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Research Article

Constrained Uncertain System Stabilization with Enlargement of Invariant Sets

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An enhanced method able to perform accurate stability of constrained uncertain systems is presented. The main objective of this method is to compute a sequence of feedback control laws which stabilizes the closed-loop system. The proposed approach is based on robust model predictive control (RMPC) and enhanced maximized sets algorithm (EMSA), which are applied to improve the performance of the closed-loop system and achieve less conservative results. In fact, the proposed approach is split into two parts. The first is a method of enhanced maximized ellipsoidal invariant sets (EMES) based on a semidefinite programming problem. The second is an enhanced maximized polyhedral set (EMPS) which consists of appending new vertices to their convex hull to minimize the distance between each new vertex and the polyhedral set vertices to ensure state constraints. Simulation results on two examples, an uncertain nonisothermal CSTR and an angular positioning system, demonstrate the effectiveness of the proposed methodology when compared to other works related to a similar subject. According to the performance evaluation, we recorded higher feedback gain provided by smallest maximized invariant sets compared to recently studied methods, which shows the best region of stability. Therefore, the proposed algorithm can achieve less conservative results.

1. Introduction

Model predictive control (MPC) is a main concern for control design applied in different systems such as linear or nonlinear [1–3], continuous or discrete [4], and monovariable or multivariable [5]. Actually, MPC is a common technique for the dynamical systems' stabilization. This method is applicable already in numerous domains in industry [6] as regulation and control. Generally, real processes are nonlinear, complex, and uncertain [7–10]. Therefore, a robust model predictive control (RMPC) has been introduced to guarantee robustness as well as constraint satisfaction against uncertainty. Moreover, model predictive control is an interesting approach to represent systems using fuzzy logic for designing controllers. Several works have focused on the use of fuzzy-model-based sliding mode control of nonlinear systems in combination with MPC algorithms [11–14]. In fact, the fuzzy logic technique is quite attractive in terms of time, simplicity of implementation, relatively low cost, and ability to rapidly model complex systems.

For constrained control problems processing, robust MPC is an effectual stabilization algorithm. This technique employs a specific model procedure based on input and output constraints, for each sampling time, in order to optimize system behavior through the prediction horizon. The controller implements merely the initial calculated input and reproduces these computations at the next sampling time, despite the fact that more than one input shift is calculated [15]. The major aim is to determine the state feedback control law to facilitate the minimization of the worst-case performance cost.

At each time phase, the convex problem is considered as an optimization problem including linear matrix inequalities (LMI). The main common current algorithm for RMPC is demonstrated to guarantee robust stability. But, due to the fact that the optimization problem is truly settled at each sampling time, it needs high computational time in online

implementation. On the other hand, such problems rise appreciably with the size of the polytopic uncertainty set [16].

Many efforts have been made to design the state feedback control law which minimizes the worst-case performance cost. However, in future RMPC research [17], some constructive simulation experiences still remain. Several techniques have been performed in this field where practical treatments of RMPC are still a challenging task for model predictive control. Wan and Kothare [15] proposed an algorithm based on an offline robust constrained MPC by the use of ellipsoidal invariant sets subject to linear matrix inequality (LMI). This algorithm provides a detailed explicit control laws sequence corresponding to stable invariant ellipsoidal sequence asymptotically constructed offline in the state space. In the work of Bumroongsri and Kheawhom [18], the algorithm of Wan and Kothare [15] is developed in order to ensure the performance of the closed-loop system focused on polyhedral invariant sets. An offline approach for the stabilization of constrained uncertain system is presented in this study. Various approaches have been proposed to investigate, estimate, or enlarge the maximum region of the state space where the system can operate without violating state and stabilization constraints. In fact, the obtainable difficulty is associated with the determination of controlled invariant sets [19, 20]. The computation of the maximal controlled invariant set process introduced in [21] and the corresponding state feedback control laws for linear systems subject to polyhedral input and state constraints have been studied in [22, 23]. Kouvaritakis et al. [24] developed an advanced method to enlarge the terminal invariant set using a linear programming approach. In the study by Henrion et al. [25], convex optimization problems are formulated for the region enlargement and hence tuning parameters for the positively invariant set improvement.

Many researchers [26–28] were interested in an automatic enlargement of invariant sets. In the work by Li and Lin [26], the characterization of the maximal contractively invariant ellipsoid associated with a given positive definite matrix is proposed for discrete-time linear systems. This description can be used to establish an algebraic computational approach and thus determine such maximal contractively invariant ellipsoids based on inputs from saturated linear feedback. In this field, the authors first divide the state space into several regions according to the saturation status of each input. Second, the possible maximal contractively invariant ellipsoids are computed in each region. Note that if none of the inputs saturate on their intersections, no region has been calculated. The minimal one among these possible maximal contractively invariant ellipsoids is the maximal contractively invariant ellipsoids of the system.

In this work, a new approach for maximizing ellipsoidal and polyhedral invariant sets associated with the determination of the corresponding state feedback control laws is developed. The contributions of this paper are twofold: firstly, to highlight the robust control of states, an RMPC algorithm [15, 18] was applied. This approach is based on a computation method of maximal controlled invariant sets [21]. Secondly, the combination of MPC method and maximized invariant sets procedure is proposed in order to precisely advance the performance of the employed system. The considered techniques are realized to enlarge ellipsoidal and polyhedral invariant sets. For invariant ellipsoidal sets maximization, a semidefinite problem is used based on quadratic Lyapunov function. Besides, the proposed method for polyhedral sets enlargement consists of the iterative expansion of an initial invariant set precomputed by the LMI method, adding new vertices to its convex hull. This is achieved by minimizing the distance between each new vertex from the vertices of the polyhedral set. Finally, an online implementation strategy has been applied.

