graph theory. However, we have not known the values of these two parameters of the (n, k)-star graph since it was proposed. In this paper, we show a maximum independent sets partition of (n, k)-star graph. From that, we can immediately deduce the exact value of the independent number and chromatic number of (n, k)-star graph.

1. Introduction

For graph-theoretical notation and terminology not defined here, we follow [1]. In particular, let $G = (V, E)$ be a simple undirected graph without loops and multiedges, where $V = V(G)$ is the *vertex set* and $E = E(G)$ is the *edge set*. If $xy \in E(G)$, we call that two vertices x and y are *adjacent*. For a vertex x, all the vertices adjacent to it are the *neighbors* of x.

A subset S of V is said to be an *independent set* if no two of vertices are adjacent in S of a graph G. The cardinality of a maximum independent set in a graph G is called the *independent number* of G and is denoted by $\alpha(G)$. Let C be a set of k colours. A *k-vertex-colouring* (simply a *k-colouring*) is a mapping $c : V \longrightarrow C$ such that any two adjacent vertices are assigned the different colours of graph G. A graph is *k-colourable* if it has a k-colouring. The *chromatic number*, which is denoted by $\chi(G)$, is the minimum k, for which graph G is k-colourable.

As we know, the interconnection networks take an important part in the parallel computing/communication systems. An interconnection network can be modeled by a graph, where the processors are the vertices and the edges are the communication links.

In 1989, Akers and Krishnamurthy [2] introduced the n-dimensional star graph S_n, which has superior degree and diameter compared to the hypercube and it is highly hierarchical and symmetrical [3]. However, the vertex cardinality of the n-dimensional star is $n!$. The gap between $n!$ and $(n+1)!$ is very large when S_n is extended to S_{n+1}. Chiang and Chen [4] in 1995 generalized the star graph S_n to the (n, k)-star graph, which preserves many good properties of the star graph and has smaller scale. Since the (n, k)-star graph was introduced, it has received great attention in the literature [4–21].

The independent number and chromatic number of a graph are two important parameters in graph theory. However, we did not know the values of these two parameters of the (n, k)-star graph as far as we know. In this paper, we show a maximum independent sets partition and determine the exact value of the independent number and chromatic number of (n, k)-star graph.

2. Preliminary Results

We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$, where n is a positive integer. A *permutation* of $[n]$ is a sequence of n distinct symbols of $u_i \in [n]$, $u_1u_2 \ldots u_n$. The n-dimensional *star network*, denoted by S_n, is a graph with the vertex set

$$V(S_n) = \left\{u_1u_2 \ldots u_n : u_i \in [n], u_i \neq u_j \text{ for } i \neq j\right\}. \quad (1)$$

The edges are specified as follows:

$E(S_n)$: $u_1u_2 \ldots u_n$ is adjacent to $v_1v_2 \ldots v_n$ if there exists i with $2 \leq i \leq n$ such that $v_j = u_j$ for $j \notin \{1, i\}$, $v_1 = u_i$, and $v_i = u_1$.

The star graphs are vertex-transitive $(n-1)$-regular of order $n!$.

Let n and k be two positive integers with $k \in [n-1]$, and let $\Gamma_{n,k}$ be the set of all k-permutations on $[n]$; that is,

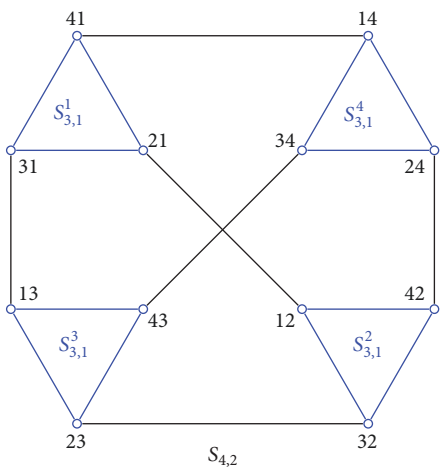

FIGURE 1: The $(4,2)$-star graph $S_{4,2}$.

$\Gamma_{n,k} = \{p_1p_2\dots p_k : p_i \in [n]$ and $p_i \neq p_j$ for $i \neq j\}$. In 1995, Chiang and Chen [4] generalized the star graph to (n,k)-star graph denoted by $S_{n,k}$ with vertex set $V(S_{n,k}) = \Gamma_{n,k}$. The adjacency is defined as follows: $p_1p_2\dots p_i\dots p_k$ is adjacent to

(1) $p_ip_2\dots p_1\dots p_k$, where $2 \le i \le k$;

(2) $xp_2\dots p_i\dots p_k$, where $x \in [n]\backslash\{p_i : i \in [k]\}$.

