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Research Article

Optimality Conditions and Scalarization of Approximate Quasi Weak Efficient Solutions for Vector Equilibrium Problem

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This paper is devoted to the investigation of optimality conditions for approximate quasi weak efficient solutions for a class of vector equilibrium problem (VEP). First, a necessary optimality condition for approximate quasi weak efficient solutions to VEP is established by utilizing the separation theorem with respect to the quasirelative interior of convex sets and the properties of the Clarke subdifferential. Second, the concept of approximate pseudoconvex function is introduced and its existence is verified by a concrete example. Under the assumption of introduced convexity, a sufficient optimality condition for VEP in sense of approximate quasi weak efficiency is also presented. Finally, by using Tammer's function and the directed distance function, the scalarization theorems of the approximate quasi weak efficient solutions of the VEP are proposed.

1. Introduction

Vector equilibrium, which is closely related to complementarity problems, variational inequalities, and fixed point theory, is one of the momentous contents in the field of applied mathematics. The characteristics and optimality conditions of various solutions are the key study of vector equilibrium problems. For instance, the optimality conditions for efficient solutions to vector equilibrium problem were presented in [1]; the literatures [2, 3] derived the optimality conditions of weakly efficient solutions; some optimality conclusions related to several properly efficient solutions were established in [4–7]. In practical applications, the majority of solutions obtained by numerical algorithms are approximate solutions. Undoubtedly, it is of great theoretical and practical significance to study the approximate solutions of vector equilibrium problem. In recent years, the concept of approximate weak efficient solutions for vector equilibrium problem was introduced and its properties were discussed in [8, 9]. Das and Nahak [10] presented the concept of approximate quasi weak efficient solutions to vector equilibrium problem and examined its optimality conditions by generalized derivatives. One of the main purposes of this paper is to establish the necessary optimality condition for approximate quasi weak efficient solutions to vector equilibrium problem via the quasirelative interior-type separation theorem of convex sets. It is worth mentioning that our method is different from that of Das and Nahak [10].

Convexity and its generalization play a critical role in optimization and vector equilibrium theory, especially in establishing the sufficient optimality conditions. For instance, Gong [11, 12] derived the sufficient optimality condition to approximate efficient solutions for vector equilibrium problem under the cone convexity; under the assumptions of arcwise connected functions, the sufficient optimality conditions with regard to properly efficient solutions to vector equilibrium problem are presented in the literature [13]; based on the assumption of generalized cone subconvexlikeness, the literature [14] proposed the properties of globally efficient solutions to vector equilibrium problem. In this paper, we will introduce notion of approximate quasi-pseudoconvex function in terms of *Clarke* subdifferential, and under its assumption, we establish the sufficient optimality condition of approximate quasi weak efficient solutions to vector equilibrium problem, which is another aim of this paper.

Scalarization is to transform a vector problem into a numerical (scalar) problem which is equivalent to primal vector problem under mild conditions. There is no doubt that scalarization is one of the core topics in the study of vector equilibrium problem. In present paper, we will utilize Tammer's nonlinear scalar function and the directed distance function to deal with the scalarization theorems for the approximate quasi weak efficient solutions to vector equilibrium problem.

In the view of the above discussion, the paper will examine the optimality conditions and scalarization theorems in sense of approximate quasi weak efficient solutions to vector equilibrium problem. The article is arranged as follows: in section 2, some symbols, concepts, and lemmas will be presented, which will be used in the subsequent sections; Section 3 is devoted to establish the optimality conditions for approximate quasi weak efficient solutions to the discussed vector equilibrium problem; in section 4, the scalarization theorems will be proven.

2. Preliminaries

Throughout the paper, we set

$$\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n):\ x_i \geq 0, \quad i = 1, 2, \ldots, n\}. \tag{1}$$

Let X and Y be real Banach spaces with topological dual spaces X^* and Y^*, respectively, and $\mathbb{B}(\overline{x}, r)$ stands for the open ball of radius $r > 0$ around $\overline{x} \in X$. For all $x \in X$ and $x^* \in X^*$, the value of linear functional x^* at x be denoted by $\langle x^*,\ x\rangle$. Let Q be a pointed closed convex cone in Y, then the dual cone of Q be defined as (see [15])

$$Q^* = \{y^* \in Y^*:\ \langle y^*, y\rangle \geq 0, \quad \forall\, y \in Q\}. \tag{2}$$

Without other specifications, we always suppose that Q is a pointed closed convex cone in Y. We will use the following properties of Q.

