Research Article

Multiplicity Results to a Conformable Fractional Differential Equations Involving Integral Boundary Condition

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In this article, by using topological degree theory coupled with the method of lower and upper solutions, we study the existence of at least three solutions to Riemann-Stieltjes integral initial value problem of the type

\[ D_\alpha x(t) = f(t, x), \quad t \in [0,1], \]

\[ x(0) = \int_0^1 x(t) dA(t), \]

where \( D_\alpha x(t) \) is the standard conformable fractional derivative of order \( \alpha \), \( 0 < \alpha \leq 1 \), and \( f \in C([0, 1] \times \mathbb{R}, \mathbb{R}) \). Simultaneously, the fixed point theorem for set-valued increasing operator is applied when considering the given problem.

1. Introduction

In recent years, fractional differential equations have exerted tremendous influence on some mathematical models of research processes and phenomena in many fields such as electrochemistry, heat conduction, underground water flow, and porous media. A growing number of papers deal with the existence or multiplicity of solutions of initial value problem and boundary value problem for fractional differential equations [1–11]. Recently, the authors [12] give an interesting fractional derivative called the “conformal fractional derivative”, which depends on the limit definition of the function derivative. Moreover, readers can find in [13] the properties of conformable fractional derivatives that are similar to ordinary differential ones. Other related work on conformable fractional differential equation can be found in [12–17] and the references therein.

Whether it is ordinary differential equations or fractional differential equations, the existence of solutions to boundary value problems with integral boundary conditions has been studied in the applied sciences and physics. For more important content of the typical theory of differential equations and integral equations with boundary value problems are obtained [18, 19]. Lots of results have been established for differential equations and differential systems with integral boundary conditions by using upper and lower solution, fixed point theory and fixed point theorem; see [20–28].

Now there are more and more articles to prove that there are multiple solutions for integral boundary [29–35]. For example, [31] introduced the system of fractional differential equations

\[ D_\alpha^n u(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \quad n - 1 < \alpha \leq n, \]

\[ D_\beta^m v(t) + f(t, v(t)) = 0, \quad t \in (0, 1), \quad m - 1 < \beta \leq m, \]  

(1)

where \( n, m \in \mathbb{N} \), \( n, m \geq 2 \), \( D_\alpha^n \), \( D_\beta^m \) are the standard the Riemann-Liouville derivatives of orders \( \alpha \), \( \beta \). By means of the Guo-Krasnoselskii fixed point theorem and the property of degree to obtain the existence and multiplicity of positive solutions to integral boundary value conditions

\[ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \]

\[ u(1) = \int_0^1 u(s) dH(s). \]  

(2)

and

\[ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \]

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Lemma 3

where \( H, K : [0,1] \rightarrow \mathbb{R} \) are nondecreasing functions.

Inspired by the above work, we consider the existence and multiple solutions of the following fractional differential equation involving integral boundary condition:

\[
D_\alpha x(t) = f(t, x(t)), \quad t \in [0,1],
\]

\[
x(0) = \int_0^1 x(t) \, dA(t),
\]

where \( f \in C([0,1] \times \mathbb{R}, \mathbb{R}) \), \( \int_0^1 x(t) \, dA(t) \) denotes the Riemann-Stieltjes integral with positive Stieltjes measure. \( D_\alpha f(t) \) is the standard conformable fractional derivative of order \( 0 < \alpha \leq 1 \) of \( f \) at \( t > 0 \), defined by

\[
D_\alpha f(t) = \lim_{\rho \to 0} \frac{f(t + \rho t^{1-\alpha}) - f(t)}{\rho}, \quad t \in (0,1].
\]

If \( D_\alpha f(t) \) exists on \((0,1)\), then \( D_\alpha f(0) = \lim_{x \to 0} D_\alpha f(x) \). Obviously, \( D_\alpha f(t) = t^{1-\alpha} f'(t) \) if \( f \) is differentiable.