So, in summary, using this proposed approach, we recorded these two contributions:

Maximization of the invariant ellipsoidal and polyhedral sets in order to increase the region of stability

Providing less conservative results and efficient system performance in terms of computational time

This paper is organized as follows. Section 2 describes the proposed methodology based on robust model predictive control. In Section 3, simulation results and discussions of the whole proposed approach are reported using two examples: an uncertain nonisothermal CSTR and an angular positioning system. The conclusion is provided in Section 4. All preliminaries and notations used in this paper are revealed in Table 1.

Schur's Lemma 1 (see [16]). Let R, S, T be given matrices with appropriate sizes and assume that $Q > 0$; then the LMI $\begin{bmatrix} R & S \\ S^T & Q \end{bmatrix} > 0$ (respectively, ≥ 0) is feasible if and only if the nonlinear constraint $R - SQ^{-1}ST > 0$ (respectively, ≥ 0) is feasible.

2. Methods

2.1. Description of Robust Model Predictive Control. In this work, robust model predictive control (RMPC) analysis is the employed procedure to emphasize stability and effectively improve the performance of the uncertain discrete-time linear systems. RMPC method is a typical scheme for minimizing the worst-case performance cost in order to determine the state feedback control law. This technique consists of two tasks: (i) offline part is introduced to search the feedback gain K_i based on the resolution of Bumroongsri and Kheawhom problem [18]; (ii) online part, at each sampling time, determines the smallest invariant set containing the measured state and implements the corresponding state feedback control law to the process. In general, RMPC preprocessing strategy is suitable for stabilization process, decreasing computational time. Here, the regulated output is demonstrated to considerably evolve the system state faster to the origin. The step-by-step method of RMPC is described as follows.

Step 1. The linear discrete-time system described by Wan and Kothare [15] is considered with the following polytopic uncertainty:

Table 1: Preliminaries and notation.

Notation	Signification
Capital letters	Real matrices
A^T	Transpose of matrix A
$\det(A)$	Determinant of matrix A
$A \geq 0$	Symmetric matrix A is positive and semidefinite
$A > 0$	Symmetric matrix A is positive and definite
x_i	The i th element of x
$S = \mathrm{conv}\{v_1, \ldots, v_q\}$	The convex hull of $\{v_1, \ldots, v_q\}$

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k), \\ y(k) = C(k)x(k), \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$, $u(k) \in \mathbb{R}^{n_u}$, and $y(k) \in \mathbb{R}^{n_y}$ are state, control, and output variables of the system, respectively. Then,

$$\begin{gathered} [A(k), B(k)] \in \Omega, \\ \Omega = \mathrm{conv}\{[A_1, B_1], [A_2, B_2], \ldots, [A_L, B_L]\}, \end{gathered} \tag{2}$$

where conv is the convex hull, Ω is a polytope, and $[A_j, B_j]$ are vertices of the polytope, where $j = 1, 2, \ldots, L$.

Step 2. Research to the feedback control law is as follows:

$$u\left(\frac{k+i}{k}\right) = Kx(k+i). \tag{3}$$

Equation (3) stabilizes system (1) with the following cost:

$$\min_{u(k+i/k)} \max_{[A(k+i), B(k+i)] \in \Omega, i \geq 0} J_\infty(k),$$

$$J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x\left(\frac{k+i}{k}\right) \\ u\left(\frac{k+i}{k}\right) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x\left(\frac{k+i}{k}\right) \\ u\left(\frac{k+i}{k}\right) \end{bmatrix}, \tag{4}$$

subject to

$$\begin{gathered} \left| u_h \frac{k+1}{k} \right| \leq u_{h,\max}, \quad h = 1, 2, \ldots, n_u, \\ \left| y_r \frac{k+1}{k} \right| \leq y_{r,\max}, \quad r = 1, 2, \ldots, n_y, \end{gathered} \tag{5}$$

where $\Theta > 0$ and $R > 0$ are symmetric weighting matrices.

Step 3. Choose a state sequence x_i, $i = 1, 2, \ldots, N$, and solve problem (6)–(10) to get the state feedback gains $K_i = Y_i Q_i^{-1}$, where Y_i and Q_i, $i = 1, 2, \ldots, N$, are solutions of the following problem:

$$\min_{\gamma_i, Q_i, Y_i} \gamma_i, \tag{6}$$

subject to

$$\begin{bmatrix} Q_i & Q_i A_j^T + Y_i^T B_j^T & Q_i \Theta^{1/2} & Y_i^T R^{1/2} \\ A_j Q_i + B_j Y_i & Q_i & 0 & 0 \\ \Theta^{1/2} Q_i & 0 & \gamma_i I & 0 \\ R^{1/2} Y_i & 0 & 0 & \gamma_i I \end{bmatrix} \geq 0, \tag{7}$$

$$\forall j = 1, 2, \ldots, L,$$

$$\begin{bmatrix} 1 & x_i^T \\ x_i & Q_i \end{bmatrix} \geq 0, \tag{8}$$

$$\begin{bmatrix} X & Y_i \\ Y_i^T & Q_i \end{bmatrix} \geq 0, \tag{9}$$

$$X_{hh} \leq u_{h,\max}^2, \quad h = 1, 2, \ldots, n_u,$$

$$\begin{bmatrix} S & C(A_j Q_i + B_j Y_i) \\ (A_j Q_i + B_j Y_i)^T C^T & Q_i \end{bmatrix} \geq 0, \tag{10}$$

$$S_{rr} \leq y_{r,\max}^2, \quad r = 1, 2, \ldots, n_y, \ \forall j = 1, 2, \ldots, L,$$

where Q is a symmetric matrix.