Lemma 1 (see [16]). *If $y^* \in Q^*/\{0\}$ and $y \in \mathrm{int}Q$, then $\langle y^*, y\rangle > 0$, where* int *represents the interior of a set.*

Let K be a nonempty subset of X, and the Clarke contingent cone (see [1]) to set K at point $\overline{x} \in K$ is defined as

$$T(\overline{x}; K) = \{y \in X:\ \exists\, t_n \longrightarrow 0,\ y_n \longrightarrow y, \quad \text{s.t}\, \overline{x} + t_n y_n \in K\}. \tag{3}$$

The Clarke normal cone (see [1]) associated with $T(\overline{x}; K)$ is denoted by

$$N(\overline{x}; K) = \{\xi \in X^*:\ \langle \xi, y\rangle \leq 0, \quad \forall\, y \in T(\overline{x}; K)\}. \tag{4}$$

Especially when K be a convex set, the Clarke contingent cone to set K at $\overline{x}$ is given by (see [15])

$$T(\overline{x}; K) = \mathrm{cl}\{y \in X:\ y = \beta(x - \overline{x}), \quad x \in K, \beta > 0\}. \tag{5}$$

The Clarke normal cone to set K at $\overline{x}$ is

$$N(\overline{x}; K) = \{\xi \in X^*:\ \langle \xi, x - \overline{x}\rangle \leq 0, \quad \forall\, x \in K\}, \tag{6}$$

where cl stands for the closure of a set.

Let F: $X \longrightarrow Y$ be a mapping. F is said to be locally Lipschitz at $\overline{x} \in X$, if there exist constant $L > 0$ and $r > 0$ such that

$$\|F(x_1) - F(x_2)\| \leq L\|x_1 - x_2\|, \quad \forall\, x_1, x_2 \in \mathbb{B}(\overline{x}, r). \tag{7}$$

If for any $x \in X$, F is locally Lipschitz at x, then F is called locally Lipschitz mapping. In particular, for a real-valued locally Lipschitz function f: $X \longrightarrow \mathbb{R}$ ($\mathbb{R}$ denotes real number), the Clarke generalized directional derivative of f at $\overline{x} \in X$ in the direction $d \in X$ is given by (see [15])

$$\begin{aligned} f^{\circ}(\overline{x}; d) &= \lim_{y \longrightarrow \overline{x}} \sup_{\lambda \longrightarrow 0^+} \frac{f(y + \lambda d) - f(y)}{\lambda}, \\ \partial f(\overline{x}) &= \left\{\xi \in X^*:\ f^{\circ}(\overline{x}; d) \geq \langle \xi, d\rangle, \quad \forall d \in X\right\}, \end{aligned} \tag{8}$$

which is defined as the Clarke subdifferential of f at $\overline{x}$.

We present below some significant properties of locally Lipschitz function that we shall use in the sequel.

Lemma 2 (see [15,17]). *Let function f: $K \subset X \longrightarrow \mathbb{R}$ is locally Lipschitz at $\overline{x} \in K$, if $\overline{x}$ is the minimum value point of f on K, then*

$$0 \in \partial f(\overline{x}) + N(\overline{x}; K). \tag{9}$$

Lemma 3 (see [15]). *Let f_i: $X \longrightarrow \mathbb{R}$, $i = 1, \ldots, m$, be locally Lipschitz at $\overline{x} \in X$, then function $\varphi(\cdot)$: $= \max\{f_i(\cdot):\ i = 1, \ldots, m\}$ is also locally Lipschitz at $\overline{x}$, and*

$$\begin{aligned} \partial \varphi(\overline{x}) &\subset \bigcup \left\{\sum_{i=1}^{m} \lambda_i \partial f_i(\overline{x}):\ \lambda_i \geq 0, \quad i = 1, 2, \ldots m, \ \sum_{i=1}^{m} \lambda_i = 1, \quad \lambda_i(f_i(\overline{x}) - \varphi(\overline{x})) = 0\right\}, \\ \partial(f_1 + \cdots + f_m)(\overline{x}) &\subset \partial f_1(\overline{x}) + \cdots + \partial f_m(\overline{x}). \end{aligned} \tag{10}$$

