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Research Article

Analytical Insights into a Generalized Semidiscrete System with Time-Varying Coefficients: Derivation, Exact Solutions, and Nonlinear Soliton Dynamics

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In this paper, a new generalized semidiscrete integrable system with time-varying coefficients is analytically studied. Firstly, the generalized semidiscrete system is derived from a semidiscrete matrix spectral problem by embedding finite time-varying coefficient functions. Secondly, exact and explicit N-soliton solutions of the semidiscrete system are obtained by using the inverse scattering analysis. Finally, three special cases when $N = 1, 2, 3$ of the obtained N-soliton solutions are simulated by selecting some appropriate coefficient functions. It is shown that the time-varying coefficient functions affect the spatio-temporal structures and the propagation velocities of the obtained semidiscrete one-soliton solutions, two-soliton solutions, and three-soliton solutions.

1. Introduction

Semidiscrete systems keep some or all their spatial variables discrete while time continuous systems have an important role in simulating complex phenomena in many fields, for instance; they often arise in high energy physics as approximations of continuum models for the numerical simulation of nonlinear soliton dynamics [1]. It is Toda [2] who first derived the classical semidiscrete system—Toda lattice when the lattice with exponential interaction was considered. Like the famous Kortweg-de Vries equation, the classical Toda lattice is one of the most important completely integrable systems, and the multisoliton solutions of which have very important significance to the study of many nonlinear problems in atomic and particle physics. The semidiscrete sine-Gordon equation can build up a connection with the dynamics of circular arrays of Josephson junctions [3]. For the important role of the semidiscrete nonlinear Schrödinger equation, it is worth mentioning that this equation has been used to describe transport on a dimer [4] and beam propagation in Kerr nonlinear media [5].

Compared with differential or difference equations, it is difficult to derive or solve the nonlinear semidiscrete systems [6–15]. In [9, 11], an effective algorithm of the Jacobian elliptic function method devised by Dai and Zhang [6] is extended to solve the semidiscrete sine-Gordon equation and the semidiscrete nonlinear Schrödinger equation, respectively. Zhu's exp-function ansatz [7] applied to the hybrid-lattice system and the lattice equation [8] is improved for N-soliton solutions of the Toda lattice equation [12]. Egorova, Michor, and Teschl [10] used the double commutation method and the inverse scattering transform to solve the Toda hierarchy on a quasiperiodic finite-gap background. In [13, 15], the mixed Toda lattice hierarchy and the variable-coefficient non-isospectral Toda lattice

hierarchy are derived and solved by means of the inverse scattering transform. In 2014, Zhang and Liu [14] obtained N-soliton solutions of a $(2+1)$-dimensional variable-coefficient Toda lattice equation through Hirota's bilinear method.

Recently, some new analytical approaches [16–25], novel nonlinear results [26–34], and soliton dynamics in inhomogeneous media [35–37] for differential equations have been reported. Generally speaking, extending the existing methods for differential equations encounters the difficulty in searching for the iterative relations of discrete variables. When the nonuniformities of boundaries or the inhomogeneities of media are taken into consideration, the systems with variable coefficients [38–43] are much better than their idealized counterparts—constant-coefficient models in describing more realistic phenomena. In 2011, Liu et al. [38] show that when such nonuniformities of boundaries and inhomogeneities of media are depicted by variable coefficients, the soliton behavior they found may be helpful to study the internal solitary waves in ocean- or atmosphere-stratified fluid. In 2017, Dai et al. [40] pointed out that the analysis and results they first reported for the Peregrine solution and breather solution of a nonlinear Schrödinger equation with variable coefficients could be expected to give new insight into higher-dimensional localized rogue waves in nonlocal media. It is shown in [39, 41–43] that the variable coefficients influence the spatial structures and the dynamical evolutions of solutions.

In this article, we analytically study a new generalized semidiscrete system of the form:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = \sum_{m=0}^{k} \alpha_m L^m \begin{pmatrix} u_n(v_n - v_{n-1}) \\ u_{n+1} - u_n \end{pmatrix}, \quad k = 0, 1, 2, \ldots, \tag{1}$$

where $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_k$ are all functions with respective to t and $\sum_{m=0}^{k} \alpha_m^2 \neq 0$ and L is a matrix operator in the following form:

$$L = \begin{pmatrix} u_n(S-1)v_{n-1}(S-1)^{-1}\dfrac{1}{u_n} & u_n(1+S^{-1}) \\ (Su_nS - u_n)(S-1)^{-1}\dfrac{1}{u_n} & v_n \end{pmatrix}. \tag{2}$$