In order to study the existence and multiplicity results of the problem (4), this paper is arranged as follows: after this introduction, in Section 2, we briefly show some necessary definitions and lemmas that are used to prove our main results. In Section 3, we shall define a modified bounded function to discuss the existence of solutions for a conformable fractional differential equation with initial value condition. Moreover, in this part we employ two pairs of upper and lower solutions, which are Schauder’s fixed point theorem and the fixed point theorem for set-valued increasing operator, respectively. In addition, using two pairs of upper and lower solutions and the property of degree theory, we deduce that problem (4) has at least three solutions.

2. Preliminaries

In the following, let \( E = C[0,1] \), then \( E \) is a Banach space with the norm \( \| x \| = \max_{x \in [0,1]} |x(t)| \).

Definition 1 (see [12]). Let \( \alpha \in (0,1] \); the conformable fractional integral starting from a point \( 0 \) of a function \( f : [0, +\infty) \rightarrow \mathbb{R} \) of order \( \alpha \) is defined as

\[
I_\alpha f(t) = \int_0^t s^{\alpha-1} f(s) \, ds.
\]

Lemma 2 (see [14]). Let \( f : (0, +\infty) \rightarrow \mathbb{R} \) be differentiable and \( 0 < \alpha \leq 1 \). Then, for all \( t > 0 \) we have

\[
I_\alpha D_\alpha f(t) = f(t) - f(0).
\]

Lemma 3 (see [15]). Let \( \alpha \in (0,1], k_1, k_2, c, k \in \mathbb{R} \), and the function \( f, g \) be \( \alpha \)-differentiable on \([0, +\infty)\); then

(i) \( D_\alpha c = 0 \) for all constant function \( f(t) = c \);

(ii) \( D_\alpha t^k = kt^{\alpha-1} \).

(iii) \( D_\alpha (k_1 f(t) + k_2 g(t)) = k_1 D_\alpha f(t) + k_2 D_\alpha g(t) \);

(iv) \( D_\alpha (f g) = f(t) D_\alpha g(t) + g(t) D_\alpha f(t) \);

(v) \( D_\alpha (f/g) = (g(t) D_\alpha f(t) - f(t) D_\alpha g(t))/(g(t))^2 \) for all function \( h(t) \neq 0 \).

Lemma 4 ([15] (mean value theorem)). Let an interval \([a, b] \subset [0, +\infty)\), and the function \( f : [0, +\infty) \rightarrow \mathbb{R} \), if it satisfies

(1) \( f \) is continuous on \([a, b] \);

(2) \( f \) is \( \alpha \)-differentiable for some \( \alpha \in (0,1] \) on \([a, b] \).

Then there exists a constant \( \xi \in (a, b) \), such that \( D_\alpha f(\xi) = (f(b) - f(a))/((1/\alpha)b^\alpha - (1/\alpha)a^\alpha) \).

Definition 5. A function \( v \in C([0,1], \mathbb{R}) \) is called a lower solution of problem (4), if it satisfies

\[
D_\alpha v(t) \leq f(t, v(t)), \quad t \in [0,1],
\]

\[
v(0) \leq \int_0^1 v(t) \, dA(t).
\]

If inequalities (8), (9) are reversed, then \( v \) is an upper solution of problem (4).

Definition 6. A function \( v \in C([0,1], \mathbb{R}) \) is said to be a strict lower solution of problem (4), if the inequality (8), (9) is strict for \( t \in [0,1] \).

Lemma 7 (see [13]). \( f : [a, b] \rightarrow \mathbb{R} \) is a given function that satisfies

(i) \( f \) is continuous on \([a, b] \);

(ii) \( f \) is \( \alpha \)-differentiable for some \( \alpha \in (0,1] \).

Then we have the following:

(1) \( \text{if } D_\alpha f(x) \geq 0 \text{ for all } x \in (a, b), \text{ then } f \text{ is increasing on } [a, b] \);

(2) \( \text{if } D_\alpha f(x) \leq 0 \text{ for all } x \in (a, b), \text{ then } f \text{ is decreasing on } [a, b] \).