By definition, $S_{n,k}$ is $(n-1)$-regular vertex-transitive with $n!/(n-k)!$ vertices. Moreover, $S_{n,n-1} \cong S_n$ and $S_{n,1} \cong K_n$, where K_n is the complete graph with order n.

Let $S^i_{n-1,k-1}$ denote a subgraph of $S_{n,k}$ induced by all the vertices with the same last symbol i, for each $i \in [n]$. See Figure 1 for instance.

Lemma 1 (Chiang and Chen [4], 1995). *$S_{n,k}$ can be decomposed into n subgraphs $S^i_{n-1,k-1}$, $i \in [n]$, and each subgraph $S^i_{n-1,k-1}$ is isomorphic to $S_{n-1,k-1}$.*

Lemma 2 (Li and Xu [14], 2014). *For any $\alpha = p_2p_3\dots p_k \in \Gamma_{n,k-1}$ $(k \ge 2)$, let $V_\alpha = \{p_1\alpha : p_1 \in [n],\ p_1 \neq p_i,\ 2 \le i \le k\}$. Then the subgraph of $S_{n,k}$ induced by V_α is a complete graph of order $n-k+1$, denoted by K^α_{n-k+1}.*

3. Maximum Independent Sets Partition of $S_{n,k}$

Proposition 3. *The independent number of $S_{n,k}$ is $\alpha(S_{n,k}) \le n!/(n-k+1)!$.*

Proof. This conclusion is true for $k = 1$ since $S_{n,1} \cong K_n$. Next, assume that $k \ge 2$. Let I be any maximum independent set of $S_{n,k}$. For any $\alpha = p_2p_3\dots p_k \in \Gamma_{n,k-1}$ $(k \ge 2)$, let $V_\alpha = \{p_1\alpha : p_1 \in [n],\ p_1 \neq p_i,\ 2 \le i \le k\}$. Then the subgraph of $S_{n,k}$ induced by V_α is a complete graph of order $n-k+1$, denoted by K^α_{n-k+1} by Lemma 2. Thus, I contains at most one vertex in K^α_{n-k+1}. By definition, there are exactly $n!/(n-k+1)!$ such K^α_{n-k+1}. Therefore, $\alpha(S_{n,k}) = |I| \le n!/(n-k+1)!$. □

Proposition 4. *Let $I^1_1 = \{1\}, I^1_2 = \{2\},\dots, I^1_n = \{n\}$. Then $\{I^1_1, I^1_2,\dots, I^1_n\}$ is a maximum independent sets partition of $S_{n,1}$.*

Proposition 5. *Let*

$$\begin{aligned} I^2_1 &= \{21, 32, 43,\dots, n(n-1), 1n\},\\ I^2_2 &= \{31, 42, 53,\dots, 1(n-1), 2n\},\\ &\dots\\ I^2_{n-1} &= \{n1, 12, 23,\dots, (n-2)(n-1), (n-1)n\}. \end{aligned} \tag{2}$$

Then $\{I^2_1, I^2_2,\dots, I^2_{n-1}\}$ is a maximum independent sets partition of $S_{n,2}$.

For each $j \in [n-k+1]$, we use I^2_j of $S_{n-k+2,2}$ to generate a maximum independent set I^3_j of $S_{n-k+3,3}$. Step by step, we generate a maximum independent set I^k_j of $S_{n,k}$ in the following. For each $i \ge [k]\backslash\{1,2\}$, $\pi \in I^{i-1}_j$, and $x \in [n-k+i-1]$, denote by $\pi(x, n-k+i)$ a permutation that replaces x by $n-k+i$ if $x \in \pi$ ($x \in \pi$ means x is equal to some symbol in π); otherwise $\pi(x, n-k+i) = \pi$. Let $\pi = p_1p_2\dots p_i$ be any vertex in $S_{n-k+i,i}$. Denote by $\pi' = p_2p_1\dots p_i$ the vertex by exchanging the first two symbols in π.

Step 1. By Proposition 5, denoted by $I^2_j = \{(j+1)1, (j+2)2,\dots, (n-k+2)(n-k+2-j), 1(n-k+3-j)$, and $2(n-k+4-j),\dots, j(n-k+2)\}$ is a maximum independent set of $S_{n-k+2,2}$ for each $j \in [n-k+1]$.

Step 2. Let $I^3_j(n-k+3) = \{\pi'(n-k+3) : \pi \in I^2_j\}$ and $I^3_j(x) = \{\pi(x, n-k+3)x : \pi \in I^2_j\}$ for each $x \in [n-k+2]$. Let $I^3_j = \bigcup_{x\in[n-k+3]} I^3_j(x)$.

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Step i-1. Let $I^i_j(n-k+i) = \{\pi'(n-k+i) : \pi \in I^{i-1}_j\}$ and $I^i_j(x) = \{\pi(x, n-k+i)x : \pi \in I^{i-1}_j\}$ for each $x \in [n-k+i-1]$. Let $I^i_j = \bigcup_{x\in[n]} I^i_j(x)$.