In matrix operator (2), S is a shift operator defined as $S^d u_n = u_{n+d}$ for any integer number d. With the help of the shift operator S, we can conveniently determine the operator $S - S^{-1}$ which has the inverse operator defined as

$$\left(S - S^{-1}\right)^{-1} u_n = -\sum_{m=n}^{\infty} u_{2m-n-1}, \tag{3}$$

or

$$\left(S - S^{-1}\right)^{-1} u_n = \sum_{m=-\infty}^{n} u_{2m-n-1}, \tag{4}$$

provided $u_n \longrightarrow 0$ rapidly as $|n| \longrightarrow 0$. At the same time, for $S - S^{-1}$, there holds the relation $(S-1)^{-1} = (S - S^{-1})^{-1}(1 + S^{-1})$.

It should be noted that system (1) is not only different from the known ones [2, 6–15, 44–46]:

$$m\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = \kappa\left(y_{n+1} - 2y_n + y_{n-1}\right),$$

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = \left(1 + \alpha u_n + \beta u_n^2\right)\left(u_{n-1} - u_{n+1}\right),$$

$$\frac{\mathrm{d}u_{n+1}}{\mathrm{d}t} - \frac{\mathrm{d}u_n}{\mathrm{d}t} = \sin\left(u_{n+1} + u_n\right),$$

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} - a(t)\left(g_{n+1}^{+} - g_{n+1}\right) = 0,$$

$$\frac{\mathrm{d}b(t)}{\mathrm{d}t} - b(t)\left(h_{n+1} - h_{n+1}^{-}\right) = 0,$$

$$i\frac{\mathrm{d}u_n}{\mathrm{d}t} = \left(u_{n+1} + u_{n-1} - 2u_n\right) + \varepsilon\left|u_n\right|^2\left(u_{n+1} + u_{n-1}\right),$$

$$\frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2} = e^{u_{n-1} - u_n} - e^{u_n - u_{n+1}},$$

$$\begin{pmatrix} \ln u_n \\ v_n \end{pmatrix}_{t_k} = L^k \begin{pmatrix} 2\mu + v(v_n - v_{n-1}) \\ \mu v_n + v(u_{n+1} - u_n) \end{pmatrix} + \mu \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix}, \quad k = 0, 1, 2, \ldots,$$

$$\frac{\partial^2 u_n}{\partial x \partial t} = \left(\frac{\partial u_n}{\partial t} + \alpha(t) \right) (u_{n-1} - 2u_n + u_{n+1}),$$

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L^k \begin{pmatrix} \alpha(t) u_n (v_n - v_{n-1}) + 2\beta(t) u_n \\ \alpha(t)(u_{n+1} - u_n) + \beta(t) v_n \end{pmatrix}, \quad k = 0, 1, 2, \ldots,$$

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L^k \begin{pmatrix} \beta(t) u_n (v_n - v_{n-1}) + 2\alpha'(t)\alpha(t) u_n \\ \alpha(t)\beta^{-1}(t)(u_{n+1} - u_n) + \beta'(t)\beta^{-1}(t) v_n \end{pmatrix}, \quad k = 0, 1, 2, \ldots, \tag{5}$$

but more general than the Toda lattice hierarchy [47]:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L^k \begin{pmatrix} u_n (v_n - v_{n-1}) \\ u_{n+1} - u_n \end{pmatrix}, \quad k = 0, 1, 2, \ldots, \tag{6}$$

This is because, on the one hand, system (1) is a superposition of finite terms and on the other hand, system (1) contains some time-varying coefficient functions. Obviously, a special case of system (1) is equation (6) which concludes two semidiscrete systems:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = \begin{pmatrix} u_n (v_n - v_{n-1}) \\ u_{n+1} - u_n \end{pmatrix}, \tag{7}$$

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = \begin{pmatrix} u_n (u_{n+1} - u_{n-1}) + u_n (v_n^2 - v_{n-1}^2) \\ u_{n+1}(v_{n+1} + v_n) - u_n (v_n + v_{n-1}) \end{pmatrix}. \tag{8}$$

A direct computation shows that $u_n = e^{x_{n-1} - x_n}$ and $v_n = -x_{n,t}$ can rewrite system (7) as the celebrated Toda lattice, i.e., $x_{n,tt} = e^{x_{n-1} - x_n} - e^{x_n - x_{n+1}}$. If we set $k = 1$, then system (1) gives the following new semidiscrete system:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = \begin{pmatrix} \alpha_0 u_n (v_n - v_{n-1}) + \alpha_1 u_n (u_{n+1} - u_{n-1}) + \alpha_1 u_n (v_n^2 - v_{n-1}^2) \\ \alpha_0 (u_{n+1} - u_n) + \alpha_1 u_{n+1} (v_{n+1} + v_n) - \alpha_1 u_n (v_n + v_{n-1}) \end{pmatrix}, \tag{9}$$

which cannot be included in known system (6). In the literature, there is no research result about semidiscrete system (1).

