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Research Article

Dynamical Analysis of Two-Microorganism and Single Nutrient Stochastic Chemostat Model with Monod-Haldane Response Function

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In this paper, we formulate and investigate a two-microorganism and single nutrient chemostat model with Monod-Haldane response function and random perturbation. First, for the corresponding deterministic system, we introduce the conditions of the stability of the equilibrium points. Then, using Lyapunov function and Itô's formula, we investigate the existence and uniqueness of the global positive solution of the stochastic chemostat model. Furthermore, we explore and obtain the criterions of the extinction and the permanence for the stochastic model. Finally, numerical simulations are carried out to illustrate our main results.

1. Introduction

The chemostat is a classic bioreactor for continuous microbial culture. It has been widely used in microbiology and bioengineering [1–3]. The device is mainly composed of three parts: nutrient bottle, incubator, and collector, generally by using pump or an overflow device to keep the chemostat volume constant. Chemostat model has attracted great interests of many scholars since Monod [4]. Monod [5], Novick et al. [6], and Herbert et al. [7] considered the basic theory of microbial interaction in the chemostat. The simple chemostat model described by ordinary differential equations (ODEs) takes the following form:

$$\begin{aligned} \frac{\mathrm{d}S(t)}{\mathrm{d}t} &= \left(S^0 - S(t)\right)D - \frac{mS(t)x(t)}{a+S(t)}, \\ \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \left(\frac{mS(t)}{a+S(t)} - D\right)x(t), \end{aligned} \tag{1}$$

where $S(t)$ and $x(t)$ stand for the concentrations of the nutrient and the microorganism at time t, respectively. S^0 and D are positive constants, which represent the input concentration of the nutrient and the common washout rate respectively. The function $mS(t)/(a+S(t))$ denotes the Monod growth functional response, m is called the maximal growth rate, and a is the half-saturation (or Michaelis-Menten) constant. Based on the above model, many scholars have studied the chemostat model with different functional response in which a single microbe feeds on a single nutrient [8–11].

Taylor and Williams [12] considered the coexistence of two different microbe populations feeding on single nutrient in the chemostat and proposed the following model:

$$\begin{aligned} \frac{\mathrm{d}S(t)}{\mathrm{d}t} &= D\left(S^0 - S(t)\right) - \frac{m_1S(t)x(t)}{a_1+S(t)} - \frac{m_2S(t)y(t)}{a_2+S(t)}, \\ \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \frac{m_1S(t)x(t)}{a_1+S(t)} - Dx(t), \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= \frac{m_2S(t)y(t)}{a_2+S(t)} - Dy(t), \end{aligned} \tag{2}$$

where $x(t)$ and $y(t)$ denote the concentrations of two different microorganisms at time t. And then many scholars have investigated the competition of microorganisms in chemostat [13–16].

Taking into account more complex inhibitory effects of high substrate concentrations on microbial growth, for example, nitrite and ammonia can lead to the inhibition of Nitrobacter and Nitrosomonas, respectively. Andrews [17] proposed the following nonmonotonic response function called Monod-Haldane growth rate (inhibitory rate):

$$\mu(S(t)) = \frac{mS(t)}{a + S(t) + KS^2(t)}, \tag{3}$$

where $m > 0$ is the maximal growth rate and $a > 0$ is the Michaelis-Menten constant, assuming that the term $KS^2(t)$ is an inhibitor and K is a half-saturation parameter. Based on [17], Bush and Cook [18] improved the inhibition function to a general functional response which retains the particular features of the Andrews function. Recently, Wang et al. [19] proposed and investigated the stochastic dynamic behaviors of a stochastic chemostat model with Monod-Haldane response function in which a single microbe feeds on a single nutrient.

Considering the nutritional substitution, Chi and Zhao [20] established a single microorganism and multinutrient chemostat model with impulsive toxicant input in a polluted environment as follows:

$$\begin{aligned}
\mathrm{d}S_1(t) &= \left(Q\left(S_{10} - S_1(t)\right) - \frac{\mu_1 S_1(t)\,x(t)}{\delta_1\left(a_1 + x(t)\right)}\right)\mathrm{d}t \\
&\quad - \frac{\sigma_1 S_1(t)\,x(t)}{\delta_1\left(a_1 + x(t)\right)}\mathrm{d}B_1(t), \\
\mathrm{d}S_2(t) &= \left(Q\left(S_{20} - S_2(t)\right) - \frac{\mu_2 S_2(t)\,x(t)}{\delta_2\left(a_2 + x(t)\right)}\right)\mathrm{d}t \\
&\quad - \frac{\sigma_2 S_2(t)\,x(t)}{\delta_2\left(a_2 + x(t)\right)}\mathrm{d}B_2(t), \\
\mathrm{d}x(t) &= \left(\frac{\mu_1 S_1(t)\,x(t)}{a_1 + x(t)} + \frac{\mu_2 S_2(t)\,x(t)}{a_2 + x(t)} - Qx(t)\right. \\
&\quad \left. - rP_0(t)\,x(t)\right)\mathrm{d}t + \frac{\sigma_1 S_1(t)\,x(t)}{a_1 + x(t)}\mathrm{d}B_1(t) \\
&\quad + \frac{\sigma_2 S_2(t)\,x(t)}{a_2 + x(t)}\mathrm{d}B_2(t), \\
\mathrm{d}P_0(t) &= \left(kP_e(t) - gP_0(t) - mP_0(t)\right)\mathrm{d}t, \\
\mathrm{d}P_e(t) &= -hP_e(t)\,\mathrm{d}t, \\
&\qquad t \neq nT,\ n \in Z^+, \\
\Delta S_1(t) &= 0, \\
\Delta S_2(t) &= 0, \\
\Delta x(t) &= 0, \\
\Delta P_0(t) &= 0, \\
\Delta P_e(t) &= u, \\
&\qquad t = nT,\ n \in Z^+.
\end{aligned} \tag{4}$$

In the above model, a single species ($x(t)$) feeds on two substitutable resources ($S_1(t)$ and $S_2(t)$) in a polluted environment. For the system without stochastic effect, by using the theory of impulsive differential equations, the authors proved that the model system has a globally asymptotically stable 'microorganism-extinction' periodic solution for $R_1 < 1$ and the system is permanent for $R_2 > 1$. And for the system with stochastic effect, by using the theory of stochastic differential equations, the authors obtained the conditions for the persistence and extinction of microorganisms. Their results showed that stochastic disturbance and toxicant can affect the survival of microorganism.

While, in reality, there are generally various microorganisms coexisting in lakes and oceans, which might depend on single nutrient in the region. Different from the model in Chi and Zhao [20], in present paper, we consider two different microbes to compete for a nutrient, by introducing Monod-Haldane functional response; we get the model as follows:

$$\begin{aligned}
\frac{\mathrm{d}S(t)}{\mathrm{d}t} &= D\left(S^0 - S(t)\right) - \frac{m_1 S(t)\,x(t)}{a_1 + S(t) + K_1 S^2(t)} \\
&\quad - \frac{m_2 S(t)\,y(t)}{a_2 + S(t) + K_2 S^2(t)}, \\
\frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \frac{m_1 S(t)\,x(t)}{a_1 + S(t) + K_1 S^2(t)} - Dx(t) - c_1 x^2(t), \\
\frac{\mathrm{d}y(t)}{\mathrm{d}t} &= \frac{m_2 S(t)\,y(t)}{a_2 + S(t) + K_2 S^2(t)} - Dy(t) - c_2 y^2(t),
\end{aligned} \tag{5}$$

where c_1 and c_2 denote the intraspecies competition rates. The meanings of other parameters are the same as those described above.

It is well known that nature is often affected by random factors [21–25], unavoidably the process of microbial culture is affected by the interference of random factors [26–28], such as the degradation of microbial strains, the existence of an inducer or inhibitor on the growth, cultivation temperature changes, and the species and concentrations of inorganic salts changes. To understand the phenomenon of stochastic perturbations deeply [29–31], many scholars have studied the effect of the noise on the dynamical behavior of the stochastic chemostat models [32–37].