Step 4. For each K_i, the corresponding polyhedral invariant set $S_i = \{x_i / M_i x_i \leq d_i\}$ is constructed in [18].

2.2. The Proposed Methodology. As illustrated in Figure 1, the proposed methodology is composed of three steps:

Step 1. Enhanced maximized sets algorithm: by the combination of an RMPC technique proposed by Bumroongrsi and Kheawhom [18] and the enhanced maximized invariant sets approach, a successful progress of the closed-loop system performance was obtained. Two methods are developed to maximize the ellipsoidal and polyhedral invariant sets constructed by the RMPC algorithm. The ellipsoidal invariant sets approach referred to in Section 2.2.1 is a semidefinite programming method. Based on the work of Athanasopoulos and Bitsoris [21], a second linear programming approach is used to enlarge polyhedral sets.

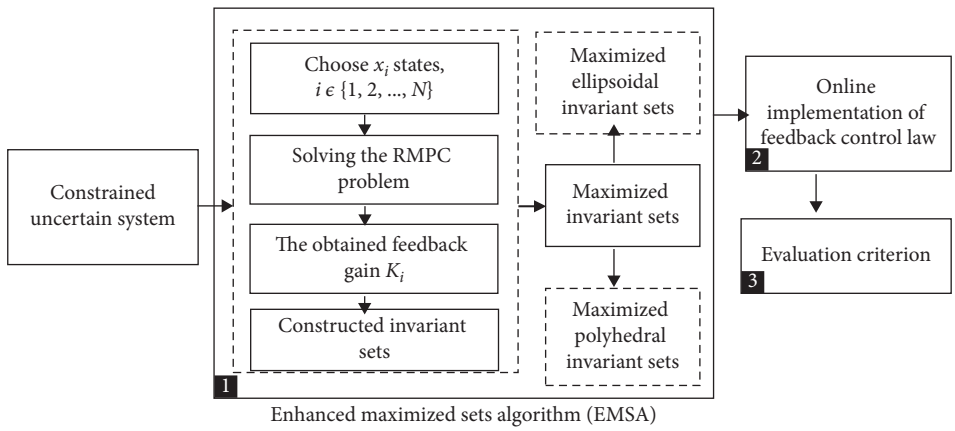

Figure 1: Flowchart of the proposed methodology.

It consists of adding new vertices to their convex hull by minimizing the distance between each new vertex and the polyhedral set vertices for securing the state constraints. The polyhedral invariant sets process is presented in Section 2.2.2.

Step 2. Online implementation of the feedback control law: at each sampling time, determine the smallest invariant set containing the measured state and implement the corresponding state feedback control law to the process.

Step 3. Evaluation criterion: the computational time (CT) required for the proposed approach has been reduced.

2.2.1. Enhanced Maximized Ellipsoidal Invariant Sets (EMES). Subsequent to the RMPC problem resolution and the feedback gains determination, an invariant ellipsoidal sets sequence is built.

Let the following inequalities be

$$x\left(\frac{k}{k}\right)^{\mathrm{T}} P x\left(\frac{k}{k}\right) \leq 1, \tag{11}$$

which is equivalent to

$$x\left(\frac{k}{k}\right)^{\mathrm{T}} Q^{-1} x\left(\frac{k}{k}\right) \leq 1, \tag{12}$$

where $P = Q^{-1}$.

To maximize the ellipsoidal region

$$\xi = \left\{x \mid x^{\mathrm{T}} P x \leq 1\right\} = \left\{x \mid x^{\mathrm{T}} Q^{-1} x \leq 1\right\}, \tag{13}$$

by guaranteeing a wider stability domain, a semidefinite programming problem will be used.

Let us consider the quadratic Lyapunov function $V(x(k)) = x(k)^{\mathrm{T}} P x(k)$. Then, we have

$$\begin{aligned} \Delta V(x(k)) &= V(x(k+1)) - V(x(k)) \leq 0 \\ &= x(k)^{\mathrm{T}}\left[(A+BK)^{\mathrm{T}} P (A+BK) - P\right] x(k) \leq 0. \end{aligned} \tag{14}$$

Condition (14) is true if and only if

$$(A+BK)^{\mathrm{T}} P (A+BK) - P \leq 0. \tag{15}$$

Using Schur's lemma, the following condition with $Q = P^{-1}$ and $K = YQ^{-1}$ is obtained:

$$\begin{bmatrix} Q & \left(A_j Q_i + B_j Y_i\right)^{\mathrm{T}} \\ A_j Q_i + B_j Y_i & Q \end{bmatrix} \geq 0. \tag{16}$$

A natural objective enables increasing the ellipsoid volume which is proportional to $\det(Q)$. Hence, if the maximal invariant ellipsoid volume corresponds to state feedback law, solving the following semidefinite programming is required:

$$\max_{Q,Y} \log(\det(Q)), \tag{17}$$

subject to

$$\begin{gathered} \begin{bmatrix} Q & \left(A_j Q_i + B_j Y_i\right)^{\mathrm{T}} \\ A_j Q_i + B_j Y_i & Q \end{bmatrix} \geq 0, \\ \begin{bmatrix} u_{\max} & Y \\ Y^{\mathrm{T}} & Q \end{bmatrix} \geq 0, \quad u_{\max} \geq |u|. \end{gathered} \tag{18}$$

2.2.2. Enhanced Maximized Polyhedral Invariant Sets (EMPS). Given the state feedback gains $K_i = Y_i Q_i^{-1}$, $i = 1, \ldots, N$, calculated from RMPC algorithm, for each K_i, the corresponding polyhedral invariant set $S_i = \{x_i / M_i x_i \leq d_i\}$ is constructed. The enhanced maximized polyhedral invariant sets (EMPS) algorithm is given as follows.