The following lemma is a direct consequence of the application of the definition of conformable fractional derivative and Lemma 7.

Lemma 8. Let \( \alpha \in (0,1] \); the function \( f(t) \) is continuous and \( \alpha \)-differentiable on \([a, b] \).

(1) \( \text{if } f \text{ has an extreme value at } t_0 \in (a, b), \text{ then } D_\alpha f(t_0) = 0 \);

(2) \( \text{if } f \text{ has a maximum (minimum) value at } t_0 = a, \text{ then } D_\alpha f(t_0) \geq 0 \) \( (D_\alpha f(t_0) \leq 0) \);

(3) \( \text{if } f \text{ has a maximum (minimum) value at } t_0 = b, \text{ then } D_\alpha f(t_0) \leq 0 \) \( (D_\alpha f(t_0) \geq 0) \).
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Theorem 9 ([36, 37] (fixed point theorem for set-valued increasing operator)). Let $X$ be a partially ordered set, $M$ be a nonempty closed set of $X$, and $L : M \rightarrow 2^M$ be a set-valued increasing operator. Assume that

(i) any totally ordered subset of $M$ is a relatively compact,
(ii) for all $x \in M$, $Lx$ is a compact set in $X$,
(iii) there exist $x_0 \in M$ and $u \in Lx_0$ such that $x_0 \leq u$.

Then $L$ has a fixed point in $M$; that is, there exists $x^* \in M$ such that $x^* \in Lx^*$.

Throughout the paper, we list some hypotheses.

$(H_1)$ $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

$(H_2)$ $\int_0^1 dA(t) > 0$, $A(t)$ is continuous at $t = 0$ and is nondecreasing at $[0, 1]$.

$(H_3)$ Assume that $v, w \in C[0, 1]$ is lower and upper solution of problem (4) with $v(t) \leq w(t)$.

3. Main Results

Based on the above preparations and the assumptions mentioned, we first consider the following initial value problem (IVP)

$$D_α^nx(t) = f(t, x(t)), \quad t \in [0, 1],$$
$$x(0) = a,$$

where $a \in \mathbb{R}$. We suppose that $(H_i)$ holds. Applying Lemma 2, it is easy to prove that (12) is equivalent to the following integral equation

$$x(t) = \int_0^t sα−1 f(s, x(s)) ds + a.$$  \hfill (13)

Define integral operator $T : E \rightarrow E$ by

$$(Tx)(t) = \int_0^t sα−1 f(s, x(s)) ds + a.$$  \hfill (14)

Then, $x$ is a solution of (12) if and only if $x \in E$ is a solution of the operator equation $(I − T)x = 0$, that is, a fixed point of operator $T$.

Theorem 10. Assume that $(H_4)$-$(H_5)$ hold. Then for a fixed number $a \in [v(0), w(0)]$, there exists a solution $x(t)$ of problem (12) such that $v(t) \leq x(t) \leq w(t)$, $t \in [0, 1]$.

Proof. Firstly we define the following modified function:

$$F_{vw}(t, x) = \begin{cases} f(t, x(t)) + \frac{x - w(t)}{1 + |x - w(t)|}, & x > w(t), \\ f(t, x(t)), & v(t) \leq x(t) \leq w(t), \\ f(t, v(t)) + \frac{v(t) - x}{1 + |v(t) - x|}, & x < v(t). \end{cases}$$  \hfill (15)

Obviously $F_{vw}$ is continuous and bounded on $[0, 1] \times \mathbb{R}$, so there exists $M$ such that $\max\{|a| + 1, |F_{vw}(t, x)|\} \leq M$.