A parameter of the system is often subject to random disturbance [38–42]; thus, in this paper, we assume that the maximal growth rates $m_i (i = 1, 2)$ are perturbed by environment noise on the basis of the approaches used in [20, 26, 43]. In this case, $m_i (i = 1, 2)$ change to random variables $\widetilde{m}_i (i = 1, 2)$, and $\widetilde{m}_i = m_i + \sigma_i \dot{B}_i(t) (i = 1, 2)$, where $B_i(t) (i = 1, 2)$ are independent standard Brownian motions with intensity $\sigma_i \geq 0 (i = 1, 2)$. In summary, we replace $m_i(i =$

1, 2) in deterministic model (5) with $m_i + \sigma_i \dot{B}_i(t) (i = 1, 2)$ to get the following stochastic differential equations model:

$$\begin{aligned}
\mathrm{d}S(t) &= \left[D\left(S^0 - S(t)\right) - \frac{m_1 S(t) x(t)}{a_1 + S(t) + K_1 S^2(t)} \right. \\
&\quad \left. - \frac{m_2 S(t) y(t)}{a_2 + S(t) + K_2 S^2(t)}\right] \mathrm{d}t \\
&\quad - \frac{\sigma_1 S(t) x(t)}{a_1 + S(t) + K_1 S^2(t)} \mathrm{d}B_1(t) \\
&\quad - \frac{\sigma_2 S(t) y(t)}{a_2 + S(t) + K_2 S^2(t)} \mathrm{d}B_2(t), \\
\mathrm{d}x(t) &= \left[\frac{m_1 S(t) x(t)}{a_1 + S(t) + K_1 S^2(t)} - Dx(t) - c_1 x^2(t)\right] \mathrm{d}t \\
&\quad + \frac{\sigma_1 S(t) x(t)}{a_1 + S(t) + K_1 S^2(t)} \mathrm{d}B_1(t), \\
\mathrm{d}y(t) &= \left[\frac{m_2 S(t) y(t)}{a_2 + S(t) + K_2 S^2(t)} - Dy(t) - c_2 y^2(t)\right] \mathrm{d}t \\
&\quad + \frac{\sigma_2 S(t) y(t)}{a_2 + S(t) + K_2 S^2(t)} \mathrm{d}B_2(t).
\end{aligned} \tag{6}$$

The remaining part of this paper is organized as follows. In Section 2, we give some notations, definitions, and lemmas which will be used in the following section. In Section 3, we explore the sufficient conditions of system (6) for the extinction and persistence of the two microorganisms. Finally, some conclusions and numerical simulations are given in Section 4.

2. Preliminary Knowledge

In this paper, we denote $R_+^3 = \{(S, x, y) \in R^3 : S > 0, x > 0, y > 0\}$. For an integrable function $\mathscr{V}$ on $[0, +\infty)$, define $\langle \mathscr{V}(t) \rangle = (1/t) \int_0^t \mathscr{V}(\theta) d\theta$.

Definition 1. (i) The microorganism $x(t)$ and $y(t)$ are said to be extinctive if $\lim_{t \to +\infty} x(t) = 0$ and $\lim_{t \to +\infty} y(t) = 0$.

(ii) The microorganism $x(t)$ and $y(t)$ are said to be permanent in mean if there exist two positive constants λ_1 and λ_2 such that $\liminf_{t \to +\infty} \langle x(t) \rangle \geq \lambda_1$ and $\liminf_{t \to +\infty} \langle y(t) \rangle \geq \lambda_2$.

Lemma 2. *For any positive solution $(S(t), x(t), y(t))$ of systems (5) or (6) with initial value $(S(0), x(0), y(0)) \in R_+^3$, we have*

$$\max\left\{\limsup_{t \to +\infty} S(t), \limsup_{t \to +\infty} x(t), \limsup_{t \to +\infty} y(t)\right\} \leq S^0, \quad a.s. \tag{7}$$

Proof. From system (5) or (6), we have

$$\begin{aligned}
&\frac{\mathrm{d}(S(t) + x(t) + y(t))}{\mathrm{d}t} \\
&\quad \leq DS^0 - D(S(t) + x(t) + y(t)).
\end{aligned} \tag{8}$$

This implies that

$$\lim_{t \to +\infty} (S(t) + x(t) + y(t)) \leq S^0. \tag{9}$$

Then obviously we obtain

$$\begin{aligned}
&\limsup_{t \to +\infty} S(t) \leq S^0, \\
&\limsup_{t \to +\infty} x(t) \leq S^0, \\
&\limsup_{t \to +\infty} y(t) \leq S^0, \\
&\qquad a.s.
\end{aligned} \tag{10}$$

This completes the proof of Lemma 2. □

Lemma 3. *(a) System (5) has at most three equilibrium points on the boundary of the positive quadrant. The trivial equilibrium point $E_0(S^0, 0, 0)$ always exists. It is a locally asymptotically stable 'microorganism-extinction' equilibrium point when $m_1 S^0/(a_1 + S^0 + K_1 S^{02}) - D < 0$ and $m_2 S^0/(a_2 + S^0 + K_2 S^{02}) - D < 0$.*

(b) If $m_1 S^0/(a_1 + S^0 + K_1 S^{02}) - D > 0$, then there exists a equilibrium point $E_1(\theta_0, \theta_1, 0)$ with $\theta_1 > 0$, and θ_0 and θ_1 satisfy the equation $c_1 \theta_1^2 + D\theta_1 - D(S^0 - \theta_0) = 0$. It is a locally asymptotically stable equilibrium point when $m_2 \theta_0/(a_2 + \theta_0 + K_2 \theta_0^2) - D < 0$ and $(\theta_1 - \theta_0)(a_1 - K_1 \theta_0^2) - \theta_0^2 > 0$.

(c) If $m_2 S^0/(a_2 + S^0 + K_2 S^{02}) - D > 0$, then there exists a equilibrium point $E_2(\alpha_0, 0, \alpha_2)$ with $\alpha_2 > 0$, and α_0 and α_2 satisfy the equation $c_2 \alpha_2^2 + D\alpha_2 - D(S^0 - \alpha_0) = 0$. It is a locally asymptotically stable equilibrium point when $m_1 \alpha_0/(a_1 + \alpha_0 + K_1 \alpha_0^2) - D < 0$ and $(\alpha_2 - \alpha_0)(a_2 - K_2 \alpha_0^2) - \alpha_0^2 > 0$.

Proof. (I) Firstly, we prove the existence of boundary equilibrium point. In system (5), let

$$\begin{aligned}
f_1 &:= D\left(S^0 - S(t)\right) - \frac{m_1 S(t) x(t)}{a_1 + S(t) + K_1 S^2(t)} \\
&\quad - \frac{m_2 S(t) y(t)}{a_2 + S(t) + K_2 S^2(t)} = 0, \\
f_2 &:= \frac{m_1 S(t) x(t)}{a_1 + S(t) + K_1 S^2(t)} - Dx(t) - c_1 x^2(t) = 0, \\
f_3 &:= \frac{m_2 S(t) y(t)}{a_2 + S(t) + K_2 S^2(t)} - Dy(t) - c_2 y^2(t) = 0.
\end{aligned} \tag{11}$$

By the above formula, we get the following equilibria:

$$E_0 : \left(S^0, 0, 0\right). \tag{12}$$

Let $y(t) = 0, x(t) \neq 0$ and $S(t) \neq 0$; we get the following equation through system (5):

$$\begin{aligned}
&D\left(S^0 - S(t)\right) - \frac{m_1 S(t) x(t)}{a_1 + S(t) + K_1 S^2(t)} = 0, \\
&\frac{m_1 S(t) x(t)}{a_1 + S(t) + K_1 S^2(t)} - Dx(t) - c_1 x^2(t) = 0.
\end{aligned} \tag{13}$$

Let $m_1S(t)/(a_1+S(t)+K_1S^2(t)) = Z(t)$; then, by simplifying (13) we get

$$\begin{gathered}\frac{m_1S(t)}{a_1+S(t)+K_1S^2(t)} = Z(t),\\ -\frac{1}{c_1}Z^2(t)+\frac{D}{c_1}Z(t)+D\left(S^0-S(t)\right)=0.\end{gathered} \tag{14}$$

For system (14), if $m_1S^0/(a_1+S^0+K_1S^{02})-D>0$, we can see that the two functional images have unique intersection point, which proves that system (13) has a positive solution (θ_0,θ_1) and satisfies the equation $c_1\theta_1^2+D\theta_1-D(S^0-\theta_0)=0$. So it proves the existence of $E_1(\theta_0,\theta_1,0)$.