Step 1. Let the polyhedral invariant sets S_i be the convex hull of its vertices:

$$S_i = \operatorname{conv}\left\{v_i^1, \ldots, v_i^q\right\}, \quad i = 1, 2, \ldots, N. \tag{19}$$

Step 2. Consider new sets $S_i' = \operatorname{conv}\left\{v_i^1, \ldots, v_i^q, v_{\sup}\right\}$, $i = 1, 2, \ldots, N$, $q = 1, 2, \ldots, n$, and choose a point $v^{\mathrm{ch}} \notin S_i$.

Step 3. Solve the following EMPS problem:

$$\min_{v_{\sup},p,u_{\sup},\varepsilon}\left\{\left\|v_{\sup}-v^{\mathrm{ch}}\right\|_{\infty}\right\}, \tag{20}$$

subject to

$$Av_{\sup}+Bu_{\sup}=\sum_{i=1}^{j}p_iv_i^j+p_{j+1}v_{\sup}, \tag{21}$$

$$p_i\geq 0,\quad i=1,\ldots,j+1, \tag{22}$$

$$\sum_{i=1}^{j+1}p_i\leq\varepsilon, \tag{23}$$

$$0<\varepsilon<1, \tag{24}$$

$$M_iv_{\sup}\leq d_i, \tag{25}$$

where $p_{j+1}\in[0,1]$.

Once the problem is solved, an optimal vertex $v_{\sup}$ is obtained, and thus, the following maximized polyhedral set is constructed:

$$S_i'=\mathrm{conv}\left\{v_i^1,\ldots,v_i^q,v_{\sup}\right\}. \tag{26}$$

Relations (21)–(23) imply the positive invariance and attractivity of S_i', while (24) and (25) guarantee constraint satisfaction.

Step 4. At each sampling time, determine the smallest maximized polyhedral invariant set containing the measured state and implement the corresponding state feedback control law $u(k/k)=K_ix(k/k)$ to the process.

3. Results

3.1. Example 1. An uncertain nonisothermal CSTR [15] is considered where the exothermic reaction $A\longrightarrow B$ takes place. The reaction is irreversible and the rate of reaction is primary order with respect to component A. A cooling coil is employed to eliminate heat which is released in the exothermic reaction. The uncertain parameters are the reaction rate constant k_0 and the heat of reaction H_{rxn}. The linearized model focused on the component balance and the energy balance is given by the following state equations:

$$\begin{cases}\dot{x}(t)=Ax(t)+Bu(t),\\ y(t)=Cx(t),\end{cases} \tag{27}$$

where $\begin{bmatrix}C_A\\T\end{bmatrix}$ is the state vector $x(t)$ and $\begin{bmatrix}C_{A,F}\\F_C\end{bmatrix}$ is the input control vector $u(t)$. Matrices are defined by

$$\begin{aligned}A&=\begin{bmatrix}0.85-0.0986\alpha(k) & -0.0014\alpha(k)\\ 0.9864\alpha(k)\beta(k) & 0.0487+0.01403\alpha(k)\beta(k)\end{bmatrix},\\ B&=\begin{bmatrix}0.15 & 0\\ 0 & -0.912\end{bmatrix},\\ C&=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\end{aligned} \tag{28}$$

where C_A is the concentration of A in the reactor, $C_{A,F}$ presents the feed concentration of A, T denotes the reactor temperature, and F_C is the coolant flow. The operating parameters are as follows:

$$\begin{aligned}F&=1\,\mathrm{m}^3/\mathrm{min},V=1\,\mathrm{m}^3\\ k_0&=10^9-10^{10}\,\mathrm{min}^{-1}\\ E/R&=8330.1\,\mathrm{K}\\ -\Delta H_{rxn}&=10^7-10^8\,\mathrm{cal/kmol}\\ \rho&=10^6\,\mathrm{g/m}^3\\ U_A&=5.3410^6\,\mathrm{cal/(K\,min)}\\ C_p&=1\,\mathrm{cal/(g\,K)}\end{aligned}$$

Let $\overline{C}_A=C_A-C_{A,\mathrm{eq}}$, $\overline{C}_{A,F}=C_{A,F}-C_{A,F,\mathrm{eq}}$, and $\overline{F}_C=F_C-F_{C,\mathrm{eq}}$, where the subscript eq is used to denote the corresponding variable at equilibrium condition. By discretization, using a sampling time (ST = 0.15 min), the discrete-time model with $\begin{bmatrix}\overline{C}_A(k)\\ \overline{T}(k)\end{bmatrix}$ and $\begin{bmatrix}\overline{C}_{A,F}\\ \overline{F}_C(k)\end{bmatrix}$, as state and control vectors, respectively, is given as follows:

$$\begin{cases}x(k+1)=Ax(k)+Bu(k),\\ y(k)=Cx(k),\end{cases} \tag{29}$$

$$\begin{cases}x(k+1)=\begin{bmatrix}\overline{C}_A(k+1)\\ \overline{T}(k+1)\end{bmatrix}\\ =\begin{bmatrix}0.85-0.0986\alpha(k) & -0.0014\alpha(k)\\ 0.9864\alpha(k)\beta(k) & 0.0487+0.01403\alpha(k)\beta(k)\end{bmatrix}\begin{bmatrix}\overline{C}_A(k)\\ \overline{T}(k)\end{bmatrix}\\ +\begin{bmatrix}0.15 & 0\\ 0 & -0.912\end{bmatrix}\begin{bmatrix}\overline{C}_{A,F}\\ \overline{F}_C(k)\end{bmatrix},\\ y(k)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\overline{C}_A(k)\\ \overline{T}(k)\end{bmatrix},\end{cases} \tag{30}$$

where $1\leq\alpha(k)=k_0/10^9\leq 10$ and $1\leq\beta(k)=-\Delta H_{rxn}/10^7\leq 10$.