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Step k-1. Let $I^k_j(n) = \{\pi' n : \pi \in I^{k-1}_j\}$ and $I^k_j(x) = \{\pi(x,n)x : \pi \in I^{k-1}_j\}$ for each $x \in [n-1]$. Let $I^k_j = \bigcup_{x\in[n]} I^k_j(x)$.

Let $j \in [n-k+1]$ and $i \in [k]\backslash\{1,2\}$. By the above construction, it is easy to see that $I^i_j(x)$ is one corresponding to one with I^{i-1}_j for each $x \in [n-k+i]$ and $I^i_j(x) \cap I^i_j(x') = \emptyset$ for every $x' \in [n-k+i]\backslash\{x\}$. We can easily show the following conclusion by induction on i.

Proposition 6. *For each $j \in [n-k+1]$ and $i \in [k]\backslash\{1\}$, $|I^i_j| = (n-k+i)!/(n-k+1)!$ and $I^i_j \cap I^i_{j'} = \emptyset$ for any $j' \in [n-k+1]\backslash\{j\}$. Therefore $\{I^i_1, I^i_2,\dots, I^i_{n-k+1}\}$ is a vertex sets partition of $S_{n-k+i,i}$.*

In the following, we show that I^i_j is an independent set of $S_{n-k+i,i}$ for each $i \in \{3,4,\dots,k\}$, $k \ge 3$, and $j \in [n-k+1]$.

Lemma 7. *Let $i \in \{3,4,\dots,k\}$, $k \ge 3$, and $j \in [n-k+1]$. If I^i_j is an independent set of $S_{n-k+i,i}$, then I^{i-1}_j is an independent set of $S_{n-k+i-1,i-1}$.*

Proof. By Proposition 5, I_j^2 is an independent set of $S_{n-k+i-1,i-1}$ for each $j \in [n-k+1]$. Next assume that $i \geq 4$. Suppose to the contrary that I_j^{i-1} is not an independent set of $S_{n-k+i-1,i-1}$. Firstly, assume that $p_1p_2\ldots p_s\ldots p_{i-1}$ and $p_sp_2\ldots p_1\ldots p_{i-1}$ are two adjacent vertices in I_j^{i-1} of $S_{n-k+i-1,i-1}$. If $s = 2$, then $p_1(n-k+i)\ldots p_{i-1}p_2$ and $(n-k+i)p_1\ldots p_{i-1}p_2$ belong to $I_j^i(p_2) \subseteq I_j^i$ by the construction of I_j^i; the one to one correspondence is

$$\begin{aligned} p_1p_2\cdots p_{i-1} &\longleftrightarrow p_1\,(n-k+i)\ldots p_{i-1}p_2, \\ p_2p_1\ldots p_{i-1} &\longleftrightarrow (n-k+i)\,p_1\ldots p_{i-1}p_2. \end{aligned} \tag{3}$$

However, $p_1(n-k+i)\ldots p_{i-1}p_2$ and $(n-k+i)p_1\ldots\ldots p_{i-1}p_2$ are adjacent in $S_{n-k+i,i}$, a contradiction. If $s > 2$, then $p_1(n-k+i)\ldots p_s\ldots p_{i-1}p_2$ and $p_s(n-k+i)\ldots p_1\ldots p_{i-1}p_2$ belong to $I_j^i(p_2) \subseteq I_j^i$ by the construction of I_j^i; the one to one correspondence is

$$\begin{aligned} p_1p_2\cdots p_s\cdots p_{i-1} &\longleftrightarrow p_1\,(n-k+i)\ldots p_s\ldots p_{i-1}p_2, \\ p_sp_2\ldots p_1\ldots p_{i-1} &\longleftrightarrow p_s\,(n-k+i)\ldots p_1\ldots p_{i-1}p_2. \end{aligned} \tag{4}$$

However, $p_1(n-k+i)\ldots p_s\ldots p_{i-1}\mathrm{p}_2$ and $p_s(n-k+i)\ldots p_1\ldots p_{i-1}p_2$ are adjacent in $S_{n-k+i,i}$, a contradiction.