Similarly, we can prove the existence of $E_2(\alpha_0,0,\alpha_2)$.

(II) Secondly, we prove the stability of the boundary equilibrium point. The Jacobian matrix associating to the equilibrium of system (5) is

$$J=\begin{pmatrix}\gamma_{11} & \gamma_{12} & \gamma_{13}\\ \gamma_{21} & \gamma_{22} & \gamma_{23}\\ \gamma_{31} & \gamma_{32} & \gamma_{33}\end{pmatrix}, \tag{15}$$

where

$$\begin{aligned}\gamma_{11} &= -D-\frac{a_1m_1x-K_1m_1xS^2}{(a_1+S+K_1S^2)^2}\\ &\quad -\frac{a_2m_2y-K_2m_2yS^2}{(a_2+S+K_2S^2)^2},\\ \gamma_{12} &= -\frac{m_1S}{a_1+S+K_1S^2},\\ \gamma_{13} &= -\frac{m_2S}{a_2+S+K_2S^2},\\ \gamma_{21} &= \frac{a_1m_1x-K_1m_1xS^2}{(a_1+S+K_1S^2)^2},\\ \gamma_{22} &= \frac{m_1S}{a_1+S+K_1S^2}-D-2c_1x,\\ \gamma_{23} &= 0,\\ \gamma_{31} &= \frac{a_2m_2y-K_2m_2yS^2}{(a_2+S+K_2S^2)^2},\\ \gamma_{32} &= 0,\\ \gamma_{33} &= \frac{m_2S}{a_2+S+K_2S^2}-D-2c_2y.\end{aligned} \tag{16}$$

(i) The stability of the 'microorganism-extinction' equilibrium point $(S^0,0,0)$ of system (5) is determined by the Jacobian

$$J_0 = \begin{pmatrix}-D & -\dfrac{m_1S^0}{a_1+S^0+K_1S^{02}} & -\dfrac{m_2S^0}{a_2+S^0+K_2S^{02}}\\ 0 & \dfrac{m_1S^0}{a_1+S^0+K_1S^{02}}-D & 0\\ 0 & 0 & \dfrac{m_2S^0}{a_2+S^0+K_2S^{02}}-D\end{pmatrix}, \tag{17}$$

which has the following eigenvalues

$$\begin{aligned}\lambda_1 &= -D<0,\\ \lambda_2 &= \frac{m_1S^0}{a_1+S^0+K_1S^{02}}-D,\\ \lambda_3 &= \frac{m_2S^0}{a_2+S^0+K_2S^{02}}-D.\end{aligned} \tag{18}$$

According to stability theory [44, 45], $(S^0,0,0)$ is stable if $\lambda_2<0$ and $\lambda_3<0$; i.e., $m_1S^0/(a_1+S^0+K_1S^{02})-D<0$ and $m_2S^0/(a_2+S^0+K_2S^{02})-D<0$.

(ii) The functional matrix at $(\theta_0,\theta_1,0)$ is equal to

$$J_1=\begin{pmatrix}-D-\dfrac{a_1m_1\theta_1-K_1m_1\theta_1\theta_0^2}{(a_1+\theta_0+K_1\theta_0^2)^2} & -\dfrac{m_1\theta_0}{a_1+\theta_0+K_1\theta_0^2} & -\dfrac{m_2\theta_0}{a_2+\theta_0+K_2\theta_0^2}\\ \dfrac{a_1m_1\theta_1-K_1m_1\theta_1\theta_0^2}{(a_1+\theta_0+K_1\theta_0^2)^2} & \dfrac{m_1\theta_0}{a_1+\theta_0+K_1\theta_0^2}-D-2c_1\theta_1 & 0\\ 0 & 0 & \dfrac{m_2\theta_0}{a_2+\theta_0+K_2\theta_0^2}-D\end{pmatrix}, \tag{19}$$

which has the following eigenvalue:

$$\lambda_1=\frac{m_2\theta_0}{a_2+\theta_0+K_2\theta_0^2}-D. \tag{20}$$

The other eigenvalues satisfy the following equations:

$$\lambda^2+\gamma\lambda+\nu=0, \tag{21}$$

where

$$\begin{aligned}\gamma &= 2D + 2c_1\theta_1 + \frac{a_1 m_1 \theta_1 - K_1 m_1 \theta_1 \theta_0^2}{\left(a_1 + \theta_0 + K_1\theta_0^2\right)^2} \\ &\quad - \frac{m_1\theta_0}{a_1 + \theta_0 + K_1\theta_0^2}, \\ \nu &= D^2 + 2c_1\theta_1 D + 2c_1\theta_1 \frac{a_1 m_1 \theta_1 - K_1 m_1 \theta_1 \theta_0^2}{\left(a_1 + \theta_0 + K_1\theta_0^2\right)^2} \\ &\quad + D\left[\frac{a_1 m_1 \theta_1 - K_1 m_1 \theta_1 \theta_0^2}{\left(a_1 + \theta_0 + K_1\theta_0^2\right)^2} - \frac{m_1\theta_0}{a_1 + \theta_0 + K_1\theta_0^2}\right].\end{aligned} \tag{22}$$

If $(\theta_1 - \theta_0)(a_1 - K_1\theta_0^2) - \theta_0^2 > 0$, then $\gamma > 0$ and $\nu > 0$. Hence, it follows that any root of (21) has negative real part owing to the Routh-Hurwitz criterion. According to stability theory [44], $E_1(\theta_0, \theta_1, 0)$ is stable if $m_2\theta_0/(a_2 + \theta_0 + K_2\theta_0^2) - D < 0$ and $(\theta_1 - \theta_0)(a_1 - K_1\theta_0^2) - \theta_0^2 > 0$.

Similarly, we can prove the stability of $E_2(\alpha_0, 0, \alpha_2)$.

This finishes the proof of Lemma 3. □

Remark 4. We also can obtain the conditions $m_1 S^0/(a_1 + S^0 + K_1 S^{02}) - D > 0$ and $m_2 S^0/(a_2 + S^0 + K_2 S^{02}) - D > 0$ for the existence of equilibrium points $E_1(\theta_0, \theta_1, 0)$ and $E_2(\alpha_0, 0, \alpha_2)$ from the Proposition 1 and Proposition 2 in [34].

Lemma 5. *For any initial value $(S_0, x_0, y_0) \in R_+^3$, system (6) has a unique solution $(S(t), x(t), y(t))$ on $t \geq 0$, and the solution remains in R_+^3 with probability one; namely, $(S(t), x(t), y(t)) \in R_+^3$ for all $t \geq 0$ almost surely.*

Proof. Since the coefficients of system (6) are locally Lipschitz continuous, for any given initial value $(S_0, x_0, y_0) \in R_+^3$, there is a unique local solution $(S(t), x(t), y(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. If we can prove $\tau_e = \infty$ a.s., then the solution will be global.

First we set $\inf \emptyset = \infty$ and $\emptyset$ is an empty set. Select $\psi_0 > 0$ such that $S_0 > \psi_0, x_0 > \psi_0, y_0 > \psi_0$. For any positive $\psi (\psi \leq \psi_0)$, define the stopping time as follows:

$$\tau_\psi = \inf\left\{t \in [0, \tau_e) : S(t) \leq \psi \text{ or } x(t) \leq \psi \text{ or } y(t) \leq \psi\right\}. \tag{23}$$

Clearly τ_ψ is increasing as $\psi \longrightarrow 0$. Assuming $\tau_0 = \lim_{\psi \longrightarrow 0}\tau_\psi$, obviously $\tau_0 \leq \tau_e$. We declare that $\tau_0 = \infty$ holds a.s.; thus, $\tau_e = \infty$ a.s. and $(S(t), x(t), y(t)) \in R_+^3$ for all $t \geq 0$. So we only need to prove $\tau_0 = \infty$ a.s.