Secondly, we consider the modified initial value problem

$$\begin{cases} D_α^nx(t) = F_{vw}(t, x(t)), & t \in [0, 1], \\ x(0) = a. \end{cases}$$  \hfill (16)

The above initial value problem (16) is equivalent to the following integral equation:

$$x(t) = \int_0^t sα−1 F_{vw}(s, x(s)) ds + a.$$  \hfill (17)

For any $x \in E$, we have

$$\left| \int_0^t sα−1 F_{vw}(s, x(s)) ds + a \right| \leq M \int_0^t sα−1 ds + |a| < M \frac{1}{α} + M = \frac{M(1 + α)}{α}.$$  \hfill (18)

Choose $M_1 > \frac{M(1 + α)}{α}$, and let

$$\Omega = \{x \in E : \|x\| < M_1\}.$$  \hfill (19)

Obviously, $\Omega$ is a bounded convex subset in $E$. From the above argument, the operator $\overline{T} : \Omega \rightarrow \Omega$ defined as follows is well defined,

$$\overline{T}x(t) = \int_0^t sα−1 F_{vw}(s, x(s)) ds + a.$$  \hfill (20)

It is easy to verify that $\overline{T} : \Omega \rightarrow \Omega$ is compact. By Schauder’s fixed point theorem, $\overline{T}$ has a fixed point $x$ in $\Omega$. Subsequently, $x$ is the solution of problem (16). Also, by the definition of $F_{vw}$ and Definition 5, we have

$$F_{vw}(t, v(t)) = f(t, v(t)) \geq D_αv(t), \quad t \in [0, 1],$$
$$v(0) \geq a,$$  \hfill (21)

and

$$F_{vw}(t, w(t)) = f(t, w(t)) \leq D_αw(t), \quad t \in [0, 1],$$
$$w(0) \leq a.$$  \hfill (22)

Hence $v$ and $w$ are lower and upper solutions of (16).

Our final job is to apply the method in [38] to illustrate that if (16) has upper and lower solutions, hence any solution $x$ of differential equation (16) must satisfy $v(t) \leq x(t) \leq w(t)$, $t \in [0, 1]$ and then is a solution of (12).

We now prove that any solution $x$ of (16) does satisfy

$$v(t) \leq x(t) \leq w(t), \quad t \in [0, 1].$$  \hfill (23)

We need to show that $v(t) \leq x(t), t \in [0, 1]$. Analogously we can also show that $x(t) \leq w(t)$. If $v(t) \notin x(t)$ on $[0, 1]$, let $h(t) = v(t) - x(t)$, then $h(t)$ has a positive maximum at some $t_0 \in [0, 1]$. By the initial boundary conditions, we derive $h(0) = v(0) - x(0) \leq 0$, so $t_0 \neq 0$. Then there will be two cases:

(i) $t_0 \in (0, 1)$, then by Lemma 8, we have $h(t_0) > 0$, that is $v(t_0) > x(t_0)$ and

$$D_αh(t_0) = 0;$$  \hfill (24)

(ii) $t_0 = 0$, then by Lemma 8, we have $h(t_0) = 0$, that is $v(t_0) = x(t_0)$. We complete the proof for (12).
(ii) \( t_0 = 1 \), then by Lemma 8, we have \( h(1) > 0 \) and \( D_h h(1) \geq 0 \).

For case (i), by the definition of \( F_{v,w} \) and Definition 5, we can get

\[
D_h h(t_0) = D_v v(t_0) - D_x x(t_0)
\]
\[
\leq f(t_0, v(t_0)) - F_{v,w} (t_0, x(t_0))
\]
\[
= f(t_0, v(t_0))
\]
\[
- \left[ f(t_0, v(t_0)) + \frac{v(t_0) - x(t_0)}{1 + |x(t_0) - v(t_0)|} \right] < 0,
\]

which contradicts with (24). This implies that \( h(t) \) has no positive local maximum on \([0,1]\).