Secondly, assume that $p_1p_2\ldots p_{i-1}$ and $p_ip_2\ldots p_{i-1}$ are two adjacent vertices in I_j^{i-1} of $S_{n-k+i-1,i-1}$. By the construction of I_j^i, $p_1(n-k+i)\ldots p_{i-1}p_2$ and $p_i(n-k+i)\ldots p_{i-1}p_2$ belong to $I_j^i(p_2) \subseteq I_j^i$; the one to one correspondence is

$$\begin{aligned} p_1p_2\cdots p_{i-1} &\longleftrightarrow p_1\,(n-k+i)\ldots p_{i-1}p_2, \\ p_ip_2\cdots p_{i-1} &\longleftrightarrow p_i\,(n-k+i)\ldots p_{i-1}p_2. \end{aligned} \tag{5}$$

However, $p_1(n-k+i)\ldots p_s\ldots p_{i-1}p_2$ and $p_i(n-k+i)\ldots p_s\ldots p_{i-1}p_2$ are adjacent in $S_{n-k+i,i}$, a contradiction. □

Lemma 8. *Let $i \in \{3,\ldots,k\}$, $k \geq 3$, and $j \in [n-k+1]$. If I_j^{i-1} is an independent set of $S_{n-k+i-1,i-1}$, then the two vertices $p\pi_1$ and $p\pi_2$ can not both belong to I_j^i of $S_{n-k+i,i}$, where π_1 and π_2 are adjacent in $S_{n-k+i-1,i-1}$.*

Proof. By Proposition 5, $I_j^2 = \{(j+1)1, (j+2)2, \ldots, (n-k+2)(n-k+2-j), 1(n-k+3-j), \text{and } 2(n-k+4-j), \ldots, j(n-k+2)\}$ is an independent set of $S_{n-k+2,2}$.

We consider the case for $i = 3$. Suppose to the contrary that there exist two vertices $p\pi_1$ and $p\pi_2$ in I_j^3 but π_1 and π_2 are adjacent in $S_{n-k+2,2}$.

Assume that $\pi_1 = p_1p_2$ and $\pi_2 = p_2p_1$. Then $pp_1p_2 \in I_j^3(p_2)$ and $pp_2p_1 \in I_j^3(p_1)$. If $p = n-k+3$, then p_2p_1 and p_1p_2 are in I_j^2 by the construction of I_j^3; the one to one correspondence is

$$\begin{aligned} (n-k+3)\,p_1p_2 &\longleftrightarrow p_2p_1, \\ (n-k+3)\,p_2p_1 &\longleftrightarrow p_1p_2. \end{aligned} \tag{6}$$

However, p_2p_1 and p_1p_2 are adjacent in $S_{n-k+2,2}$, a contradiction with I_j^2 being an independent set of $S_{n-k+2,2}$. If $p < n-k+3$, then pp_1 and pp_2 are two vertices in I_j^2 by the construction of I_j^3; the one to one correspondence is

$$\begin{aligned} pp_1p_2 &\longleftrightarrow pp_1, \\ pp_2p_1 &\longleftrightarrow pp_2, \end{aligned} \tag{7}$$

a contradiction with the construction of I_j^2.

Now, assume that $\pi_1 = p_1p_2$ and $\pi_2 = p_3p_2$. Then $pp_1p_2 \in I_j^3(p_2)$ and $pp_3p_2 \in I_j^3(p_2)$. If $p = n-k+3$, then p_2p_1 and p_2p_3 are in I_j^2 by the construction of I_j^3; the one to one correspondence is

$$\begin{aligned} (n-k+3)\,p_1p_2 &\longleftrightarrow p_2p_1, \\ (n-k+3)\,p_3p_2 &\longleftrightarrow p_2p_3, \end{aligned} \tag{8}$$

a contradiction with the construction of I_j^2. If $p < n-k+3$, then pp_1 and pp_3 are two vertices in I_j^2 by the construction of I_j^3; the one to one correspondence is

$$\begin{aligned} pp_1p_2 &\longleftrightarrow pp_1, \\ pp_2p_1 &\longleftrightarrow pp_2, \end{aligned} \tag{9}$$

a contradiction with the construction of I_j^2.

Therefore, the conclusion is true for $i = 3$.

We prove this Lemma by induction on i. Assume that the induction hypothesis is true for $i-1$ with $i \geq 4$. We prove the case for $i \geq 4$. Assume that I_j^{i-1} is an independent set of $S_{n-k+i-1,i-1}$. Then I_j^{i-2} is an independent set of $S_{n-k+i-2,i-2}$ by Lemma 7. Suppose to the contrary that there exist two vertices $p\pi_1$ and $p\pi_2$ in I_j^i but π_1 and π_2 are adjacent in $S_{n-k+i-1,i-1}$.

Firstly, assume that $\pi_1 = p_2p_3\ldots p_s\ldots p_{i-1}p_i$ and $\pi_2 = p_sp_3\ldots p_2\ldots p_{i-1}p_i$. Suppose that $s = i$. If $p = n-k+i$, then $p_ip_2p_3\ldots p_{i-1}$ and $p_2p_ip_3\ldots p_{i-1}$ are in I_j^{i-1} by the construction of I_j^i; the one to one correspondence is

$$\begin{aligned} (n-k+i)\,p_2p_3\ldots p_{i-1}p_i &\longleftrightarrow p_ip_2p_3\ldots p_{i-1}, \\ (n-k+i)\,p_ip_3\ldots p_{i-1}p_2 &\longleftrightarrow p_2p_ip_3\ldots p_{i-1}. \end{aligned} \tag{10}$$

However, $p_ip_2p_3\ldots p_{i-1}$ and $p_2p_ip_3\ldots p_{i-1}$ are adjacent in $S_{n-k+i-1,i-1}$, a contradiction.