Constructing or solving semidiscrete systems is a significant work in the theory of nonlinear lattices [2]. In 1990, Tu [48] proposed a scheme for generating hierarchies of semidiscrete integrable systems. In 2005, Yang and Xu [49] developed a direct method for constructing integrable expanding models for semidiscrete systems. In soliton theory, there are many methods like those in [50–58] besides the inverse scattering method [59] for solving nonlinear differential equations. In this paper, we shall analytically derive and solve system (1) by means of the inverse scattering analysis. Employing other methods to construct new solutions of system (1) is worthy of studying. In many research fields, more and more attention has been paid to fractional-order models [60–76]. Recently, Aslan [77–80] has successfully extended analytical methods-combined symbolic computation to solve fractional semidiscrete equations. How to extend the method used in this paper to such fractional semidiscrete equations is also worthy of studying.

The rest of this paper is organized as follows. In Section 2, we derive system (1). In Section 3, we construct the formulae of the N-soliton solutions of system (1). In Section 4, we analyze the dynamical evolutions of the obtained soliton solutions. In Section 5, we conclude this paper.

2. Derivation

To derive system (1), we consider the following semidiscrete matrix spectral problem [13, 15, 44, 47]:

$$S\Psi_n = U\Psi_n, \quad U = \begin{pmatrix} 0 & 1 \\ -u_n & \lambda - v_n \end{pmatrix}, \quad \Psi_n = \begin{pmatrix} \Psi_{1,n} \\ \Psi_{2,n} \end{pmatrix}, \tag{10}$$

$$\Psi_{n,t} = V\Psi_n, \quad V = \begin{pmatrix} X_n & Y_n \\ Z_n & W_n \end{pmatrix}, \tag{11}$$

where the potential functions u_n and v_n are smooth enough and $(u_n, v_n) \longrightarrow (1, 0)$ rapidly as $|n| \longrightarrow 0$, the spectral parameter $\lambda_t = 0$, and X_n, Y_n, Z_n, and W_n are the functions of u_n, v_n, and λ to be determined later. Then, we employ spectral problem (10) and its time evolution equation (11)

equipped with different isospectral parameter λ and functions X_n, Y_n, Z_n, and W_n to derive system (1).

Theorem 1. *System (1) is Lax integrable, which can be derived from*

$$U_t = (SV)U - UV, \tag{12}$$

by introducing different isospectral parameter λ and functions X_n, Y_n, Z_n, and W_n.

Proof. Substituting the matrixes U and V in equations (10) and (11) into equation (12) yields

$$u_n Y_{n+1} + Z_n = 0, \tag{13}$$

$$X_{n+1} + (\lambda - v_n)Y_{n+1} - W_n = 0, \tag{14}$$

$$u_{n,t} = u_n W_{n+1} - u_n X_n + (\lambda - v_n)Z_n, \tag{15}$$

$$v_{n,t} = -Z_{n+1} - (\lambda - v_n)W_{n+1} - u_n Y_n + (\lambda - v_n)W_n, \tag{16}$$

which can be simplified as

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L_1 \begin{pmatrix} W_n \\ Y_n \end{pmatrix} - \lambda L_2 \begin{pmatrix} W_n \\ Y_n \end{pmatrix}, \tag{17}$$

by introducing two operators L_1 and L_2:

$$\begin{aligned} L_1 &= \begin{pmatrix} u_n(S - S^{-1}) & u_n(S-1)v_{n-1} \\ v_n(S-1) & Su_nS - u_n \end{pmatrix}, \\ L_2 &= \begin{pmatrix} 0 & u_n(S-1) \\ S-1 & 0 \end{pmatrix}. \end{aligned} \tag{18}$$