For case (ii), if \( t_0 = 1 \), then \( h(1) > 0, h(0) \leq 0 \). Notice that \( h(t) \) has no positive local maximum on \([0,1]\), and there exists \( t_1 \in [0,1] \) such that

\[
h(t_1) = 0, \quad h(t) \leq 0, \quad t \in [0, t_1],
\]

and

\[
D_h h(t) \geq 0, \quad h(t) \geq 0, \quad t \in (t_1, 1).
\]

But for every \( t \in (t_1, 1) \), we have

\[
D_h h(t) = D_v v(t) - D_x x(t)
\]
\[
\leq f(t, v(t)) - F_{v,w} (t, x(t))
\]
\[
= f(t, v(t))
\]
\[
- \left[ f(t, v(t)) + \frac{v(t) - x(t)}{1 + |x(t) - v(t)|} \right] < 0,
\]

again a contradiction. We complete the proof. \( \square \)

**Theorem 11.** Assume that \((H_1)-(H_3)\) hold, then problem (4) has a solution \( x(t) \) with \( v(t) \leq x(t) \leq w(t) \), \( \forall t \in [0,1] \).

**Proof.** It follows from Theorem 10 that for any fixed \( a \in [v(0), w(0)] \), IVP (12) have a solution \( x(t) \); we can also mark it as \( x_a(t) \) satisfies \( v(t) \leq x_a(t) \leq w(t), t \in [0,1] \). The following two cases are considered:

(i) \( v(0) = 0 = a \). It is easy to obtain that \( \int_0^1 v(t) \, dA(t) \leq \int_0^1 x_a(t) \, dA(t) \leq \int_0^1 w(t) \, dA(t) \). Using Definition 5, we can conclude that

\[
\int_0^1 x_a(t) \, dA(t) \leq \int_0^1 w(t) \, dA(t) \leq w(0) = x_a(0)
\]
\[
= v(0) \leq \int_0^1 v(t) \, dA(t)
\]
\[
\leq \int_0^1 x_a(t) \, dA(t),
\]

i.e., \( x_a(0) = \int_0^1 x_a(t) \, dA(t) \), then \( x_a(t) \) is the solution of problem (4).

(ii) \( v(0) < w(0) \). Define the set-valued operator on \([v(0), w(0)]\) as follows:

\[
L(a) = \left\{ \int_0^1 x_a(t) \, dA(t) : x_a(t) \right\}
\]

is the solution of IVP (12).

It can be seen from Theorem 10 that for any fixed \( a \in [v(0), w(0)] \), \( L(a) \subset \int_0^1 v(t) \, dA(t) \subset \int_0^1 w(t) \, dA(t) \subset [v(0), w(0)] \), this shows that \( L : [v(0), w(0)] \rightarrow 2^{[v(0), w(0)]} \). Moreover, from the definition of \( L \), we can obtain that (4) has a solution \( x(t) \) with \( v(t) \leq x(t) \leq w(t) \), \( \forall t \in [0,1] \) if \( L \) has a fixed point. In order to apply the fixed point theorem for set-valued increasing operator, we shall prove the following:

**Step 1.** The sequence \( \{x_n\} \) of \( L(a) \) must have subsequence \( \{x_{n_k}\} \) converging to \( x^* \in L(a) \) for \( \forall a \in [v(0), w(0)] \).

Assume that \( x_{n_k} \) converges to \( x^* \). By using Theorem 10, we know that the solution \( x_{n_k}(t) \) of (12) related to \( x_{n_k} \) satisfies \( v(t) \leq x_{n_k}(t) \leq w(t) \), that is to say \( \{x_{n_k}(t)\} \) is uniformly bounded. Moreover, by the continuity of \( f \), we conclude that \( |f(t, x_{n_k}(t))| \) is bounded on \([0,1]\) for all \( n \). Let \( t_1, t_2 \in [0,1] \) with \( t_1 < t_2 \). Using Lemma 4, we can conclude that

\[
|x_{n_k}(t_1) - x_{n_k}(t_2)| \leq \frac{1}{a} |D_a x_{n_k}(\xi)| |t_1 - t_2|
\]
\[
\leq \frac{1}{a} |f(\xi, x_{n_k}(\xi))| |t_1 - t_2|,
\]

\( \xi \in (t_1, t_2) \).