Now assume that $p < n-k+i$; then $pp_2p_3\ldots p_{i-1}$ and $pp_ip_3\ldots p_{i-1}$ are two vertices in I_j^{i-1} by the construction of I_j^i; the one to one correspondence is

$$\begin{aligned} pp_2p_3\cdots p_{i-1}p_i &\longleftrightarrow pp_2p_3\cdots p_{i-1}, \\ pp_ip_3\cdots p_{i-1}p_2 &\longleftrightarrow pp_ip_3\cdots p_{i-1}. \end{aligned} \tag{11}$$

However, $p_2p_3\ldots p_{i-1}$ and $p_ip_3\ldots p_{i-1}$ are adjacent in $S_{n-k+i-2,i-2}$, a contradiction with the induction hypothesis.

Next, suppose that $s \neq i$. If $p = n-k+i$, then $p_ip_2p_3\ldots p_s\ldots p_{i-1}$ and $p_ip_sp_3\ldots p_2\ldots p_{i-1}$ are two vertices in I_j^{i-1} by the construction of I_j^i. Now assume that

$p < n-k+i$. Then $pp_2p_3\dots p_s\dots p_{i-1}$ and $pp_sp_3\dots p_2\dots p_{i-1}$ are two vertices in I_j^{i-1} by the construction of I_j^i. However, $p_2p_3\dots p_s\dots p_{i-1}$ and $p_sp_3\dots p_2\dots p_{i-1}$ are two adjacent vertices in $S_{n-k+i-2,i-2}$, a contradiction with the induction hypothesis.

Secondly, assume that $\pi_1 = p_2p_3\dots p_{i-1}p_i$ and $\pi_2 = p_{i+1}p_3\dots p_{i-1}p_i$. If $p = n-k+i$, then $p_ip_2p_3\dots p_{i-1}$ and $p_ip_{i+1}p_3\dots p_{i-1}$ are two vertices in I_j^{i-1} by the construction of I_j^i. Now suppose that $p < n-k+i$. Then $pp_2p_3\dots p_{i-1}$ and $pp_{i+1}p_3\dots p_{i-1}$ are two vertices in I_j^{i-1} by the construction of I_j^i. However, $p_2p_3\dots p_{i-1}$ and $p_{i+1}p_3\dots p_{i-1}$ are two adjacent vertices in $S_{n-k+i-2,i-2}$, a contradiction with the induction hypothesis.

By the principle of induction, this Lemma completes. □

Lemma 9. *Let $i \in \{3,\dots,k\}$, $k \geq 3$, and $j \in [n-k+1]$. If I_j^{i-1} is an independent set of $S_{n-k+i,i}$, then $I_j^i(x)$ is an independent set of $S_{n-k+i,i}$ for each $x \in [n-k+i]$.*

Proof. Assume that $i = 3$. Suppose to the contrary that $I_j^3(x)$ is not an independent set of $S_{n-k+3,3}$. Assume that p_1p_2x and p_2p_1x are two adjacent vertices in $I_j^3(x)$. By the construction of $I_j^3(x)$, $p_1p_2, p_2p_1 \in I_j^2$ if $n-k+3 \notin \{p_1,p_2\}$ and $xp_2, p_2x \in I_j^2$ if $p_1 = n-k+3$ (the case for $p_2 = n-k+3$ is similar), a contradiction with the construction of I_j^2 in Proposition 5. Assume that p_1p_2x and p_3p_2x are two adjacent vertices in $I_j^3(x)$. By the construction of $I_j^3(x)$, we have

$$\begin{aligned} p_2p_1, p_2p_3 &\in I_j^2, \quad \text{if } x = n-k+3,\\ p_1x, p_3x &\in I_j^2, \quad \text{if } p_2 = n-k+3,\\ xp_2, p_3p_2 &\in I_j^2, \quad \text{if } p_1 = n-k+3,\\ p_1p_2, xp_2 &\in I_j^2, \quad \text{if } p_3 = n-k+3,\\ p_1p_2, p_3p_2 &\in I_j^2, \quad \text{otherwise.} \end{aligned} \tag{12}$$

Any case above makes a contradiction with the construction of I_j^2 in Proposition 5.

We proceed by induction on $i \geq 3$. Assume that the induction hypothesis is true for $i-1$ with $i \geq 4$. Next we prove that $I_j^i(x)$ is an independent set of $S_{n-k+i,i}$ for each $x \in [n-k+i]$. Suppose to the contrary that $I_j^i(x)$ is not an independent set of $S_{n-k+i,i}$.