In order to derive system (1), we equivalently rewrite equation (17) as

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L_1 \begin{pmatrix} W_n \\ Y_n \end{pmatrix} - \lambda L_2 \begin{pmatrix} W_n \\ Y_n \end{pmatrix} + \sum_{l=0}^{k} \alpha_l \lambda^{l+1} L_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{19}$$

and suppose that Y_n and W_n can be expressed by

$$\begin{pmatrix} W_n \\ Y_n \end{pmatrix} = \sum_{j=0}^{k} \begin{pmatrix} w_{n,j} \\ y_{n,j} \end{pmatrix} \lambda^{k-j}, \tag{20}$$

with the conditions

$$W_n = Y_n = \sum_{l=0}^{k} \alpha_l \lambda^l, \text{ when } (u_n, v_n) = (1, 0). \tag{21}$$

We substitute equation (20) into equation (19) and compare the coefficients of λ, then the following equations are obtained:

$$\lambda^0 : \begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L_1 \begin{pmatrix} w_{n,k} \\ y_{n,k} \end{pmatrix}, \tag{22}$$

$$\lambda^{k-j} : \begin{pmatrix} w_{n,j+1} \\ y_{n,j+1} \end{pmatrix} = L_2^{-1} L_1 \begin{pmatrix} w_{n,j} \\ y_{n,j} \end{pmatrix} + \alpha_{k-j-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad j = 0, 1, 2, \ldots, k-1, \tag{23}$$

$$\lambda^{k+1} : \begin{pmatrix} w_{n,0} \\ y_{n,0} \end{pmatrix} = \alpha_k \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{24}$$

With the help of equations (2), (18), and (24), from equation (23), we have

$$\begin{pmatrix} w_{n,k} \\ y_{n,k} \end{pmatrix} = \left(\sum_{m=1}^{k} \alpha_m L_2^{-1} L^{m-1} \right) L_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{25}$$

Substituting equation (25) into equation (22) yields

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = \left(\sum_{m=0}^{k} \alpha_m L^m \right) L_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{26}$$

which can be rewritten as system (1) by employing the operator L_1. Thus, system (1) is integrable in Lax's sense. In view of equations (10) and (11), the Lax pair of system (1) can be written as

$$\Psi_{n+1}(z) + u_n \Psi_{n-1}(z) + v_n \Psi_n(z) = \left(z + \frac{1}{z} \right) \Psi_n(z), \tag{27}$$

$$\Psi_{n,t}(z) = Z_n \Psi_{n-1}(z) + W_n \Psi_n(z), \tag{28}$$

by cancelling $\Psi_{1,n}$ and setting $\Psi_{2,n} = \Psi_n$ and $\lambda = (z+1)/z$. □

3. Exact Solutions

In this section, we would like to solve system (1) through the inverse scattering method. To avoid duplication, we omit here the similar direct and inverse scattering analysis [47].

Theorem 2. *Suppose the time evolutions of u_n and v_n obey system (1), then the scattering data*

$$\left\{ |z| = 1, R(z) = \frac{b(z)}{a(z)}, z_j, c_j, \ j = 1, 2, \ldots, N \right\}, \tag{29}$$

for the isospectral problem (27) have the following time dependences:

$$R(z,t) = R(z,0) \exp\left\{ \left[\sum_{l=0}^{k} \left(z + \frac{1}{z} \right)^l \int_0^t \alpha_l(\tau) \mathrm{d}\tau \right] \left(z - \frac{1}{z} \right) \right\}, \tag{30}$$

$$\begin{aligned} z_j(t) &= z_j(0), \\ c_j^2(t) &= c_j^2(0) \exp\left\{ \left[\sum_{l=0}^{k} \left(z_j + \frac{1}{z_j} \right)^l \int_0^t \alpha_l(\tau) \mathrm{d}\tau \right] \left(z_j - \frac{1}{z_j} \right) \right\}, \end{aligned} \tag{31}$$

where $c_j(0)$ and $R(z,0)$ are the scattering data when $u_n(t) = u_n(0)$ and $v_n(t) = v_n(0)$.

Proof. For spectral problem (27), there exist two pairs of Jost solutions [47]:

$$\Psi_n(z) \sim z^n, \overline{\Psi}_n(z) \sim z^{-n}, \text{when } n \longrightarrow +\infty, \tag{32}$$

$$\Phi_n(z) \sim z^{-n}, \overline{\Phi}_n(z) \sim z^n, \text{ when } n \longrightarrow -\infty, \tag{33}$$

where $\Psi_n(z)$ and $\Phi_n(z)$ are analytic for $|z| \le 1$ on the complex plane of z, $\overline{\Psi}_n(z)$ and $\overline{\Phi}_n(z)$ are analytic for $|z| > 1$, and $\overline{\Psi}_n(z) = \Psi_n^*(z)$ and $\overline{\Phi}_n(z) = \Phi_n^*(z)$ for $|z| = 1$.