Therefore, \( \{x_{n_k}(t)\} \) is equicontinuous. It means that \( \{x_{n_k}(t)\} \) is completely continuous sequences; by Arzela-Ascoli theorem, we get \( \{x_{n_k}(t)\} \) has subsequences and \( \{x_{n_{k_n}}\} \rightarrow x^* \in E \) when \( k \rightarrow \infty \). Note that \( x_{n_{k_n}} \) satisfy

\[
x_{n_k}(t) = \int_0^t s^{a-1} f(s, x_{n_k}(s)) \, ds + a, \quad t \in [0,1],
\]

\[
z_{n_k} = \int_0^1 x_{n_k}(t) \, dA(t),
\]

Letting \( k \rightarrow \infty \), we obtain that

\[
x^*(t) = \int_0^t s^{a-1} f(s, x^*(s)) \, ds + a, \quad t \in [0,1],
\]

\[
z^* = \int_0^1 x^*(t) \, dA(t),
\]

This shows that \( x^*(t) \) is a solution of problem (4); hence, \( x^* \in L(a) \).

**Step 2.** \( L \) is set-valued increasing operator, that is, \( \forall a, b \in [v(0), w(0)], a \leq b \) and \( \forall c \in L(a), \exists \delta \in L(b) \) such that \( c \leq \delta \). Without loss of generality, we assume that \( a < b \), for \( \forall c \in L(a) \) such that \( c = \int_0^1 x_a(t) \, dA(t) \) and \( v(t) \leq x_a(t) \leq w(t) \), where \( x_a(t) \) is the solution of (16) related to \( a \). Therefore,
It is easy to obtain that $x_a$, $w$ are the lower and upper solutions of initial value problem (34). By the help of Theorem 10, we deduce that problem (34) has existence solution $x_b(t)$ with $x_a(t) \leq x_b(t) \leq w(t)$. Subsequently, $\int_0^1 x_a(t) dA(t) \leq \int_0^1 x_b(t) dA(t)$. Let $d = \int_0^1 x_b(t) dA(t)$, then $c \leq d$. So the theorem is now proved.

Above we have studied the solution of (4), and then we present the main results of this paper.

**Theorem 12.** Suppose that $(H_1)$, $(H_2)$ and the following conditions hold.

$(H_1)$ $v_1, v_2 \in E$ are two lower solutions and $w_1, w_2 \in E$ are two upper solutions of (4) such that $v_1 \leq v_2 \leq w_2, v_1 \leq w_1 \leq w_2$, and $v_2 \notin w_1$.

$(H_2)$ $v_2, w_1$ are strict lower and upper solutions of problem (4).

$(H_3)$ $k_1 \neq 0$, where $k_1 = 1 - \int_0^1 dA(t)$.

Then problem (4) has at least three solutions $x_i (i = 1, 2, 3)$, such that $v_1 \leq x_1 \leq w_1, v_2 \leq x_2 \leq w_2, x_2 \notin w_1$, and $x_2 \notin w_2$ on $[0,1]$.

**Proof.** Similar to Theorem 10, we define the following modified function $F_{v_1 w_2}$:

$$F_{v_1 w_2}(t, x) = \begin{cases} f(t, w_2(t)) + \frac{x - w_2(t)}{1 + |x - w_2(t)|}, & x > w_2(t), \\ f(t, x(t)), & v_1(t) \leq x(t) \leq w_2(t), \\ f(t, v_1(t)) + \frac{v_1(t) - x}{1 + |v_1(t) - x|}, & x < v_1(t). \\ \end{cases}$$  \hspace{1cm} (35)

Obviously $F_{v_1 w_2}$ is continuous and bounded on $[0,1] \times \mathbb{R}$, so there exists $M > 0$ such that $|F_{v_1 w_2}(t, x)| \leq M$.

Let $M_2 > (M/\alpha)(1 + k_2/|k_1|)$, where $k_2 = \int_0^1 t^\alpha dA(t)$. We consider the modified boundary value problem

$$D_0^\alpha x(t) = F_{v_1 w_2}(t, x), \quad t \in [0,1],$$  \hspace{1cm} (36)

$$x(0) = \int_0^1 x(t) dA(t).$$

It is clear that the solution $x(t)$ of the above problem satisfies $v_1(t) \leq x(t) \leq w_2(t)$ and then is a solution of problem (4).