The two parameters $\alpha(k)$ and $\beta(k)$ are independent of each other. Then, we consider the following polytopic uncertain model with four vertices:

$$\Omega=\mathrm{conv}\left\{\begin{bmatrix}0.751 & -0.0014\\ 0.986 & 0.063\end{bmatrix},\begin{bmatrix}0.751 & -0.0014\\ 9.864 & 0.189\end{bmatrix},\begin{bmatrix}-0.136 & -0.014\\ 9.864 & 0.189\end{bmatrix},\begin{bmatrix}-0.136 & -0.014\\ 98.644 & 1.451\end{bmatrix}\right\}. \tag{31}$$

By manipulating $\overline{C}_{A,F}$ and $\overline{F}_C$, the control of concentration $\overline{C}_A$ and the reactor temperature $\overline{T}$ return to the origin. These variables are constrained having $|\overline{C}_{A,F}| \leq 0.5\,\text{kmol/m}^3$ and $|\overline{F}_C| \leq 1.5\,\text{m}^3/\text{min}$.

The cost function is given by (4) with $\Theta = I$ and $R = 0.1I$.

The sequence of the chosen states is

$$x_i = \left\{ \begin{array}{ll} (0.0525, 0.0525), & (0.0475, 0.0475) \\ (0.0425, 0.0425), & (0.0375, 0.0375) \\ (0.0325, 0.0325), & (0.0275, 0.0275) \end{array} \right\}. \tag{32}$$

These sequences are used to compute six offline feedback gains K_i, $i = 1, 2, \ldots, 6$. This allows building an ellipsoidal and polyhedral invariant sets sequences.

Focused on the EMSA method, the maximized ellipsoidal and polyhedral invariant sets are larger compared to invariant sets [15, 18]. The difference between these sets is, respectively, shown in Figures 2 and 3.

For both techniques, the invariant sets, ellipsoidal (Figure 2) and polyhedral ones (Figure 3), are constructed based on the choice of the same states sequence x_i, $i = 1, \ldots, 6$.

The maximized polyhedral invariant sets enable us to obtain an appreciably larger domain of stability compared to the polyhedral invariant ones in [18], for each chosen state x_i. This is due to the additional vertex of the obtained sets that have been added by the EMPS approach. Figure 3 reveals the comparison between the stabilizable sets of two feedback gains in terms of A and B points. As shown in Figure 3, it is clear that the maximized polyhedral invariant sets stabilize the states at point A by the use of feedback gain K_1 since the states are contained in the maximized sets S_1. Contrariwise, the polyhedral sets [18] are not able to stabilize the states at point A because they are not contained in the initial invariant set. As illustrated in Figures 2 and 3, beginning at the point B, the polyhedral set can stabilize the states to the origin taking on the lowest feedback gain K_1. In brief, the proposed approach EMPS algorithm can regulate the states at point B to the origin using a higher feedback gain K_6 in the fact that the points are contained in S_6. In this case, EMPS method achieves the higher feedback gain, when compared to previous studies. Consequently, the proposed maximized approach attains less conservative results. To significantly clarify our results, Figures 4 and 5 demonstrate the regulated outputs. Here, we report that the considered EMSA method provides less conservative results and efficient system performance, when the state evolves faster to the origin. Compared to the previous work [15, 18], the proposed strategy seems to be helpful for uncertain system control. As demonstrated in Table 2, we can deduce from the stabilization validation results that the EMSA technique is more efficient compared to the other model predictive control methods [15, 18] in terms of stabilizable region and computational time (CT). EMSA strategy provides rigorous results in terms of CT (4.951 s) and larger stabilization region in different points. Although the construction of maximized polyhedral invariants sets requests more computational time than the standard ellipsoidal and polyhedral invariants sets, it is still more precise in enlargement of stability domain. Table 3 summarizes the cumulative cost obtained in Example 1.

3.2. Example 2. We consider the angular positioning system described by the following discrete-time equation [29]:

$$\left\{ \begin{array}{l} \begin{bmatrix} \theta(k+1) \\ \dot{\theta}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 - 0.1\alpha(k) \end{bmatrix} \begin{bmatrix} \theta(k) \\ \dot{\theta}(k) \end{bmatrix} \\ + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u(k), \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta(k) \\ \dot{\theta}(k) \end{bmatrix}, \end{array} \right. \tag{33}$$

where $\theta(k)$ is the angular position of the antenna, $\dot{\theta}(k)$ is the angular velocity, and $u(k)$ is the input voltage of the motor. It is assumed that the uncertain parameter is arbitrarily time varying: $0.1 \leq \alpha(k) \equiv k_0/10^9 \leq 10$.