With the above preparations, we first consider the continuous spectrum. Since $\Phi_{n,t}(z) - Z_n\Phi_{n-1}(z) - W_n\Phi_n(z)$ is also a solution of equation (27), there exist two functions $\gamma(t)$ and $\vartheta(t)$ such that

$$\Phi_{n,t}(z) - Z_n\Phi_{n-1}(z) - W_n\Phi_n(z) = \gamma(t)\overline{\Phi}_n(z) + \vartheta(t)\Phi_n(z). \tag{34}$$

Letting $n \longrightarrow -\infty$ and using equations (13), (21), (32), and (33), from equation (34), we have

$$-nz_t + \left[\sum_{l=0}^{k} \alpha_l \left(z + \frac{1}{z}\right)^l\right]\left(z^2 - z\right) = \gamma(t)z^{2n+1} + \vartheta(t)z. \tag{35}$$

In view of the arbitrariness of n, from equation (35), we obtain

$$\begin{aligned} z_t &= 0, \\ \gamma(t) &= 0, \\ \vartheta(t) &= \left[\sum_{l=0}^{k} \alpha_l \left(z + \frac{1}{z}\right)^l\right](z-1). \end{aligned} \tag{36}$$

Thus, equation (34) is simplified as

$$\begin{aligned} &\Phi_{n,t}(z) - Z_n\Phi_{n-1}(z) - W_n\Phi_n(z) \\ &\quad = \left[\sum_{l=0}^{k} \alpha_l \left(z + \frac{1}{z}\right)^l\right](z-1)\Phi_n(z). \end{aligned} \tag{37}$$

Substituting the linear relation $\Phi_n(z) = a(z)\overline{\Psi}_n(z) + b(z)\Psi_n(z)$ into equation (37) and setting $n \longrightarrow +\infty$ yields

$$a_t(z,t)z^{-n} + b_t(z,t)z^n = \left[\sum_{l=0}^{k} \alpha_l \left(z + \frac{1}{z}\right)^l\right]\left(z - \frac{1}{z}\right)b(z,t)z^n, \tag{38}$$

from which we have

$$\begin{aligned} a_t(z,t) &= 0, \\ b_t(z,t) &= \left[\sum_{l=0}^{k} \alpha_l \left(z + \frac{1}{z}\right)^l\right]\left(z - \frac{1}{z}\right)b(z,t), \end{aligned} \tag{39}$$

and then we obtain

$$\begin{aligned} a(z,t) &= a(z,0), \\ b(z,t) &= b(z,0)\exp\left\{\left[\sum_{l=0}^{k}\left(z + \frac{1}{z}\right)^l \int_0^t \alpha_l(\tau)\mathrm{d}\tau\right]\left(z - \frac{1}{z}\right)\right\}. \end{aligned} \tag{40}$$

We next consider the time dependences of the discrete scattering data. For such purpose, we take the Jost solution $\Psi_n(z)$ which satisfies the asymptotic property in equation (34) and set $z = z_j$; then, the linear relation $\Psi_{n,t}(z_j) - Z_n\Psi_{n-1}(z_j) - W_n(z_j)$ is also a solution of equation (27). Thus, there exist two undetermined functions $\delta(t)$ and $\omega(t)$ such that

$$\begin{aligned} &\Psi_{n,t}(z_j) - Z_n\Psi_{n-1}(z_j) - W_n\Psi_n(z_j) \\ &\quad = \delta(t)\overline{\Psi}_n(z_j) + \omega(t)\Psi_n(z_j), \end{aligned} \tag{41}$$

where $\Psi_n(z_j) \sim z_j^n$ and $\overline{\Psi}_n(z_j) \sim z_j^{-n}$ as $n \longrightarrow +\infty$. Therefore, $\delta(t) = 0$ and equation (41) have a simplified form as follows:

$$\Psi_{n,t}(z_j) - Z_n\Psi_{n-1}(z_j) - W_n\Psi_n(z_j) = \omega(t)\Psi_n(z_j). \tag{42}$$