By Lemma 2, it is easy to know that (36) is equivalent to the following integral equation:

$$x(t) = \int_0^t s^{-1} F_{v_1 w_2}(s, x(s)) ds + \int_0^1 x(t) dA(t).$$  \hspace{1cm} (37)

Integrating (37) with respect to $A(t)$ form 0 to 1,

$$\int_0^1 x(t) dA(t) = \int_0^1 \int_0^1 s^{-1} F_{v_1 w_2}(s, x(s)) ds dA(t) + (1 - k_1) \int_0^1 x(t) dA(t).$$  \hspace{1cm} (38)

Combining with (37) and (38), we reduce (36) to the following equivalent integral equation:

$$x(t) = \int_0^t s^{-1} F_{v_1 w_2}(s, x(s)) ds + k_1^{-1} \int_0^1 \int_0^t s^{-1} F_{v_1 w_2}(s, x(s)) ds dA(t).$$  \hspace{1cm} (39)

Now, we prove the existence of at least three solutions of the problem (36). Note that $\Omega_1 = \{ x \in C[0,1] : ||x|| < M_2 \}$ is a bounded convex subset in $C[0,1]$. Define the following operator $T_1 : E \rightarrow E$

$$(T_1 x)(t) = \int_0^t s^{-1} F_{v_1 w_2}(s, x(s)) ds + k_1^{-1} \int_0^1 \int_0^t s^{-1} F_{v_1 w_2}(s, x(s)) ds dA(t).$$

Then $x$ is a solution of (36) if and only if $x \in E$ is a solution of the equation $(I - T_1)x = 0$, that is, a fixed point of $T_1$. For $x \in \Omega_1$, we get

$$[T_1 (x)(t)]$$

$$\leq \int_0^t s^{-1} |F_{v_1 w_2}(s, x(s))| ds + k_1^{-1} \int_0^1 \int_0^t s^{-1} |F_{v_1 w_2}(s, x(s))| ds dA(t)$$

$$\leq M_2 \left( 1 + \frac{k_2}{|k_1|} \right) < M_2.$$  \hspace{1cm} (41)

So $T_1(\Omega_1) \subset \Omega_1$ and $T_1 : \Omega_1 \rightarrow \Omega_1$ is compact. By the topological degree theory, we have

$$d(I - T_1, \Omega_1, \theta) = 1.$$  \hspace{1cm} (42)

Let

$$\Omega_{v_2} = \{ x \in \Omega_1 : x > v_2 \}$$  \hspace{1cm} (43)

and

$$\Omega_{w_1} = \{ x \in \Omega_1 : x < w_1 \}.$$  \hspace{1cm} (44)

It follows from the fact $v_2 \notin w_1$ that

$$\Omega_{w_1} \neq \emptyset \neq \Omega_{v_2},$$

$$\Omega_{v_2} \cap \Omega_{w_1} = \emptyset,$$

then $\Omega_1 \setminus (\Omega_{v_2} \cup \Omega_{w_1}) \neq \emptyset.$
Next, by $(H_2)$, we conclude that (36) has no solution on $\partial \Omega_{\nu_2} \cup \partial \Omega^{w_1}$. Without loss of generality, we assume that (36) has a solution $\tilde{x}(t)$ on $\partial \Omega_{\nu_2}$. Thus there exists $t_0 \in [0, 1]$ such that $\tilde{x}(t_0) = \nu_2(t_0)$ and $\tilde{x}(t) \geq \nu_2(t)$ for $t \in [0, 1]$. It follows from Lemma 8 that $D_{\alpha}^\nu \tilde{x}(t_0) - D_{\alpha}^\nu \nu_2(t_0) \leq 0$ when $t_0 \in (0, 1]$. But we noticed that $\nu_2$ is a strict lower solution of (36), thus

$$D_{\alpha}^\nu \tilde{x}(t_0) - D_{\alpha}^\nu \nu_2(t_0) > F_{\nu_{w_1}}(t_0, \tilde{x}(t_0)) - F_{\nu_{w_1}}(t_0, \nu_2(t_0))$$

$$= F_{\nu_{w_1}}(t_0, \tilde{x}(t_0)) - F_{\nu_{w_1}}(t_0, \tilde{x}(t_0)) - F_{\nu_{w_1}}(t_0, \tilde{x}(t_0)) = 0.$$