Let $K \subset X$ be a nonempty subset, and F: $K \times K \longrightarrow Y$ be a mapping. Consider the following vector equilibrium problem (VEP):

$$\text{(VEP) find } \overline{x} \in K, \text{ such that } F(\overline{x}, x) \notin -Q\backslash\{0\}, \quad \forall\, x \in K. \tag{11}$$

Given $\overline{x} \in K$, $F_{\overline{x}}$: $K \longrightarrow Y$ be vector-valued mapping of one variable, which is defined by

$$F_{\overline{x}}(x) := F(\overline{x}, x), \quad \forall x \in K. \tag{12}$$

Throughout this paper, it is always assumed that $F_{\overline{x}}(\overline{x}) = 0$ and

$$F_{\overline{x}}(K) = F(\overline{x}, K) = \bigcup_{x \in K} F(\overline{x}, x). \tag{13}$$

Definition 1 (see [10]). Let $K \subset X$ be a nonempty subset, $\varepsilon \geq 0$, $e \in \mathrm{int}Q$. $\overline{x} \in K$ is called an εe-quasi weak efficient solution to VEP, if

$$F(\overline{x}, x) + \varepsilon \|x - \overline{x}\| e \notin -\mathrm{int}Q, \quad \forall x \in K. \tag{14}$$

The notion of εe-quasi weak efficient solution is illustrated by the following example.

Example 1. Let $X = Y = \mathbb{R}$, $K = \mathbb{R}_+$, $Q = \mathbb{R}^2_+$, and $\overline{x} \in K$. Consider the following questions:

$$F(\overline{x}, x) = (-|x - \overline{x}|, |x - \overline{x} - |x - \overline{x}|), \quad \forall x \in K. \tag{15}$$

Taking $\varepsilon = 1$ and $e = (1, 1)$, then

$$\begin{aligned} F(\overline{x}, x) + \varepsilon \|x - \overline{x}\| e &= (-|x - \overline{x}|, |x - \overline{x} - |x - \overline{x}|) \\ &\quad + (|x - \overline{x}|, |x - \overline{x}|) \\ &= (0, x - \overline{x}). \end{aligned} \tag{16}$$

Taking $\overline{x} = 0$, for all $x \in K$, we obtain

$$F(\overline{x}, x) + \varepsilon \|x - \overline{x}\| e = (0, x) \notin -\mathrm{int}Q. \tag{17}$$

Hence, 0 is an εe-quasi weak efficient solution of VEP.

It is well known that, for a nonempty convex set, its interior may be empty, but its quasirelative interior is always nonempty (see [18]). In this paper, we will prove the optimality condition of VEP by the separation theorem with respect to the quasirelative interior of convex sets (see [19]).

Definition 2 (see [18]). Let $K \subset X$ is a convex subset; the quasirelative interior of K denoted by $\mathrm{qri}K$ is defined as

$$\mathrm{qri}K = \{x \in K: \ \mathrm{clcone}(K - x) \text{ is a linear subset of } X\}, \tag{18}$$

where cl and cone stand for closure and cone hull.

Lemma 4 (see [19]). *Let M and N be nonempty convex subsets of Y, $qriM \neq \varnothing$ and $qriN \neq \varnothing$, and clcone $(qriM - qriN)$ is not a linear subset of Y, then there exists $\lambda \in Y^*\{0\}$ such that*

$$\langle \lambda, m \rangle \leq \langle \lambda, n \rangle, \quad \forall m \in M, \forall n \in N. \tag{19}$$

3. Optimality Conditions

In this section, first, we propose a necessary optimality condition for εe-quasi weak efficient solutions to VEP by using separation theorem in terms of quasirelative interiors of a convex set. Second, the concept of approximate quasi-pseudoconvex function is introduced and a sufficient optimality conditions is established under the introduced generalized convexity. Throughout this section, let $K \subset X$ be a nonempty convex set.