Firstly, assume that $p_1p_2\dots p_s\dots p_{i-1}x$ and $p_sp_2\dots p_1\dots p_{i-1}x$ are two adjacent vertices in $I_j^i(x)$. On one hand, suppose that $x = n-k+i$. By the construction of $I_j^i(n-k+i)$, we have

$$\begin{aligned} &p_2p_1\dots p_s\dots p_{i-1}, p_2p_s\dots p_1\dots p_{i-1} \in I_j^{i-1}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{if } s \geq 3,\\ &p_2p_1\dots p_{i-1}, p_1p_2\dots p_{i-1} \in I_j^{i-1}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{if } s = 2. \end{aligned} \tag{13}$$

The first case makes a contradiction with the result in Lemma 8, since $p_1p_3\dots p_s\dots p_{i-1}$ and $p_sp_3\dots p_1\dots p_{i-1}$ are adjacent in $S_{n-k+i-2,i-2}$. The second case makes a contradiction with I_j^{i-1} being an independent set. On the other hand, suppose that $x < n-k+i$. By the construction of $I_j^i(x)$, we have $p_1p_2\dots p_s\dots p_{i-1}, p_sp_2\dots p_1\dots p_{i-1} \in I_j^{i-1}$ if $p_t \neq n-k+i$ for each $t = 1,2,\dots,i-1$ (if $p_t = n-k+i$, for some $t \in \{1,2,\dots,i\}$, we just replace p_t by x). However, $p_1p_2\dots p_s\dots p_{i-1}$ and $p_sp_2\dots p_1\dots p_{i-1}$ are adjacent in $S_{n-k+i-1,i-1}$, a contradiction with I_j^{i-1} being an independent set.

Secondly, assume that $p_1p_2\dots p_{i-1}x$ and $p_ip_2\dots p_{i-1}x$ are two adjacent vertices in $I_j^i(x)$. If $x = n-k+i$, then $p_2p_1\dots p_{i-1}$ and $p_2p_i\dots p_{i-1}$ are two vertices in I_j^{i-1} by the construction of $I_j^i(n-k+i)$. However, $p_1\dots p_{i-1}$ and $p_i\dots p_{i-1}$ are two adjacent vertices in I_j^{i-2}, a contradiction with the result in Lemma 8. Assume that $x < n-k+i$. By the construction of $I_j^i(x)$, we have $p_1p_2\dots p_{i-1}, p_ip_2\dots p_{i-1} \in I_j^{i-1}$ if $p_t \neq n-k+i$ for each $t = 1,2,\dots,i-1,i$ (if $p_t = n-k+i$, for some $t \in \{1,2,\dots,i\}$, we just replace p_t by x). However, $p_1p_2\dots p_{i-1}$ and $p_ip_2\dots p_{i-1}$ are adjacent in $S_{n-k+i-1,i-1}$, a contradiction with I_j^{i-1} being an independent set.

By the principle of induction, this Lemma completes. □

Theorem 10. *The vertex set I_j^i is an independent set of $S_{n-k+i,i}$ for each $j \in [n-k+1]$ and $i \in \{2,3,\dots,k\}$.*

Proof. We proceed by induction on $i \geq 2$. By Proposition 5, I_j^2 is an independent set of $S_{n-k+2,2}$. Assume that the induction hypothesis is true for $i-1$ with $i \geq 3$.

Assume that I_j^{i-1} is an independent set of $S_{n-k+i-1,i-1}$. By Lemma 9, $I_j^i(x)$ is an independent set of $S_{n-k+i,i}$ for each $x \in [n-k+i]$. Suppose to the contrary that I_j^i is not independent. Assume that $p_1p_2\dots p_{i-1}p_i$ and $p_ip_2\dots p_{i-1}p_1$ are two adjacent vertices in I_j^i (this is the only possible case since $I_j^i(x)$ is an independent set of $S_{n-k+i,i}$ for each $x \in [n-k+i]$). If $p_1 = n-k+i$, then $p_ip_2\dots p_{i-1}$ and $p_2p_i\dots p_{i-1}$ should be in I_j^{i-1} by the construction of I_j^i, but they are two adjacent vertices in I_j^{i-1}, a contradiction (the case for $p_i = n-k+i$ is similar). If $p_s = n-k+i$, for some s with $2 \leq s \leq i-1$, then $p_1p_2\dots p_i\dots p_{i-1}$ (replace $n-k+i$ by p_i) and $p_ip_2\dots p_1\dots p_{i-1}$ (replace $n-k+i$ by p_1) should be in I_j^{i-1} by the construction of I_j^i, but they are two adjacent vertices in I_j^{i-1}, a contradiction. Now assume that $p_s \neq n-k+i$ for each $s \in [i]$. Then $p_1p_2\dots p_{i-1}$ and $p_ip_2\dots p_{i-1}$ should be in I_j^{i-1} by the construction of I_j^i, but they are two adjacent vertices in I_j^{i-1}, a contradiction. This Theorem completes. □