This is a contradiction. If $t_0 = 1$ and $\tilde{x}(t) > \nu_2(t)$ for $t \in (0, 1]$, from the boundary condition and $(H_2)$, we have

$$0 = \tilde{x}(0) - \nu_2(0) \geq \int_0^1 (\tilde{x}(t) - \nu_2(t)) \, dA(t) > 0. \quad (47)$$

This is a contradiction again. So (36) has no solution on $\partial \Omega_{\nu_2} \cup \partial \Omega^{w_1}$.

The additivity of degree implies that

$$d(I - T_1, \Omega_{\nu_2}, \theta) = d(I - T_1, \Omega_{w_1}, \theta) + d(I - T_1, \Omega_{\nu_2} \cup \Omega_{w_1}, \theta)$$

which we deduce that

$$d(I - T_1, \Omega_{\nu_2}, \theta) = 1. \quad (50)$$

and there are solutions in $\Omega_{\nu_2}, \Omega_{w_1}, \Omega \setminus (\Omega_{\nu_2} \cup \Omega_{w_1})$, respectively.

Now we show that $d(I - T_1, \Omega_{\nu_2}, \theta) = d(I - T_1, \Omega_{w_1}, \theta) = 1$.

Firstly, we show that $d(I - T_1, \Omega_{\nu_2}, \theta) = 1$.

Define functions $F_{\nu_{w_1}}$, in a similar way, and we consider the modified boundary value problem

$$D_{\alpha} x(t) = F_{\nu_{w_1}}(t, x), \quad t \in [0, 1],$$

$$x(0) = \int_0^1 x(t) \, dA(t). \quad (51)$$

Equation (51) is equivalent to the operator equation $(I - T_2)x = 0$, where

$$(T_2 x)(t) = \int_0^t s^{a-1} F_{\nu_{w_1}}(s, x(s)) \, ds$$

$$+ k_1 \int_0^1 \int_0^t s^{a-1} F_{\nu_{w_1}}(s, x(s)) \, ds \, dA(t). \quad (52)$$

Similar to the above argument, we can conclude that any solution $x(t)$ of (51) satisfies $x(t) \geq \nu_2(t)$, which in view of $(H_2)$ leads to $x(t) \geq \nu_2(t)$ on $(0, 1)$; therefore,

$$x \in \Omega_{\nu_2}.$$

Moreover, since $T_2(\Omega_{\nu_2}) \subseteq \Omega_1$, we have

$$d(I - T_2, \Omega_{\nu_2}, \theta) = 1. \quad (54)$$

It follows from (53) and (54) that

$$1 = d(I - T_2, \Omega_{\nu_2}, \theta) = d(I - T_2, \Omega_{w_1}, \theta) + d(I - T_2, \Omega_{\nu_2} \setminus \Omega_{w_1}, \theta) \quad (55)$$

Taking into account of $F_{\nu_{w_1}} = F_{\nu_{w_2}}$ on $\Omega_{\nu_2}$ we deduce that

$$d(I - T_1, \Omega_{\nu_2}, \theta) = d(I - T_2, \Omega_{\nu_2}, \theta) = 1 \quad (56)$$

Similarly, we can show that $d(I - T_1, \Omega_{w_1}, \theta) = 1$. Therefore, there exist three solutions $x_1 \in \Omega_{\nu_2}, x_2 \in \Omega_{w_1}, x_3 \in \Omega \setminus (\Omega_{\nu_2} \cup \Omega_{w_1})$ of problem (4). The proof is finished.

Data Availability

The datasets used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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