Theorem 1. *In VEP, let $\overline{x} \in K$, $\varepsilon \geq 0$, and $e \in \mathrm{int}Q$. Assume that $\overline{x}$ be an εe-quasi weak efficient solution of VEP and $F_{\overline{x}}$: $X \longrightarrow Y$ is locally Lipschitz mapping at $\overline{x}$. In addition, $qriF_{\overline{x}}(K) \neq \varnothing$ and clcone $[qri(coF_{\overline{x}}(K)) + qriQ]$ is not a linear subspace of Y. Then, there exist $\lambda \in Q^*\{0\}$ such that*

$$0 \in \partial(\lambda \circ F_{\overline{x}}(\overline{x})) + N(\overline{x}; K) + \langle \lambda, e \rangle \varepsilon \mathbb{B}, \tag{20}$$

where $co(\cdot)$ stands for the convex hull, $\mathbb{B}: = \mathbb{B}(0,1)$, and $\lambda^\circ F_{\overline{x}}(\cdot): = \langle \lambda, F_{\overline{x}}(\cdot) \rangle$.

Proof 1. Since

$$\mathrm{qri}Q = -\mathrm{qri}(-Q), \tag{21}$$

and clcone $[\mathrm{qri}(\mathrm{co}F_{\overline{x}}(K)) + \mathrm{qri}Q]$ is not a linear subspace of Y, then clcone $[\mathrm{qri}(\mathrm{co}F_{\overline{x}}(K)) - \mathrm{qri}(-Q)]$ is not a linear subspace of Y. Moreover,

$$\mathrm{qri}F_{\overline{x}}(K) \neq \varnothing. \tag{22}$$

Thus,

$$\mathrm{qri}[\mathrm{co}F_{\overline{x}}(K)] \neq \varnothing. \tag{23}$$

Noticing that $\mathrm{qri}Q \neq \varnothing$, it follows from Lemma 4 that there exists $\lambda \in Y^*\backslash\{0\}$ such that

$$\langle \lambda, q \rangle \leq \langle \lambda, x \rangle, \quad \forall q \in -Q, \forall x \in \mathrm{co}F_{\overline{x}}(K), \tag{24}$$

which means

$$\langle \lambda, q \rangle \leq \langle \lambda, F_{\overline{x}}(x) \rangle, \quad \forall q \in -Q, \forall x \in K. \tag{25}$$

Taking $x = \overline{x}$ in the above formula, we obtain

$$\langle \lambda, q \rangle \leq 0, \quad \forall q \in -Q. \tag{26}$$

Hence, $\lambda \in Q^*\backslash\{0\}$. Since

$$F(\overline{x}, x) + \varepsilon \|x - \overline{x}\| e \notin -\mathrm{int}Q, \quad \forall x \in K, \tag{27}$$

and $F(\overline{x}, \overline{x}) = 0$, it leads to

$$F(\overline{x}, x) - F(\overline{x}, \overline{x}) + \varepsilon \|x - \overline{x}\| e \notin -\mathrm{int}Q, \quad \forall x \in K. \tag{28}$$

It follows from $\lambda \in Q^*\backslash\{0\}$ and equation (28) that

$$\langle \lambda, F(\overline{x}, x) - F(\overline{x}, \overline{x}) + \varepsilon \|x - \overline{x}\| e \rangle \geq 0, \quad \forall x \in K, \tag{29}$$

that is

$$\langle \lambda, F_{\overline{x}}(x) \rangle - \langle \lambda, F_{\overline{x}}(\overline{x}) \rangle + \langle \lambda, e \rangle \varepsilon \|x - \overline{x}\| \geq 0, \quad \forall x \in K. \tag{30}$$

On the other hand, let $f := \lambda \circ F_{\overline{x}}$. Since $F_{\overline{x}}$ is locally Lipschitz at $\overline{x}$, it is obvious that f is a locally Lipschitz function at $\overline{x}$. We set

$$\varphi(x) = f(x) - f(\overline{x}) + \langle \lambda, e \rangle \varepsilon \|x - \overline{x}\|, \quad \forall x \in K. \tag{31}$$

It follows from equation (29) that

$$\varphi(x) \geq 0 = \varphi(\overline{x}), \quad \forall x \in K, \tag{32}$$

which shows that $\overline{x}$ is the minimum point of $\varphi(x)$ on K. Taking account of Lemma 2, we arrive at

$$0 \in \partial\varphi(\overline{x}) + N(\overline{x}; K). \tag{33}$$

Since f is a locally Lipschitz function at $\overline{x}$, by Lemma 3, we have

$$\begin{aligned} \partial\varphi(\overline{x}) &\subset \partial\left(f + \langle\lambda, e\rangle\varepsilon\|\cdot - \overline{x}\|\right)(\overline{x}), \\ &\subset \partial\left(\lambda \circ F_{\overline{x}}(\overline{x})\right) + \langle\lambda, e\rangle\varepsilon\mathbb{B}. \end{aligned} \tag{34}$$