Theorem 11. *The constructed I_j^k is a maximum independent set of $S_{n,k}$ for each $j \in [n-k+1]$ and $\alpha(S_{n,k}) = n!/(n-k+1)!$. Moreover, $\{I_1^k, I_2^k, \dots, I_{n-k+1}^k\}$ is a maximum independent sets partition of $S_{n,k}$.*

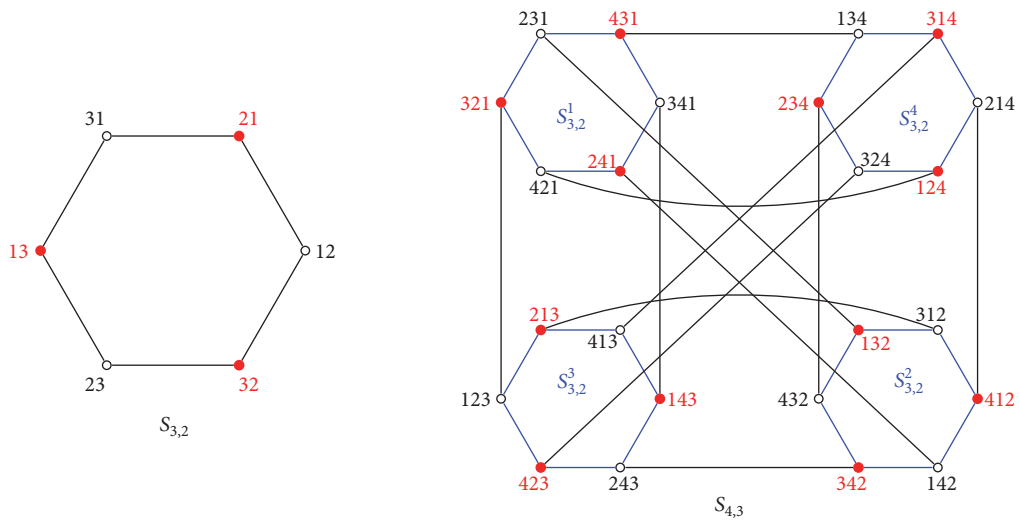

FIGURE 2: The (3, 2)-star graph $S_{3,2}$ and the (4, 3)-star graph $S_{4,3}$.

Proof. By Proposition 3, $\alpha(S_{n,k}) \leq n!/(n-k+1)!$. By Proposition 6, $I_j^k = n!/(n-k+1)!$. By Theorem 10, I_j^k is an independent set of $S_{n,k}$. Therefore, I_j^k for each $j \in [n-k+1]$ is a maximum independent set of $S_{n,k}$ and $\alpha(S_{n,k}) = n!/(n-k+1)!$. By Proposition 6, $\{I_1^k, I_2^k, \ldots, I_{n-k+1}^k\}$ is a vertex sets partition of $S_{n,k}$, so $\{I_1^k, I_2^k, \ldots, I_{n-k+1}^k\}$ is a maximum independent sets partition of $S_{n,k}$. □

Since $\{I_1^k, I_2^k, \ldots, I_{n-k+1}^k\}$ is a maximum independent sets partition of $S_{n,k}$, we immediately obtain the chromatic number of $S_{n,k}$.

Corollary 12. *The chromatic number of $S_{n,k}$ is $\chi(S_{n,k}) = n - k + 1$.*

Next, we show the maximum independent sets partition of $S_{4,3}$ by our construction.

Example 13 (see Figure 2). By Proposition 5, $I_1^2 = \{21, 32, 13\}$ and $I_2^2 = \{31, 12, 23\}$ are two maximum independent sets of $S_{3,2}$. The constructed two maximum independent sets of $S_{4,3}$ are

$$\begin{aligned} I_1^3 &= \{124, 234, 314\} \cup \{241, 321, 431\} \\ &\quad \cup \{412, 342, 132\} \cup \{213, 423, 143\} \\ &= I_1^3(4) \cup I_1^3(1) \cup I_1^3(2) \cup I_1^3(3), \\ I_2^3 &= \{134, 214, 324\} \cup \{341, 421, 231\} \\ &\quad \cup \{312, 142, 432\} \cup \{413, 123, 243\} \\ &= I_2^3(4) \cup I_2^3(1) \cup I_2^3(2) \cup I_2^3(3). \end{aligned} \tag{14}$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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References

[1] J. M. Xu, *Theory and Application of Graphs*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[2] S. Akers and B. Krishnamurthy, "A group theoretic model for symmetric interconnection networks," *IEEE Transactions on Computers*, vol. 38, no. 4, pp. 555–566.