If this assertion is false, then there is a pair of constants $T > 0$ and $\eta \in (0, 1)$ such that $P\{\tau_0 \leq T\} > \eta$. Hence, there is a positive constant $\psi_1 \leq \psi_0$ such that $P\{\tau_\psi \leq T\} > \eta$ for any positive $\psi \leq \psi_1$.

From system (6), the total biomass $N(t) = S(t) + x(t) + y(t)$ satisfies the equation $dN(t) \leq D(S^0 - N(t))dt$ then leads to that for all $t < \tau_e$,

$$N(t) \leq \max\left\{S_0 + x_0 + y_0, S^0\right\} := h_1. \tag{24}$$

Define a C^2-function $V: R_+^3 \longrightarrow R_+$ by

$$V(S(t), x(t), y(t)) = -\ln\frac{S(t)}{h_1} - \ln\frac{x(t)}{h_1} - \ln\frac{y(t)}{h_1}. \tag{25}$$

Obviously, V is positive defined. Applying Itô formula, we obtain

$$\mathrm{d}V = LV\mathrm{d}t + \frac{\sigma_1 (x(t) - S(t))}{a_1 + S(t) + K_1 S^2(t)}\mathrm{d}B_1(t) + \frac{\sigma_2 (y(t) - S(t))}{a_2 + S(t) + K_2 S^2(t)}\mathrm{d}B_2(t), \tag{26}$$

where

$$\begin{aligned}LV &= -\frac{DS^0}{S(t)} + 3D + \frac{m_1(x(t) - S(t))}{a_1 + S(t) + K_1 S^2(t)} \\ &\quad + \frac{m_2(y(t) - S(t))}{a_2 + S(t) + K_2 S^2(t)} \\ &\quad + \frac{\sigma_1^2\left(x^2(t) + S^2(t)\right)}{2\left(a_1 + S(t) + K_1 S^2(t)\right)^2} \\ &\quad + \frac{\sigma_2^2\left(y^2(t) + S^2(t)\right)}{2\left(a_2 + S(t) + K_2 S^2(t)\right)^2} + c_1 x(t) + c_2 y(t) \\ &\leq 3D + \frac{m_1 h_1}{a_1} + \frac{m_2 h_1}{a_2} + \frac{\sigma_1^2 h_1^2}{a_1^2} + \frac{\sigma_2^2 h_1^2}{a_2^2} + c_1 h_1 \\ &\quad + c_2 h_1 := h_2.\end{aligned} \tag{27}$$

So we get

$$\mathrm{d}V(t) \leq h_2\mathrm{d}t + \frac{\sigma_1 (x(t) - S(t))}{a_1 + S(t) + K_1 S^2(t)}\mathrm{d}B_1(t) + \frac{\sigma_2 (y(t) - S(t))}{a_2 + S(t) + K_2 S^2(t)}\mathrm{d}B_2(t). \tag{28}$$

Integrating both sides from 0 to $\tau_\psi \wedge T$, and then taking expectations, we have

$$EV\left(S\left(\tau_\psi \wedge T\right), x\left(\tau_\psi \wedge T\right), y\left(\tau_\psi \wedge T\right)\right) \leq V(S_0, x_0, y_0) + h_2 T. \tag{29}$$

Denote $\Omega_\psi = \{\tau_\psi \leq T\}$ and $P(\Omega_\psi) > \eta$ for any positive $\psi \leq \psi_1$. Note that for every $\varrho \in \Omega_\psi$, at least, one of $S(\tau_\psi, \varrho)$, $x(\tau_\psi, \varrho)$ or $y(\tau_\psi, \varrho)$ equals to ψ, and then

$$V\left(S\left(\tau_\psi, \varrho\right), x\left(\tau_\psi, \varrho\right), y\left(\tau_\psi, \varrho\right)\right) \geq -\ln\frac{\psi}{h_1}. \tag{30}$$

So we obtain

$$\begin{aligned} V(S_0, x_0, y_0) + h_2 T &\geq E\left[I_{\Omega_\psi}(\varrho) V\left(S\left(\tau_\psi, \varrho\right), x\left(\tau_\psi, \varrho\right), y\left(\tau_\psi, \varrho\right)\right)\right] \\ &= P\left(\Omega_\psi\right) V\left(S\left(\tau_\psi, \varrho\right), x\left(\tau_\psi, \varrho\right), y\left(\tau_\psi, \varrho\right)\right) \\ &> -\eta \ln \frac{\psi}{h_1}, \end{aligned} \tag{31}$$

where $I_{\Omega_\psi}(\varrho)$ is the indicator function of Ω_ψ. Letting $\psi \longrightarrow 0$ leads to the contradiction

$$\infty > V(S_0, x_0, y_0) + h_2 T = \infty. \tag{32}$$

So $\tau_0 = \infty$ almost surely. The proof is complete. □

3. Extinction and Persistence of System (6)

3.1. Extinction. In this section, we will try to give conditions that lead to the extinction of microorganism. Let

$$\begin{aligned} \mathcal{R}_i &= \frac{m_i}{\left(1 + 2\sqrt{a_i K_i}\right) D} - \frac{\sigma_i^2}{2D}\left(\frac{1}{1 + 2\sqrt{a_i K_i}}\right)^2, \\ \mathcal{R}_i^* &= \frac{m_i S^0}{\left(a_i + S^0 + K_i S^{02}\right) D} - \frac{\sigma_i^2 S^{02}}{2D\left(a_i + S^0 + K_i S^{02}\right)^2}, \\ & \qquad i = 1, 2. \end{aligned} \tag{33}$$

Then we get the following theorem.

Theorem 6. *Let $(S(t), x(t), y(t))$ be a solution of system (6) with initial value $(S(0), x(0), y(0)) \in R_+^3$. Then if one of the following holds*

A_{11} : $S^0 \geq \sqrt{a_1/K_1}$, $1/(1 + 2\sqrt{a_1 K_1}) \geq m_1/\sigma_1^2$ and $\sigma_1^2 > m_1^2/2D$, or

A_{21} : $S^0 < \sqrt{a_1/K_1}$, $S^0/(a_1 + S^0 + K_1 S^{02}) \geq m_1/\sigma_1^2$ and $\sigma_1^2 > m_1^2/2D$, or

A_{31} : $S^0 \geq \sqrt{a_1/K_1}$, $1/(1 + 2\sqrt{a_1 K_1}) < m_1/\sigma_1^2$ and $\mathcal{R}_1 < 1$, or

A_{41} : $S^0 < \sqrt{a_1/K_1}$, $S^0/(a_1 + S^0 + K_1 S^{02}) < m_1/\sigma_1^2$ and $\mathcal{R}_1^* < 1$,

the microorganism $x(t)$ goes to extinction almost surely, i.e., $\lim_{t\longrightarrow+\infty} x(t) = 0$, a.s.

Proof. Applying Itô's formula to system (6) yields

$$\begin{aligned} d \ln x(t) = {} & \left[\frac{m_1 S(t)}{a_1 + S(t) + K_1 S^2(t)} - D - c_1 x(t) \right. \\ & \left. - \frac{\sigma_1^2 S^2(t)}{2\left(a_1 + S(t) + K_1 S^2(t)\right)^2}\right] dt \\ & + \frac{\sigma_1 S(t)}{a_1 + S(t) + K_1 S^2(t)} dB_1(t). \end{aligned} \tag{34}$$

Integrating from 0 to t and dividing by t on both sides of (34) yields

$$\begin{aligned} \frac{\ln x(t)}{t} = {} & \frac{1}{t}\int_0^t \left[\frac{m_1 S(t)}{a_1 + S(t) + K_1 S^2(t)} - D - c_1 x(t) \right. \\ & \left. - \frac{\sigma_1^2 S^2(t)}{2\left(a_1 + S(t) + K_1 S^2(t)\right)^2}\right] dt + \frac{M(t)}{t} \\ & + \frac{\ln x(0)}{t} \leq \frac{1}{t}\int_0^t \left[-\frac{\sigma_1^2 S^2(t)}{2\left(a_1 + S(t) + K_1 S^2(t)\right)^2} \right. \\ & \left. + \frac{m_1 S(t)}{a_1 + S(t) + K_1 S^2(t)} - D\right] dt + \frac{M(t)}{t} \\ & + \frac{\ln x(0)}{t}, \end{aligned} \tag{35}$$

where the function $M(t) = \int_0^t (\sigma_1 S(\tau)/(a_1 + S(\tau) + K_1 S^2(\tau))) dB_1(\tau)$. By the strong law of large number and Lemma 2, we get

$$\lim_{t\longrightarrow+\infty} \frac{M(t)}{t} = 0, \quad a.s. \tag{36}$$

Then four cases should be discussed.