Together with equation (33), we obtain

$$0 \in \partial\left(\lambda \circ F_{\overline{x}}(\overline{x})\right) + N(\overline{x}; K) + \langle\lambda, e\rangle\varepsilon\mathbb{B}. \tag{35}$$

Next, we introduce the concept of approximate quasi-pseudoconvex function, and under the assumption of this generalized convexity, a sufficient optimality condition for εe-quasi weak efficient solutions to VEP is derived.

Definition 3. Let $\varepsilon \geq 0$ and the function $f\colon X \longrightarrow \mathbb{R}$ be locally Lipschitz at $\overline{x} \in X$. f is said to be ε-quasi-pseudoconvex at $\overline{x}$, if there exists $\xi \in \partial f(\overline{x})$ such that for each $x \in X$ satisfying

$$\langle\xi, x - \overline{x}\rangle + \varepsilon\|x - \overline{x}\| \geq 0 \Longrightarrow f(x) - f(\overline{x}) + \varepsilon\|x - \overline{x}\| \geq 0. \tag{36}$$

Example 2. Let $X = \mathbb{R}$, then $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \dfrac{2}{3}x^2 + x, & \text{if } x < 0, \\ \ln(x+1), & \text{if } x \geq 0. \end{cases} \tag{37}$$

Taking $\varepsilon = 1$ and $\overline{x} = 0$, by a simple computation, we derive $\partial f(\overline{x}) = \{1\}$. For any $x \in \mathbb{R}$, $1 = \xi \in \partial f(0)$, if

$$\langle\xi, x - \overline{x}\rangle + \varepsilon\|x - \overline{x}\| = 1 \cdot x + 1 \cdot |x| \geq 0, \tag{38}$$

then

$$f(x) - f(\overline{x}) + \varepsilon\|x - \overline{x}\| = f(x) + 1 \cdot |x| = \begin{cases} \dfrac{2}{3}x^2 \geq 0, & x < 0, \\ \ln(x+1) + x \geq 0, & x \geq 0. \end{cases} \tag{39}$$

Thus, f is a 1-quasi-pseudoconvex at 0.

Theorem 2. *In VEP, let $\varepsilon \geq 0$, $e \in \operatorname{int}Q$, $\overline{x} \in K$, and $F_{\overline{x}}\colon K \longrightarrow Y$ be locally Lipschitz at $\overline{x}$. Suppose that there exists $\lambda \in Q^*\backslash\{0\}$ such that*

$$0 \in \partial\left(\lambda \circ F_{\overline{x}}(\overline{x})\right) + N(\overline{x}; K) + \langle\lambda, e\rangle\varepsilon\mathbb{B}. \tag{40}$$

If $\lambda \circ F_{\overline{x}}\colon K \longrightarrow \mathbb{R}$ is $\langle\lambda, e\rangle\varepsilon$-quasi-pseudoconvex at $\overline{x}$, then $\overline{x}$ is εe-quasi weak efficient solutions of VEP.

Proof 2. It follows from (40) that there exist $\xi \in \partial(\lambda \circ F_{\overline{x}}(\overline{x}))$, $\sigma \in N(\overline{x}; K)$, and $b \in \mathbb{B}$ such that

$$\xi + \sigma + \langle\lambda, e\rangle\varepsilon b = 0, \tag{41}$$

which implies for each $x \in K$,

$$\langle\xi + \sigma + \langle\lambda, e\rangle\varepsilon b,\ x - \overline{x}\rangle = 0, \tag{42}$$

which is equivalent to

$$\langle\xi, x - \overline{x}\rangle + \langle\sigma, x - \overline{x}\rangle + \langle\lambda, e\rangle\varepsilon\langle b, x - \overline{x}\rangle = 0, \quad \forall x \in K. \tag{43}$$

Since K is a convex set, according to the definition of contingent cone to set K at $\overline{x}$,

$$T(\overline{x}; K) = \operatorname{cl}\{y \in X\colon\ y = \beta(x - \overline{x}), \quad x \in K, \beta > 0\}. \tag{44}$$