[3] K. Day and A. Tripathi, "A comparative study of topological properties of hypercubes and star graphs," *IEEE Transactions on Parallel and Distributed Systems*, vol. 5, no. 1, pp. 31–38.

[4] W.-K. Chiang and R.-J. Chen, "The $S_{n,k}$-star graphs: a generalized star graph," *Information Processing Letters*, vol. 56, no. 5, pp. 259–264, 1995.

[5] J.-H. Chang and J. Kim, "Ring embedding in faulty $S_{n,k}$-star graphs," in *Proceedings of the 8th Int Conf Parallel and Distributed Systems*, pp. 99–106, KyongJu, Korea, 2001.

[6] Y.-Y. Chen, D.-R. Duh, T.-L. Ye, and J.-S. Fu, "Weak-vertex-pancyclicity of $S_{n,k}$-star graphs," *Theoretical Computer Science*, vol. 396, no. 1-3, pp. 191–199, 2008.

[7] E. Cheng, J. W. Grossman, L. Lipták, K. Qiu, and Z. Shen, "Distance formula and shortest paths for the $S_{n,k}$-star graphs," *Information Sciences*, vol. 180, no. 9, pp. 1671–1680, 2010.

[8] E. Cheng and M. J. Lipman, "Unidirectional $S_{n,k}$-star graphs," *Journal of Interconnection Networks*, vol. 3, no. (1-2), pp. 19–34, 2002.

[9] E. Cheng, L. Lipták, and D. E. Steffy, "Strong local diagnosability of $S_{n,k}$-star graphs and Cayley graphs generated by 2-trees with missing edges," *Information Processing Letters*, vol. 113, no. 12, pp. 452–456, 2013.

[10] W.-K. Chiang and R.-J. Chen, "Topological properties of the $S_{n,k}$-star graph," *International Journal of Foundations of Computer Science*, vol. 9, pp. 235–248, 1998.

[11] D.-R. Duh, T.-L. Chen, and Y.-L. Wang, "$(n-3)$-edge-fault-tolerant weak-pancyclicity of $S_{n,k}$-star graphs," *Theoretical Computer Science*, vol. 516, pp. 28–39, 2014.

[12] H.-C. Hsu, Y.-L. Hsieh, J. J. Tan, and L.-H. Hsu, "Fault hamiltonicity and fault hamiltonian connectivity of the $S_{n,k}$-star graphs," *Networks. An International Journal*, vol. 42, no. 4, pp. 189–201, 2003.

[13] H.-C. Hsu, C.-K. Lin, H.-M. Hung, and L.-H. Hsu, "The spanning connectivity of the $S_{n,k}$-star graphs," *International Journal of Foundations of Computer Science*, vol. 17, no. 2, pp. 415–434, 2006.

[14] X.-J. Li and J.-M. Xu, "Fault-tolerance of $S_{n,k}$-star networks," *Applied Mathematics and Computation*, vol. 248, pp. 525–530, 2014.

[15] T.-C. Lin, D.-R. Duh, and H.-C. Cheng, "Wide diameters of $S_{n,k}$-star networks," in *Proceedings of the 2nd International Conference on Computing, Communications and Control Technologies*, pp. 160–165, Austin, USA, 2004.

[16] T.-C. Lin and D.-R. Duh, "Constructing vertex-disjoint paths in $S_{n,k}$-star graphs," *Information Sciences*, vol. 178, no. 3, pp. 788–801, 2008.

[17] Z. Shen, K. Qiu, and E. Cheng, "On the surface area of the $S_{n,k}$-star graph," *Theoretical Computer Science*, vol. 410, no. 52, pp. 5481–5490, 2009.

[18] J. Wang, X. Xu, D. Zhu, L. Gao, and J.-M. Xu, "On the bounds of feedback numbers of $S_{n,k}$-star graphs," *Information Processing Letters*, vol. 112, no. 12, pp. 473–478, 2012.

[19] Y. Xiang and I. A. Stewart, "One-to-many node-disjoint paths in $S_{n,k}$-star graphs," *Discrete Applied Mathematics: The Journal of Combinatorial Algorithms, Informatics and Computational Sciences*, vol. 158, no. 1, pp. 62–70, 2010.

[20] W. Yang, H. Li, and X. Guo, "A kind of conditional fault tolerance of $S_{n,k}$-star graphs," *Information Processing Letters*, vol. 110, no. 22, pp. 1007–1011, 2010.

[21] S. Zhou, "The conditional fault diagnosability of $S_{n,k}$-star graphs," *Applied Mathematics and Computation*, vol. 218, no. 19, pp. 9742–9749, 2012.