Let $\overline{\theta} = \theta - \theta_{\text{eq}}$, $\dot{\overline{\theta}} = \dot{\theta} - \dot{\theta}_{\text{eq}}$, and $\overline{u} = u - u_{\text{eq}}$, where the subscript eq denotes the corresponding variable at equilibrium condition Figure 6. The obtained system can be written as follows:

$$\left\{ \begin{array}{l} \begin{bmatrix} \overline{\theta}(k+1) \\ \dot{\overline{\theta}}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 - 0.1\alpha(k) \end{bmatrix} \begin{bmatrix} \overline{\theta}(k) \\ \dot{\overline{\theta}}(k) \end{bmatrix} \\ + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u(k), \\ y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{\theta}(k) \\ \dot{\overline{\theta}}(k) \end{bmatrix}. \end{array} \right. \tag{34}$$

System (34) has the following polytopic structure:

$$A(k) \in \text{conv}\left\{ \begin{bmatrix} 1 & 0.1 \\ 0 & 0.9 \end{bmatrix}, \begin{bmatrix} 1 & 0.1 \\ 0 & 0 \end{bmatrix} \right\}. \tag{35}$$

The input constraint is

$$|\overline{u}(k)| \leq 2\,\text{volts}. \tag{36}$$

The weighting matrices Θ and R are given by

$$\Theta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R = 0.00002I. \tag{37}$$

Let us choose the following seven states sequence:

$$x_i = \left\{ \begin{array}{lc} (0.35, 0.35), & (0.3, 0.3) \\ (0.25, 0.25), & (0.02, 0.02) \\ (0.15, 0.15), & (0.1, 0.1), (0.05, 0.05) \end{array} \right\}. \tag{38}$$

In this example, the sequence of seven states x_i, $i = 1, \ldots, 7$, is used to compute seven state feedback gains K_i corresponding to seven ellipsoidal and polyhedral invariant sets. Using the EMSA algorithm, the maximized ellipsoidal and polyhedral invariant sets are drawn compared to invariant sets [15, 18]. Figure 7 exemplifies the comparison between the maximized ellipsoidal and polyhedral invariant sets.

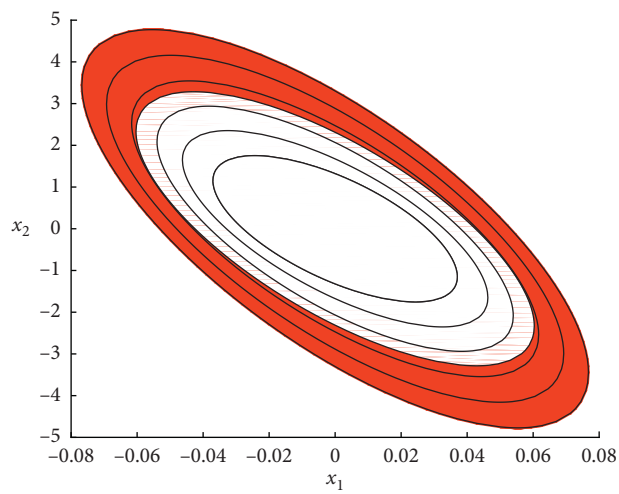

Figure 2: Resulting maximized ellipsoidal invariant sets (in red).

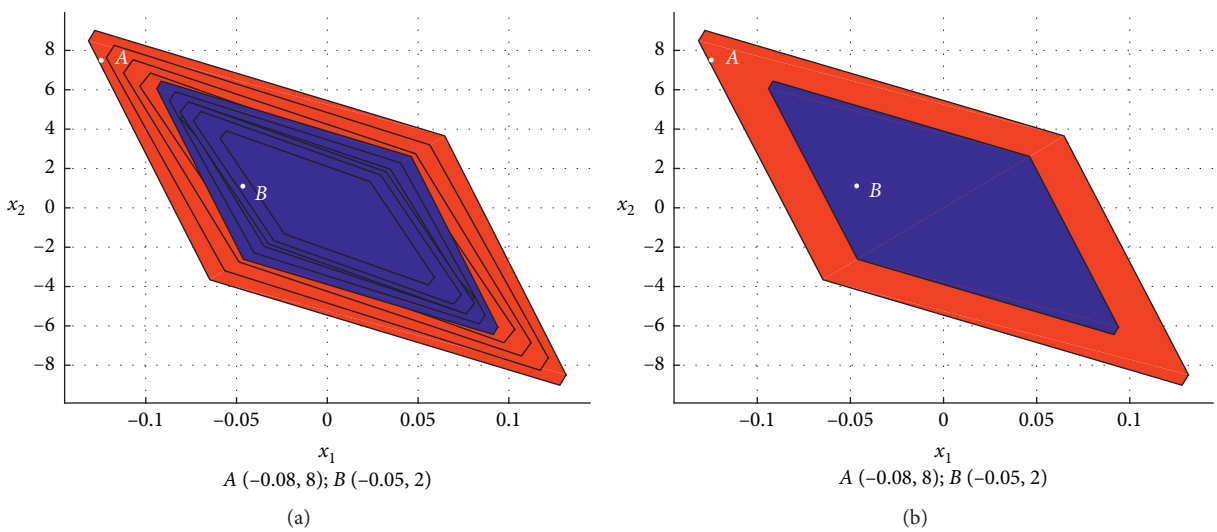

Figure 3: Examples of maximized polyhedral invariant sets from (a) six imprecated sets and (b) first set compared to [18].