A_{11} : since $S^0 \geq \sqrt{a_1/K_1}$, $1/(1 + 2\sqrt{a_1 K_1}) \geq m_1/\sigma_1^2$ and $\sigma_1^2 > m_1^2/2D$, then we can easily see from (35) that

$$\frac{\ln x(t)}{t} \leq -\left(D - \frac{m_1^2}{2\sigma_1^2}\right) + \frac{M(t)}{t} + \frac{\ln x(0)}{t}. \tag{37}$$

Taking the superior limit on both sides of (37) yields

$$\lim_{t\longrightarrow+\infty} \sup \frac{\ln x(t)}{t} \leq -\left(D - \frac{m_1^2}{2\sigma_1^2}\right) < 0, \quad a.s., \tag{38}$$

which implies $\lim_{t\longrightarrow+\infty} x(t) = 0$, a.s.

Similarly, we can prove that microorganism also tends to extinction under condition A_{21}, and we omit it here.

A_{31} : $S^0 \geq \sqrt{a_1/K_1}$, $1/(1+2\sqrt{a_1 K_1}) < m_1/\sigma_1^2$ and $\mathcal{R}_1 < 1$. From (35), we have

$$\begin{aligned} \frac{\ln x(t)}{t} \leq {} & -\frac{\sigma_1^2}{2 + 8\sqrt{a_1 K_1} + 8 a_1 K_1} + \frac{m_1}{1 + 2\sqrt{a_1 K_1}} - D \\ & + \frac{M(t)}{t} + \frac{\ln x(0)}{t}. \end{aligned} \tag{39}$$

Taking the superior limit on both sides of (39) leads to

$$\lim_{t\longrightarrow+\infty} \sup \frac{\ln x(t)}{t} \leq D\left(\mathcal{R}_1 - 1\right) < 0, \quad a.s., \tag{40}$$

which implies $\lim_{t\longrightarrow+\infty} x(t) = 0$, a.s.

A_{41} : $S^0 < \sqrt{a_1/K_1}$, $S^0/(a_1 + S^0 + K_1 S^{02}) < m_1/\sigma_1^2$ and $\mathcal{R}_1^* < 1$. In this case, we can see from (35) that

$$\begin{aligned} \frac{\ln x(t)}{t} \leq {} & -\frac{\sigma_1^2 S^{02}}{2\left(a_1 + S^0 + K_1 S^{02}\right)^2} + \frac{m_1 S^0}{a_1 + S^0 + K_1 S^{02}} \\ & - D + \frac{M(t)}{t} + \frac{\ln x(0)}{t}. \end{aligned} \tag{41}$$

Taking the superior limit on both sides of (41) yields

$$\lim_{t\longrightarrow+\infty}\sup\frac{\ln x(t)}{t}\leq D\left(\mathscr{R}_1^*-1\right)<0,\quad a.s.,\tag{42}$$

which implies $\lim_{t\longrightarrow+\infty}x(t)=0$, a.s.

This completes the proof of Theorem 6. □

The same discussion can be used in $y(t)$; we have the following.

Theorem 7. *Let $(S(t),x(t),y(t))$ be a solution of system (6) with initial value $(S(0),x(0),y(0))\in R_+^3$. Then if one of the following holds*

A_{12} : $S^0\geq\sqrt{a_2/K_2}, 1/(1+2\sqrt{a_2K_2})\geq m_2/\sigma_2^2$ *and* $\sigma_2^2>m_2^2/2D$, *or*

A_{22} : $S^0<\sqrt{a_2/K_2}, S^0/(a_2+S^0+K_2S^{02})\geq m_2/\sigma_2^2$ *and* $\sigma_2^2>m_2^2/2D$, *or*

A_{32} : $S^0\geq\sqrt{a_2/K_2}, 1/(1+2\sqrt{a_2K_2})<m_2/\sigma_2^2$ *and* $\mathscr{R}_2<1$, *or*

A_{42} : $S^0<\sqrt{a_2/K_2}, S^0/(a_2+S^0+K_2S^{02})<m_2/\sigma_2^2$ *and* $\mathscr{R}_2^*<1$,

the microorganism $y(t)$ goes to extinction almost surely, i.e., $\lim_{t\longrightarrow+\infty}y(t)=0$, *a.s.*

3.2. Permanence in Mean. For system (6), let

$$\ell_i=m_i\left(c_1+c_2\right)S^{02}+D^2\left(a_i+S^0+K_iS^{02}\right),\quad i=1,2,\tag{43}$$

and

$$\overline{\mathscr{R}}_i=\frac{Dm_iS^0}{\ell_i}-\frac{D\sigma_i^2S^{02}}{2a_i\ell_i},\quad i=1,2.\tag{44}$$

Then we get the following theorem.

Theorem 8. *Let $(S(t),x(t),y(t))$ be a solution of system (6) with initial value $(S(0),x(0),y(0))\in R_+^3$, then we have the following.*

(i) If $\overline{\mathscr{R}}_1>1$, $y(t)$ satisfies one of the conditions A_{12},A_{22},A_{32}, and A_{42} of Theorem 7, then the microorganism $y(t)$ goes extinct and the microorganism $x(t)$ is permanent in mean; moreover, $x(t)$ satisfies

$$\liminf_{t\longrightarrow+\infty}\langle x(t)\rangle\geq\frac{\ell_1\left(\overline{\mathscr{R}}_1-1\right)}{D\left[c_1\left(a_1+S^0+K_1S^{02}\right)+m_1\right]}.\tag{45}$$

(ii) If $\overline{\mathscr{R}}_2>1$, $x(t)$ satisfies one of the conditions A_{11},A_{21},A_{31}, and A_{41} of Theorem 6, then the microorganism $x(t)$ goes extinct and the microorganism $y(t)$ is permanent in mean; moreover, $y(t)$ satisfies

$$\liminf_{t\longrightarrow+\infty}\langle y(t)\rangle\geq\frac{\ell_2\left(\overline{\mathscr{R}}_2-1\right)}{D\left[c_2\left(a_2+S^0+K_2S^{02}\right)+m_2\right]}.\tag{46}$$

(iii) If $\overline{\mathscr{R}}_1>1$ and $\overline{\mathscr{R}}_2>1$, then two microorganisms $x(t)$ and $y(t)$ are permanent in mean; moreover, $x(t)$ and $y(t)$ satisfy

$$\liminf_{t\longrightarrow+\infty}\langle x(t)+y(t)\rangle\geq\frac{1}{D\Delta_{\max}}\sum_{i=1}^2\ell_i\left(\overline{\mathscr{R}}_i-1\right),\tag{47}$$

where

$$\Delta_{\max}=\max\left\{m_1+m_2+c_1\left(a_1+S^0+K_1S^{02}\right),m_1+m_2+c_2\left(a_2+S^0+K_2S^{02}\right)\right\}.\tag{48}$$

Proof. Case (i): by Theorem 7, since $y(t)$ satisfies one of the conditions A_{12},A_{22},A_{32}, and A_{42} of Theorem 7, then $\lim_{t\longrightarrow+\infty}y(t)=0$. Since $\overline{\mathscr{R}}_1>1$, for ε small enough, such that $0<y(t)<\varepsilon$ for all t large enough and

$$\frac{Dm_1\left(S^0-\varepsilon\right)}{\ell_1}-\frac{D\sigma_1^2S^{02}}{2a_1\ell_1}>1.\tag{49}$$