Therefore,

$$\langle\sigma, x - \overline{x}\rangle \leq 0, \quad \forall x \in K. \tag{45}$$

Combining (43) and (44), it is not difficult to find

$$\langle\xi, x - \overline{x}\rangle + \langle\lambda, e\rangle\varepsilon\langle b, x - \overline{x}\rangle \geq 0, \quad \forall x \in K. \tag{46}$$

Because $b \in \mathbb{B}$, we obtain $\|b\| = 1$. Hence,

$$\langle b, x - \overline{x}\rangle \leq \|x - \overline{x}\|, \quad \forall x \in K. \tag{47}$$

Together with equation (46), it leads to

$$\langle\xi, x - \overline{x}\rangle + \langle\lambda, e\rangle\varepsilon\|x - \overline{x}\| \geq 0, \quad \forall x \in K. \tag{48}$$

Since $\lambda \circ F_{\overline{x}}$ is $\langle\lambda, e\rangle\varepsilon$-quasi-pseudoconvex at $\overline{x}$, by Definition 3, we obtain

$$\lambda \circ F_{\overline{x}}(x) - \lambda \circ F_{\overline{x}}(\overline{x}) + \langle\lambda, e\rangle\varepsilon\|x - \overline{x}\| \geq 0, \quad \forall x \in K. \tag{49}$$

In view of $F_{\overline{x}}(\overline{x}) = 0$, we arrive at

$$\lambda \circ F_{\overline{x}}(x) + \langle\lambda, e\rangle\varepsilon\|x - \overline{x}\| \geq 0, \quad \forall x \in K. \tag{50}$$

Suppose that $\overline{x}$ is not εe-quasi weak efficient solutions of VEP, then there exists $\widehat{x} \in K$ such that

$$F(\overline{x}, \widehat{x}) + \varepsilon\|\widehat{x} - \overline{x}\|e \in -\operatorname{int}Q. \tag{51}$$

Since $\lambda \in Q^*\backslash\{0\}$, it yields from Lemma 1 that

$$\langle\lambda, F(\overline{x}, \widehat{x}) + \varepsilon\|\widehat{x} - \overline{x}\|e\rangle < 0, \tag{52}$$

which means

$$\langle\lambda, F(\overline{x}, \widehat{x})\rangle + \langle\lambda, \varepsilon\|\widehat{x} - \overline{x}\|e\rangle < 0. \tag{53}$$

That is,

$$\lambda \circ F_{\overline{x}}(\widehat{x}) + \langle\lambda, e\rangle\varepsilon\|\widehat{x} - \overline{x}\| < 0, \tag{54}$$

which contradicts (50). Hence, $\overline{x}$ is εe-quasi weak efficient solutions of VEP.

4. Scalarization

In this section, the scalarization theorems for approximate quasi weak efficient solutions to VEP are established by using Tammer's function and the directed distance function, respectively.

4.1. Scalarization via Tammer's Function

Lemma 5 (see [20]). *Let $Q \subset Y$ is a pointed closed convex cone and $e \in \mathrm{int}Q \neq \varnothing$ is a fixed element, then Tammer's function Ψ_e^Q: $Y \longrightarrow \mathbb{R}$ ($\mathbb{R}$ represents the set of real number) is defined by*

$$\Psi_e^Q(y) = \inf\{t \in R\colon y \in te - Q\}, \quad y \in Y. \tag{55}$$

Then, Ψ_e^Q is continuous sublinear functional and

$$\left\{y \in Y\colon \Psi_e^Q(y) < 0\right\} = -\mathrm{int}Q. \tag{56}$$

Definition 4. Let K be a nonempty subset of X, $\varepsilon \geq 0$, and $f\colon X \longrightarrow \mathbb{R}$ is a real-valued function. Define optimization problem (P) as follows:

$$(P)\inf f(x), \quad \text{s.t. } x \in K. \tag{57}$$

$\overline{x}$ is called a ε-quasi-optimality solution of (P) if

$$f(x) - f(\overline{x}) + \varepsilon\|x - \overline{x}\| \geq 0, \quad \forall x \in K. \tag{58}$$