Compared to the invariant set [18], the maximized invariant set has a significantly larger domain of stability, for each chosen state x_i, $i = 1, \ldots, 7$. Figure 7 reveals the comparison between the stabilizable sets of three feedback gains in terms of A, B, and C points. Simulation results illustrated in Figure 7 highlight the robustness of the proposed method using maximized polyhedral invariant sets which stabilize the states at point A employing the feedback gain K_1 (the states are contained in the maximized set S_1). On the other hand, the polyhedral [18] and the maximized ellipsoidal sets are not able to stabilize the states at point A (the states are not contained in the original polyhedral and maximized ellipsoidal sets). Concerning the point B, the polyhedral set [18] can stabilize the states to the origin corresponding to the lowest feedback gain K_1. In addition, the proposed EMPS approach can regulate the states at point B to the origin utilizing a higher feedback gain K_5 (points contained in S_5). On the contrary, the maximized ellipsoidal set and the ellipsoidal set [15] cannot control the states at point B because they are not situated in these invariant sets. Also, starting by the point C, it is obvious that the maximized invariant set obtained from EMES approach can stabilize the states to the origin containing the lowest feedback gain K_1. The proposed EMSA model can control the states at this point to the origin exploiting higher feedback gain K_7 (points contained in S_7). Note that previous studies [18] are capable of stabilizing these states at points C from the feedback gain K_6. Figures 8 and 9 display the regulated outputs. In this case, it is evident that the projected EMSA method supplies less conservative results. If the state evolves faster to the origin, the

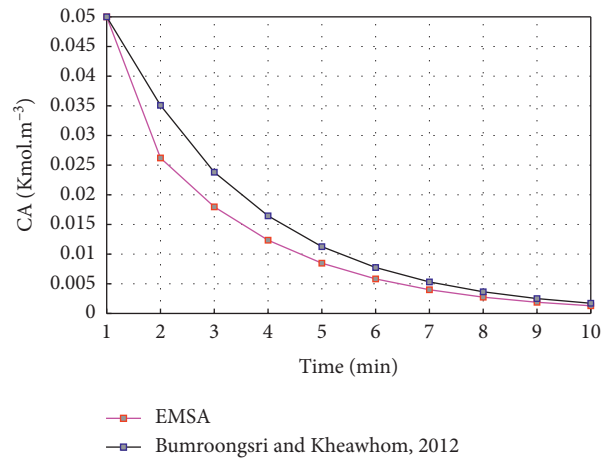

Figure 4: The concentration of A in the reactor of the regulated output obtained with EMPS algorithm.

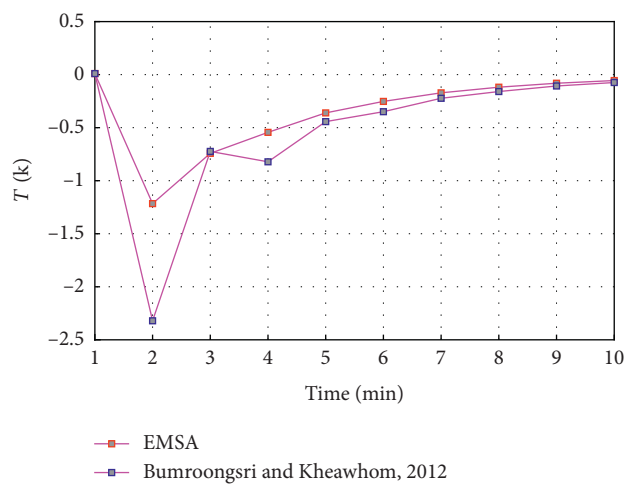

Figure 5: The reactor temperature of the regulated output obtained with EMPS algorithm.

Table 2: Performance comparison with previous works.

Methods	Years	Stabilizable region *Different points* A (−0.05, 3)	B (−0.08, 4)	C (−0.1, 8)	Stabilization domain	Invariant sets number	Maximization methods	Computational time (s)
Wan and Kothare [15]	2003	×	×	✓	Ellipsoidal invariant sets	6	×	3.672
Bumroongsri and Kheawhom [18]	2012	×	✓	✓	Polyhedral invariant sets	6	×	4.372
The proposed method		✓	✓	✓	Maximized invariant sets	6	Semidefine and linear programming	4.951

Table 3: Cumulative cost in Example 1.

Methods	Cumulative cost	Cumulative equation
Wan and Kothare [15]	20.48	$\sum_{i=0}^{\infty} x_i^T \Theta x_i + u_i^T R u_i$
Bumroongsri and Kheawhom [18]	19, 02	
Proposed approach	17, 9	

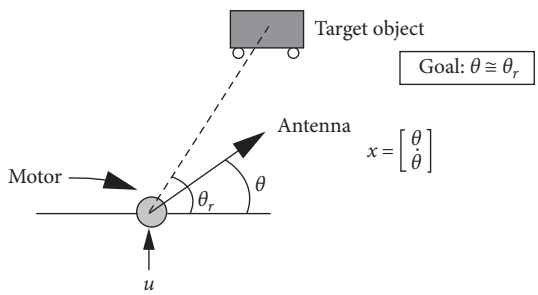

Figure 6: Angular positioning system.