Integrating from 0 to t and dividing by t on both sides of system (6) yields

$$\begin{aligned}\epsilon(t)&\triangleq\frac{S(t)-S(0)}{t}+\frac{x(t)-x(0)}{t}+\frac{y(t)-y(0)}{t}\\&=DS^0-D\langle S(t)\rangle-D\langle x(t)\rangle-D\langle y(t)\rangle\\&\quad-\frac{c_1}{t}\int_0^t x^2(t)\,dt-\frac{c_2}{t}\int_0^t y^2(t)\,dt\\&\geq DS^0-D\langle S(t)\rangle-D\langle x(t)\rangle-D\varepsilon\\&\quad-\left(c_1+c_2\right)S^{02},\end{aligned}\tag{50}$$

then one can get

$$\langle S(t)\rangle\geq S^0-\langle x(t)\rangle-\varepsilon-\frac{c_1+c_2}{D}S^{02}-\frac{\epsilon(t)}{D}.\tag{51}$$

Applying Itô's formula gives

$$\begin{aligned}&\mathrm{d}\left(\left(a_1+S^0+K_1S^{02}\right)\ln x(t)\right)\\&=\left[\frac{\left(a_1+S^0+K_1S^{02}\right)m_1S(t)}{a_1+S(t)+K_1S^2(t)}\right.\\&\quad-D\left(a_1+S^0+K_1S^{02}\right)-c_1\left(a_1+S^0+K_1S^{02}\right)x(t)\\&\quad\left.-\frac{\left(a_1+S^0+K_1S^{02}\right)\sigma_1^2S^2(t)}{2\left(a_1+S(t)+K_1S^2(t)\right)^2}\right]\mathrm{d}t\\&\quad+\frac{\left(a_1+S^0+K_1S^{02}\right)\sigma_1S(t)}{a_1+S(t)+K_1S^2(t)}\mathrm{d}B_1(t)\geq\left[m_1S(t)\right.\\&\quad-D\left(a_1+S^0+K_1S^{02}\right)-c_1\left(a_1+S^0+K_1S^{02}\right)x(t)\\&\quad\left.-\frac{\sigma_1^2S^{02}}{2a_1}\right]\mathrm{d}t+\frac{\left(a_1+S^0+K_1S^{02}\right)\sigma_1S(t)}{a_1+S(t)+K_1S^2(t)}\mathrm{d}B_1(t).\end{aligned}\tag{52}$$

Integrating from 0 to t and dividing by t on both sides of (52) yields

$$\begin{aligned}
&\frac{\left(a_1+S^0+K_1S^{02}\right)(\ln x(t)-\ln x(0))}{t}\\
&\geq m_1\langle S(t)\rangle - D\left(a_1+S^0+K_1S^{02}\right)\\
&\quad - c_1\left(a_1+S^0+K_1S^{02}\right)\langle x(t)\rangle - \frac{\sigma_1^2S^{02}}{2a_1}+\frac{M_1(t)}{t}\\
&\geq m_1\left(S^0-\langle x(t)\rangle-\varepsilon-\frac{c_1+c_2}{D}S^{02}-\frac{\epsilon(t)}{D}\right)\\
&\quad - D\left(a_1+S^0+K_1S^{02}\right)\\
&\quad - c_1\left(a_1+S^0+K_1S^{02}\right)\langle x(t)\rangle - \frac{\sigma_1^2S^{02}}{2a_1}+\frac{M_1(t)}{t}\\
&= m_1\left(S^0-\varepsilon\right)-\frac{m_1\left(c_1+c_2\right)}{D}S^{02}-\frac{m_1\epsilon(t)}{D}\\
&\quad - D\left(a_1+S^0+K_1S^{02}\right)\\
&\quad - \left[c_1\left(a_1+S^0+K_1S^{02}\right)+m_1\right]\langle x(t)\rangle-\frac{\sigma_1^2S^{02}}{2a_1}\\
&\quad + \frac{M_1(t)}{t},
\end{aligned} \tag{53}$$

where $M_1(t) = \int_0^t((a_1+S^0+K_1S^{02})\sigma_1 S(\tau)/(a_1+S(\tau)+K_1S^2(\tau)))\mathrm{d}B_1(\tau)$. The inequality (53) can be rewritten as

$$\begin{aligned}
&\frac{\left(a_1+S^0+K_1S^{02}\right)\ln x(t)}{t}\\
&\geq m_1\left(S^0-\varepsilon\right)-\frac{m_1\left(c_1+c_2\right)}{D}S^{02}-\frac{m_1\epsilon(t)}{D}\\
&\quad - D\left(a_1+S^0+K_1S^{02}\right)\\
&\quad - \left[c_1\left(a_1+S^0+K_1S^{02}\right)+m_1\right]\langle x(t)\rangle-\frac{\sigma_1^2S^{02}}{2a_1}\\
&\quad + \frac{M_1(t)}{t}+\frac{\left(a_1+S^0+K_1S^{02}\right)\ln x(0)}{t}.
\end{aligned} \tag{54}$$

By the strong law of large numbers and Lemma 2, we have $\lim_{t\longrightarrow+\infty}(M_1(t)/t) = 0$ and $\lim_{t\longrightarrow+\infty}\epsilon(t) = 0$. Taking the inferior limit of both sides of (54) and using Lemma 2.5 in [46], we can get

$$\begin{aligned}
&\liminf_{t\longrightarrow+\infty}\langle x(t)\rangle\\
&\geq \frac{\ell_1}{D\left[c_1\left(a_1+S^0+K_1S^{02}\right)+m_1\right]}\left[\frac{Dm_1\left(S^0-\varepsilon\right)}{\ell_1}\right.\\
&\quad\left.-\frac{D\sigma_1^2S^{02}}{2a_1\ell_1}-1\right]>0,\quad a.s.
\end{aligned} \tag{55}$$

Let $\varepsilon \longrightarrow 0$, then we have

$$\liminf_{t\longrightarrow+\infty}\langle x(t)\rangle \geq \frac{\ell_1\left(\overline{\mathscr{R}}_1-1\right)}{D\left[c_1\left(a_1+S^0+K_1S^{02}\right)+m_1\right]},\quad a.s. \tag{56}$$

The same discussion can be used in Case (ii), and we omit it here.

Case (iii): notice that

$$\langle S(t)\rangle \geq S^0-\langle x(t)\rangle-\langle y(t)\rangle-\frac{c_1+c_2}{D}S^{02}-\frac{\epsilon(t)}{D}. \tag{57}$$

Define

$$V(t) = \rho_1\ln x(t)+\rho_2\ln y(t), \tag{58}$$

where $\rho_1 = a_1+S^0+K_1S^{02}$ and $\rho_2 = a_2+S^0+K_2S^{02}$. Then we have

$$\begin{aligned}
D^+V(t) &= \left[\left(\frac{\left(a_1+S^0+K_1S^{02}\right)m_1}{a_1+S(t)+K_1S^2(t)}\right.\right.\\
&\quad\left.+\frac{\left(a_2+S^0+K_2S^{02}\right)m_2}{a_2+S(t)+K_2S^2(t)}\right)S(t)\\
&\quad-\sum_{i=1}^{2}D\left(a_i+S^0+K_iS^{02}\right)-c_1\left(a_1+S^0+K_1S^{02}\right)\\
&\quad\cdot x(t)-c_2\left(a_2+S^0+K_2S^{02}\right)y(t)\\
&\quad\left.-\sum_{i=1}^{2}\frac{\left(a_i+S^0+K_iS^{02}\right)\sigma_i^2S^2(t)}{2\left(a_i+S(t)+K_iS^2(t)\right)^2}\right]\mathrm{d}t\\
&\quad+\sum_{i=1}^{2}\frac{\left(a_i+S^0+K_iS^{02}\right)\sigma_iS(t)}{a_i+S(t)+K_iS^2(t)}\mathrm{d}B_i(t)\geq\left[\left(m_1\right.\right.\\
&\quad\left.+m_2\right)S(t)-\sum_{i=1}^{2}D\left(a_i+S^0+K_iS^{02}\right)-c_1\left(a_1+S^0\right.\\
&\quad\left.+K_1S^{02}\right)x(t)-\sum_{i=1}^{2}\frac{\sigma_i^2S^{02}}{2a_i}-c_2\left(a_2+S^0+K_2S^{02}\right)\\
&\quad\left.\cdot y(t)\right]\mathrm{d}t+\sum_{i=1}^{2}\frac{\left(a_i+S^0+K_iS^{02}\right)\sigma_iS(t)}{a_i+S(t)+K_iS^2(t)}\mathrm{d}B_i(t).
\end{aligned} \tag{59}$$