Let $\overline{x} \in K$ and $e \in \mathrm{int}Q$. Based on VEP and Tammer's function Ψ_e^Q, consider the following scalarization problem $(P_{\Psi_e^Q})$:

$$\left(P_{\Psi_e^Q}\right)\inf\Psi_e^Q(F_{\overline{x}}(x)), \quad \text{s.t. } x \in K. \tag{59}$$

Theorem 3. *Let $\varepsilon \geq 0$ and $e \in \mathrm{int}Q$. If $\overline{x} \in K$ is εe-quasi weak efficient solutions of VEP, then $\overline{x}$ is ε-quasi-optimality solutions of scalarization problem $(P_{\Psi_e^Q})$.*

Proof 3. Since $\overline{x} \in K$ is εe-quasi weak efficient solutions of VEP, then

$$F(\overline{x}, x) + \varepsilon\|x - \overline{x}\|e \notin -\mathrm{int}Q, \quad \forall x \in K. \tag{60}$$

Considering Tammer's nonlinear scalarization function Ψ_e^Q,

$$\Psi_e^Q(y) = \inf\{t \in \mathbb{R}\colon y \in te - Q\}, \quad y \in Y. \tag{61}$$

According to Lemma 5 and combining (60) and (61) yield that

$$\Psi_e^Q(F(\overline{x}, x) + \varepsilon\|x - \overline{x}\|e) \geq 0, \quad \forall x \in K. \tag{62}$$

Since Ψ_e^Q is continuous sublinear functional, it holds that

$$\Psi_e^Q(F(\overline{x}, x)) + \varepsilon\|x - \overline{x}\| \geq 0, \quad \forall x \in K. \tag{63}$$

Since $F(\overline{x}, \overline{x}) = 0$, then

$$\Psi_e^Q(F(\overline{x}, x)) - \Psi_e^Q(F(\overline{x}, \overline{x})) + \varepsilon\|x - \overline{x}\| \geq 0, \quad \forall x \in K. \tag{64}$$

Therefore, $\overline{x}$ is ε-quasi-optimality solutions of scalarization problem $(P_{\Psi_e^Q})$.

Theorem 4. *Let $\varepsilon \geq 0$, $e \in \mathrm{int}Q$, and $\overline{x} \in K$. Suppose function φ: $Y \longrightarrow \mathbb{R}$ satisfying the following:*

(i) φ is monotone with respect to the pointed closed convex cone Q, that is, if $y_1 - y_2 \in Q$, then $\varphi(y_1) \geq \varphi(y_2)$

(ii) $\varphi(0) = 0$, and φ is positively homogeneous functional that means $\varphi(ay) = a\varphi(y)$, $a > 0$

(iii) $\varphi(-\varepsilon e) < -\varepsilon$

Let scalarization problem (P_φ) be defined by

$$\left(P_\varphi\right)\inf\varphi(F_{\overline{x}}(x)), \quad \text{s.t. } x \in K. \tag{65}$$

If $\overline{x}$ is ε-quasi-optimality solutions of scalarization problem (P_φ), then $\overline{x}$ is εe-quasi weak efficient solutions of VEP.

Proof 4. If $\overline{x}$ is not εe-quasi weak efficient solutions of VEP, there would exist $\hat{x} \in K$ such that

$$F(\overline{x}, \hat{x}) + \varepsilon\|\hat{x} - \overline{x}\|e \in -\mathrm{int}Q, \tag{66}$$

which is equivalent to

$$-\varepsilon\|\hat{x} - \overline{x}\|e - F(\overline{x}, \hat{x}) \in \mathrm{int}Q. \tag{67}$$

Then, we have

$$-\varepsilon\|\hat{x} - \overline{x}\|e - F(\overline{x}, \hat{x}) \in Q. \tag{68}$$

Since φ is monotone with respect to Q,

$$\varphi(F(\overline{x}, \hat{x})) \leq \varphi(-\varepsilon\|\hat{x} - \overline{x}\|e). \tag{69}$$

Noticing that $\overline{x}$ is ε-quasi-optimality solutions of problem (P_φ), we obtain

$$\varphi(F(\overline{x}, \hat{x})) - \varphi(F(\overline{x}, \overline{x})) + \varepsilon\|\hat{x} - \overline{x}\| \geq 0. \tag{70}$$

Because $F(\overline{x}, \overline{x}) = 0$ and $\varphi(0) = 0$, it holds that

$$\varphi(F(\overline{x}, \hat{x})) + \varepsilon\|\hat{x} - \overline{x}\| \geq 0. \tag{71}$$

Combining (69) and (71), we obtain

$$\varphi(-\varepsilon\|\hat{x} - \overline{x}\|e) \geq -\varepsilon\|\hat{x} - \overline{x}\|. \tag{72}$$

Since φ is positively homogeneous functional,

$$\varphi(-\varepsilon e) \geq -\varepsilon, \tag{73}$$

which contradicts to condition (iii).