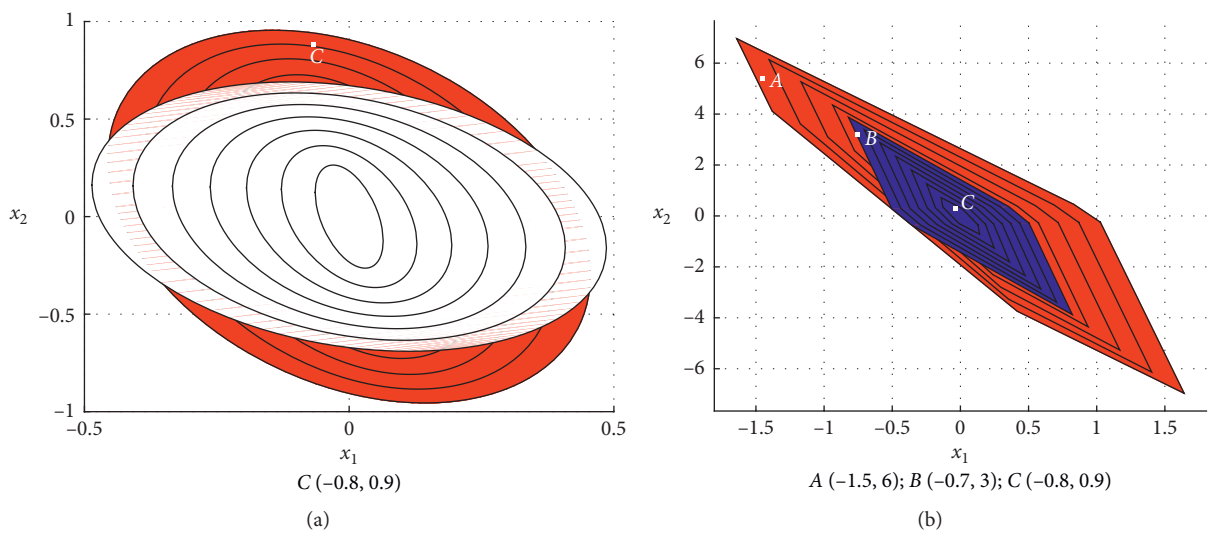

Figure 7: (a) Maximized ellipsoidal invariant sets compared to [15] and (b) maximized polyhedral invariant sets compared to [18].

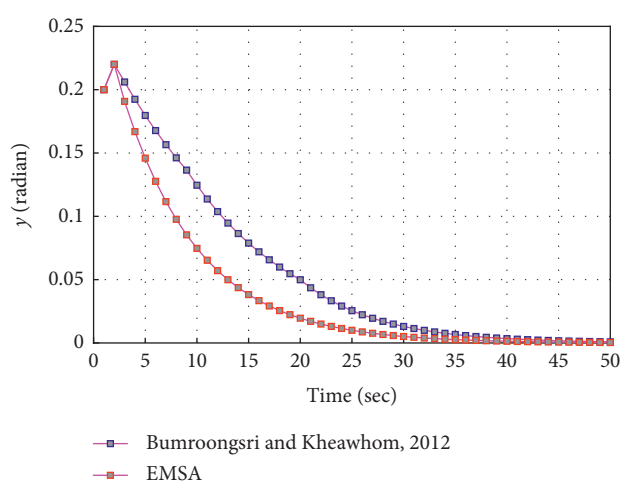

Figure 8: The regulated output obtained with EMPS approach.

applied approach reaches better performance system. The resolution of the predictive control problem based on the proposed EMSA algorithm aims to improve the uncertain system performances under consideration. Depending on the result of Table 4, we can assume that the proposed scheme is more successfully having a larger stabilizable region. Table 5 resumes the cumulative cost in the second example.

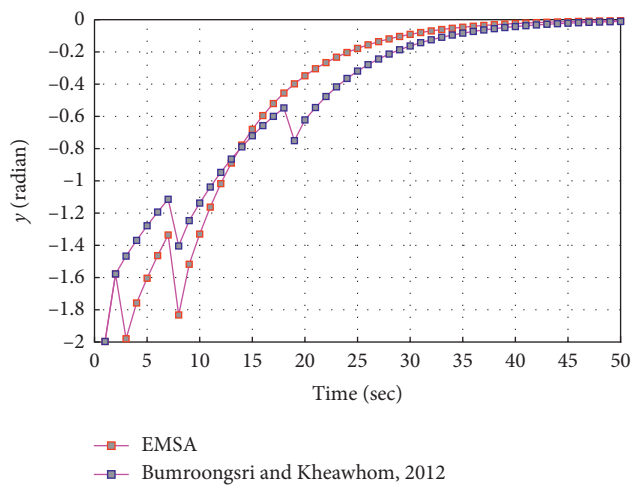

Figure 9: The control input obtained with EMPS approach.

Table 4: Performance comparison with previous works.

Methods	Years	Stabilizable region Different points A (−0.2, 0.4)	B (0, 2)	C (1.5, −3)	Stabilization domain	Invariant sets number	Maximization methods	Computational time (s)
Wan and Kothare [15]	2003	×	×	✓	Ellipsoidal invariant sets	7	×	2.831
Bumroongsri and Kheawhom [18]	2012	×	✓	✓	Polyhedral invariant sets	7	×	3.541
The proposed method		✓	✓	✓	Maximized invariant sets	7	Semidefine and linear programming	4.183

Table 5: Cumulative cost in Example 2.

Methods	Cumulative cost	Cumulative equation
Wan and Kothare [15]	0.19	
Bumroongsri and Kheawhom [18]	0.12	$\sum_{i=0}^{\infty} x_i^T \Theta x_i + u_i^T R u_i$
Proposed approach	0.9	

4. Conclusion

In this paper, we described an enhanced method which can be used for constrained uncertain discrete-time linear systems stabilization. A useful RMPC technique was applied to emphasize the robust control and improve the state stabilization. The proposed procedure gives appropriate optimization and notable precision when compared to existing model predictive control results. Then, we have suggested the combined RMPC method and maximized invariant sets process that can accurately progress the performance of the closed-loop system. The included methods are used to enlarge ellipsoidal and polyhedral invariant sets constructed by the RMPC algorithm. An online implementation for the obtained feedback control laws has been made. The proposed method has been compared with some existing algorithms in order to enlarge stability domain. Experiment results demonstrate that the proposed method can permanently control system states having a larger stabilizable region. Therefore, the performance of the proposed strategy furnishes a rigid basis in support of solving the control problem. As future works, we propose to use deep learning to obtain flexible models for nonlinear model predictive control (MPC).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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