Integrating from 0 to t and dividing by t on both sides of (59) yields

$$\begin{aligned}
&\frac{V(t)}{t}-\frac{V(0)}{t}\\
&\quad\geq\left(m_1+m_2\right)\langle S(t)\rangle-\sum_{i=1}^{2}D\left(a_i+S^0+K_iS^{02}\right)\\
&\quad-c_1\left(a_1+S^0+K_1S^{02}\right)\langle x(t)\rangle\\
&\quad-c_2\left(a_2+S^0+K_2S^{02}\right)\langle y(t)\rangle-\sum_{i=1}^{2}\frac{\sigma_i^2S^{02}}{2a_i}\\
&\quad+\sum_{i=1}^{2}\frac{M_i(t)}{t}\\
&\quad\geq\left(m_1+m_2\right)S^0-\frac{\left(m_1+m_2\right)\left(c_1+c_2\right)}{D}S^{02}\\
&\quad-\sum_{i=1}^{2}D\left(a_i+S^0+K_iS^{02}\right)-\frac{\left(m_1+m_2\right)\epsilon(t)}{D}\\
&\quad-\left[m_1+m_2+c_1\left(a_1+S^0+K_1S^{02}\right)\right]\langle x(t)\rangle\\
&\quad-\left[m_1+m_2+c_2\left(a_2+S^0+K_2S^{02}\right)\right]\langle y(t)\rangle\\
&\quad-\sum_{i=1}^{2}\frac{\sigma_i^2S^{02}}{2a_i}+\sum_{i=1}^{2}\frac{M_i(t)}{t}\\
&\quad\geq\left(m_1+m_2\right)S^0-\frac{\left(m_1+m_2\right)\left(c_1+c_2\right)}{D}S^{02}\\
&\quad-\sum_{i=1}^{2}D\left(a_i+S^0+K_iS^{02}\right)-\frac{\left(m_1+m_2\right)\epsilon(t)}{D}\\
&\quad-\Delta_{\max}\left[\langle x(t)\rangle+\langle y(t)\rangle\right]-\sum_{i=1}^{2}\frac{\sigma_i^2S^{02}}{2a_i}\\
&\quad+\sum_{i=1}^{2}\frac{M_i(t)}{t},
\end{aligned}\tag{60}$$

where $M_i(t)=\int_0^t((a_i+S^0+K_iS^{02})\sigma_iS(\tau)/(a_i+S(\tau)+K_iS^2(\tau)))\mathrm{d}B_i(\tau)$. The inequality (60) can be rewritten as

$$\begin{aligned}
&\langle x(t)\rangle+\langle y(t)\rangle\\
&\quad\geq\frac{1}{\Delta_{\max}}\left[\frac{\ell_1}{D}\left(\frac{Dm_1S^0}{\ell_1}-\frac{D\sigma_1^2S^{02}}{2a_1\ell_1}-1\right)\right.\\
&\quad-\frac{\left(m_1+m_2\right)\epsilon(t)}{D}+\sum_{i=1}^{2}\frac{M_i(t)}{t}\\
&\quad\left.+\frac{\ell_2}{D}\left(\frac{Dm_2S^0}{\ell_2}-\frac{D\sigma_2^2S^{02}}{2a_2\ell_2}-1\right)-\frac{V(t)-V(0)}{t}\right].
\end{aligned}\tag{61}$$

By the strong law of large numbers and Lemma 2, we have $\lim_{t\longrightarrow+\infty}(M_i(t)/t)=0(i=1,2)$ and $\lim_{t\longrightarrow+\infty}\epsilon(t)=0$. Taking the inferior limit of both sides of (61) yields

$$\liminf_{t\longrightarrow+\infty}\langle x(t)+y(t)\rangle\geq\frac{1}{D\Delta_{\max}}\sum_{i=1}^{2}\ell_i\left(\overline{\mathscr{R}}_i-1\right)>0,\quad a.s.\tag{62}$$

This proof is completed. □

4. Conclusions and Simulations

In this paper, we propose and analyse the dynamics of two-microorganism and single nutrient stochastic chemostat models with Monod-Haldane response function. First, we discuss the existence and locally asymptotical stability of boundary equilibria of the system neglecting stochastic effect. Then, we investigate the dynamics of the system under stochastic effect and obtain the conditions which determine the persistence and extinction of the microorganisms with stochastic effect. Our results show that large stochastic noise can lead to microbial extinction (see Theorems 6 and 7), and small stochastic noise is beneficial to the survival of microorganisms (see Theorem 8).

In order to verify the theoretical results obtained in this paper, we give some numerical simulation. We choose the parameters in model (5) and model (6) as follows:

$$\begin{aligned}
S^0&=2.1,\\
D&=1,\\
m_1&=2.2,\\
m_2&=2.2,\\
a_1&=0.1,\\
a_2&=0.1,\\
K_1&=0.1,\\
K_2&=0.2,\\
c_1&=0.05,\\
c_2&=0.08,\\
\sigma_1&=1.7,\\
\sigma_2&=1.9.
\end{aligned}\tag{63}$$

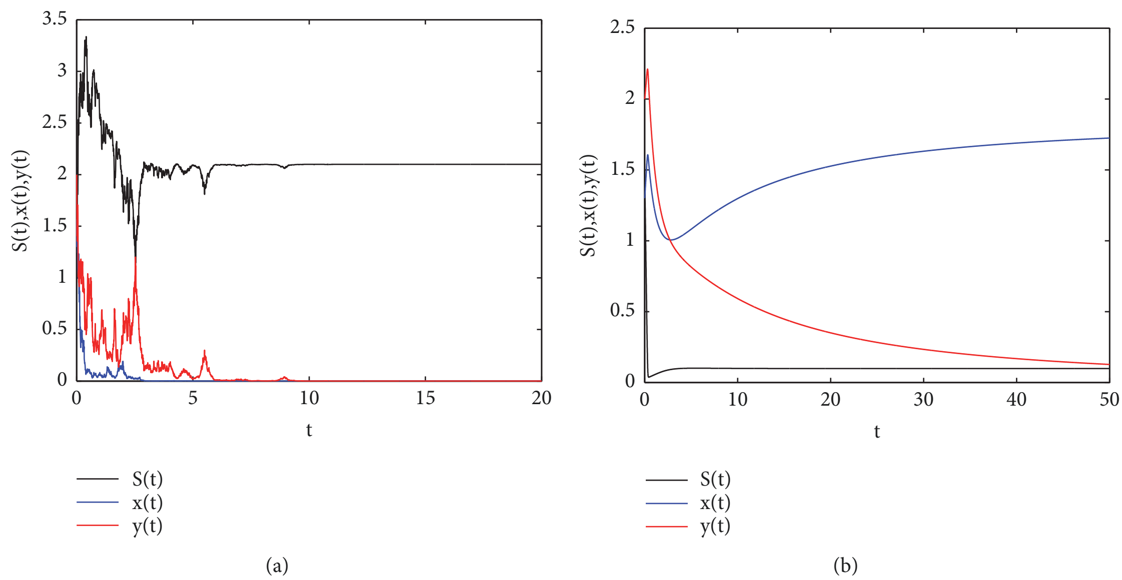

FIGURE 1: Computer simulation of the paths $S(t), x(t), y(t)$ for the chemostat model, where (a) is the stochastic system for $\sigma_1 = 1.7, \sigma_2 = 1.9$. (b) is the corresponding deterministic system of (a) for $\sigma_1 = 0, \sigma_2 = 0$.