4.2. Scalarization via the Directed Distance Function. Let us introduce the concept of directed distance function.

Definition 5 (see [21]). Let $A \subseteq Y$ is a nonempty subset, then the directed distance function Δ_A: $Y \longrightarrow \mathbb{R}$ be defined as

$$\Delta_A(y) = d_A(y) - d_{Y\backslash A}(y), \quad y \in Y, \tag{74}$$

where

$$d_A(y) = \inf_{x \in A}\|y - x\|. \tag{75}$$

Lemma 6 (see [21]). *Let $A \subseteq Y$ is a nonempty subset, then the following properties hold:*

(i) *Δ_A is real-valued Lipschitz function of rank 1.*

(ii) *If A is closed, then $A = \{y: \Delta_A(y) \leq 0\}$; if A is a cone, then Δ_A is positively homogeneous; if A is pointed closed convex cone, then Δ_A is sublinear.*

(iii) *If A is closed convex cone, then Δ_A is nonincreasing with respect to A, that is, if $y_1, y_2 \in Y$, then*

$$y_1 - y_2 \in A \Longrightarrow \Delta_A(y_1) \leq \Delta_A(y_2);\ y_1 - y_2 \in \mathrm{int}A \Longrightarrow \Delta_A(y_1) < \Delta_A(y_2). \tag{76}$$

Let $e \in \mathrm{int}Q$ and $\overline{x} \in K$. Based on VEP and the directed distance function Δ_{-Q}, consider the following scalarization problem $(P_{\Delta_{-Q}})$:

$$\left(P_{\Delta_{-Q}}\right) \inf \Delta_{-Q}(F_{\overline{x}}(x)), \quad \text{s.t. } x \in K. \tag{77}$$

Theorem 5. *Let $\varepsilon > 0$, $e \in \mathrm{int}Q$ with $\|e\| = 1$, and $\overline{x} \in K$. If $\overline{x}$ is εe-quasi weak efficient solutions to VEP, then for any $x \in K$, $\overline{x}$ is an ε-quasi-optimality solution of scalarization problem $(P_{\Delta_{-Q}})$.*

Proof 5. By the given conditions, we have

$$F(\overline{x}, x) + \varepsilon\|x - \overline{x}\|e \notin -\mathrm{int}Q, \quad \forall x \in K. \tag{78}$$

Since Q is a pointed closed convex cone, by Lemma 6, it yields that

$$\Delta_{-Q}(F(\overline{x}, x)) + \varepsilon\|x - \overline{x}\|_{-Q}(e) \geq \Delta_{-Q}(F(\overline{x}, x) + \|x - \overline{x}\|e) \geq 0, \quad \forall x \in K. \tag{79}$$

Noticing that $\Delta_{-Q}(e) = \|e\| = 1$, $F(\overline{x}, \overline{x}) = 0$, and $\Delta_{-Q}(0) = 0$, we arrive at

$$\Delta_{-Q}(F(\overline{x}, x)) - \Delta_{-Q}(F(\overline{x}, x)) + \varepsilon\|x - \overline{x}\| \geq 0, \quad \forall x \in K. \tag{80}$$

Hence, $\overline{x}$ is an ε-quasi-optimality solution of scalarization problem $(P_{\Delta_{-Q}})$.

5. Conclusions

Making use of the quasirelative interior-type separation theorem of convex set, we have examined the optimality condition of the approximate quasi weak efficient solutions of VEP. In addition, the scalarization theorems of approximate quasi weak efficient solutions to VEP are also established via using Tammer's function and directed distance function, respectively, and scalarization theorems realize the purpose that solving the approximate quasi weak efficient solutions of vector equilibrium problem is equivalent to solving the approximate quasi-optimality solution of a specific scalar optimization problem.

Abbreviations

VEP: Vector equilibrium problem.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the manuscript and read and approved the final version of the manuscript.

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