Note that

$$\begin{aligned}
2.1 = S^0 &> \sqrt{\frac{a_1}{K_1}} = 1, \\
0.8333 = \frac{1}{1 + 2\sqrt{a_1 K_1}} &> \frac{m_1}{\sigma_1^2} = 0.7612, \\
2.8900 = \sigma_1^2 &> \frac{m_1^2}{2D} = 2.4200, \\
2.1 = S^0 &> \sqrt{\frac{a_2}{K_2}} = 0.7071, \\
0.7795 = \frac{1}{1 + 2\sqrt{a_2 K_2}} &> \frac{m_2}{\sigma_2^2} = 0.6094, \\
3.6100 = \sigma_2^2 &> \frac{m_2^2}{2D} = 2.4200.
\end{aligned} \tag{64}$$

Theorems 6 and 7 show that the microorganism die out under a large white noise disturbance intensity (see Figure 1(a) with $\sigma_1 = 1.7, \sigma_2 = 1.9$). Figure 1 shows that the persistent microorganism of a deterministic system (see Figure 1(b)) can become extinct due to the white noise stochastic disturbance. Therefore, the large white noise stochastic disturbance intensity is detrimental to the survival of microorganisms.

Theorem 8 indicates that the microorganism can become extinct or persistent under a small white noise disturbance intensity. We keep the system parameters same as those in Figure 1, and let σ_1 and σ_2 take different parameter values. When σ_1 is larger and σ_2 is smaller ($\sigma_1 = 1.6, \sigma_2 = 0.1$), here $0.8594 = m_1/\sigma_1^2 > 1/(1 + 2\sqrt{a_1 K_1}) = 0.8333, R_1 = 0.9444 < 1, \overline{\mathscr{R}}_2 = 1.0129 > 1$. Thus, the microorganism $x(t)$ tends to die out, and the microorganism $y(t)$ is persistent (see Figure 2(a)). Conversely, when σ_1 is smaller and σ_2 is larger ($\sigma_1 = 0.15, \sigma_2 = 1.65$), here $0.8081 = m_2/\sigma_2^2 > 1/(1+2\sqrt{a_2 K_2}) = 0.7795, R_2 = 0.8878 < 1, \overline{\mathscr{R}}_1 = 1.0568 > 1$, Figure 2(b) shows that the microorganism $y(t)$ tends to die out, and the microorganism $x(t)$ is persistent. Furthermore, let σ_1 and σ_2 take small parameter values ($\sigma_1 = 0.05, \sigma_2 = 0.05$); here $\overline{\mathscr{R}}_1 = 1.1698 > 1, \overline{\mathscr{R}}_2 = 1.0510 > 1$, then both $x(t)$ and $y(t)$ are persistent (see Figure 2(c)); that is, the two microorganisms can coexist. Although σ_1 and σ_2 still take small parameter values ($\sigma_1 = 0.4, \sigma_2 = 0.7$), when $7.8750 = m_1/\sigma_1^2 > 1/(1+2\sqrt{a_1 K_1}) = 0.8333, R_1 = 0.9944 < 1, 3.0000 = m_2/\sigma_2^2 > 1/(1 + 2\sqrt{a_2 K_2}) = 0.7795, R_2 = 0.9970 < 1$, both $x(t)$ and $y(t)$ can become extinct (see Figure 2(d)). The above simulations support our results in the article.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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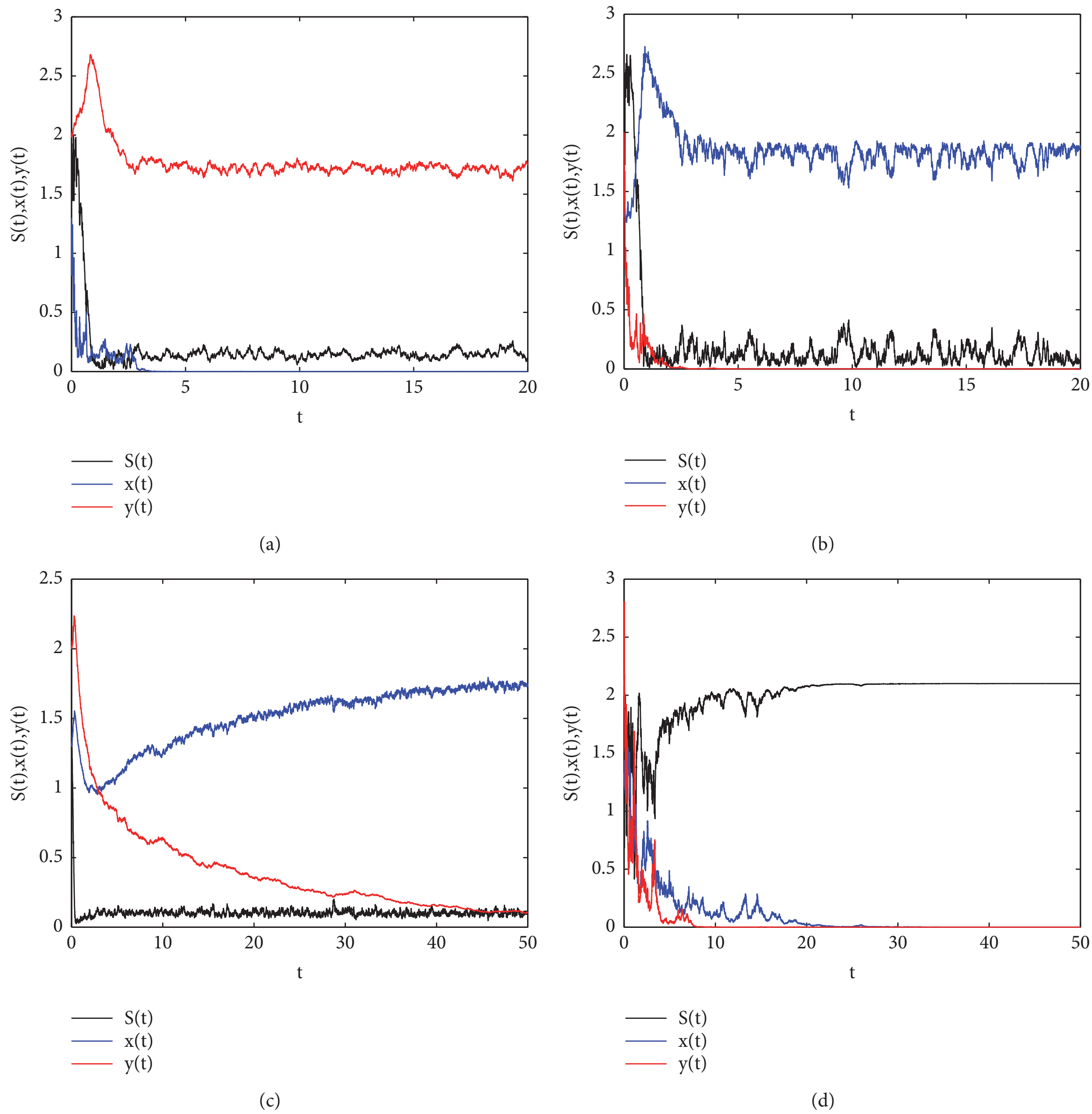

Figure 2: Numerical simulation of the paths $S(t)$, $x(t)$, $y(t)$ for the chemostat stochastic system (6), where (a) $\sigma_1 = 1.6, \sigma_2 = 0.1. R_1 = 0.9444 < 1, \overline{\mathscr{R}}_2 = 1.0129 > 1$. (b) $\sigma_1 = 0.15, \sigma_2 = 1.65. R_2 = 0.8878 < 1, \overline{\mathscr{R}}_1 = 1.0568 > 1$. (c) $\sigma_1 = 0.05, \sigma_2 = 0.05. \overline{\mathscr{R}}_1 = 1.1698 > 1, \overline{\mathscr{R}}_2 = 1.0510 > 1$. (d) $m_1 = 1.26, m_2 = 1.47. \sigma_1 = 0.4, \sigma_2 = 0.7. R_1 = 0.9944 < 1, R_2 = 0.9970 < 1$.

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