Letting $Q_n(z_j) = \sqrt{S_n}\Psi_n(z_j)$, here $S_n = \prod_{j=n+1}^{\infty} u_j$, we have

$$\sqrt{u_{n+1}}Q_{n+1}(z_j) + \sqrt{u_n}Q_{n-1}(z_j) + v_nQ_n(z_j) = \lambda_jQ_n(z_j). \tag{43}$$

Then, equation (42) can be rewritten as

$$\begin{aligned} &Q_{n,t}(z_j) + \frac{1}{2}(W_{n+1} - W_n)Q_n(z_j) - \frac{1}{2}(\lambda_j - v_n)Y_{n+1}Q_n(z_j) \\ &\quad + \sqrt{u_n}Y_{n+1}Q_{n-1}(z_j) \\ &= \left[\omega(t) + \left(1 - \frac{1}{2}\lambda_j\right)\sum_{l=0}^{k}\alpha_l\lambda_j^l\right]Q_n(z_j). \end{aligned} \tag{44}$$

With the help of equation (43), we simplify equation (44) as

$$\begin{aligned} &Q_{n,t}(z_j) + \frac{1}{2}(W_{n+1} - W_n)Q_n(z_j) \\ &\quad + \frac{1}{2}(Y_{n+1} - Y_n)\sqrt{u_n}Q_{n-1}(z_j) \\ &\quad + \frac{1}{2}\left(Y_n\sqrt{u_n}Q_{n-1}(z_j) - Y_{n+1}\sqrt{u_{n+1}}Q_{n+1}(z_j)\right) \\ &= \left[\omega(t) + \left(1 - \frac{1}{2}\lambda_j\right)\sum_{l=0}^{k}\alpha_l\lambda_j^l\right]Q_n(z_j). \end{aligned} \tag{45}$$

Multiplying equation (45) by $2Q_n(z_j)$ and then summing it, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=-\infty}^{\infty}Q_n^2(z_j)&+\sum_{n=-\infty}^{\infty}(W_{n+1}-W_n)Q_n^2(z_j)+\sum_{n=-\infty}^{\infty}(Y_{n+1}-Y_n)\sqrt{u_n}Q_{n-1}(z_j)Q_n(z_j)\\&+\sum_{n=-\infty}^{\infty}\left(Y_n\sqrt{u_n}\,Q_{n-1}(z_j)Q_n(z_j)-Y_{n+1}\sqrt{u_{n+1}}\,Q_n(z_j)Q_{n+1}(z_j)\right)\\&=2\left[\omega(t)+\left(1-\frac{1}{2}\lambda_j\right)\sum_{l=0}^{k}\alpha_l\lambda_j^l\right]\sum_{n=-\infty}^{\infty}Q_n^2(z_j),\end{aligned}\tag{46}$$

which implies

$$\omega(t)=-\left(1-\frac{1}{2}\lambda_j\right)\sum_{l=0}^{k}\alpha_l\lambda_j^l,\tag{47}$$

Here, the assumption that $Q_n(z_j)$ is a normalized eigenfunction, i.e., $\sum_{n=-\infty}^{\infty}Q_n^2(z_j)=1$, and the following two results have been used:

$$\sum_{n=-\infty}^{\infty}\left[(W_{n+1}-W_n)Q_n^2(z_j)+(Y_{n+1}-Y_n)\sqrt{u_n}\,Q_{n-1}(z_j)Q_n(z_j)\right]=0,\tag{48}$$

$$\sum_{n=-\infty}^{\infty}\left(Y_n\sqrt{u_n}\,Q_{n-1}(z_j)Q_n(z_j)-Y_{n+1}\sqrt{u_{n+1}}\,Q_n(z_j)Q_{n+1}(z_j)\right)=0.\tag{49}$$

It is easy to see that equation (49) is obvious. For equation (48), inspired by the inner product [47], we have

$$\begin{aligned}&(W_{n+1}-W_n)Q_n^2(z_j)+(Y_{n+1}-Y_n)\sqrt{u_n}Q_{n-1}(z_j)Q_n(z_j)\\&=\left((S-1)\begin{pmatrix}W_n\\Y_n\end{pmatrix},\begin{pmatrix}Q_n^2(z_j)\\\sqrt{u_n}\,Q_{n-1}(z_j)Q_n(z_j)\end{pmatrix}\right)\\&=\sum_{s=1}^{k}s\alpha_s\lambda_j^{s-1}\left(\begin{pmatrix}u_{n+1}-u_n\\v_n-v_{n-1}\end{pmatrix},\begin{pmatrix}Q_n^2(z_j)\\\sqrt{u_n}\,Q_{n-1}(z_j)Q_n(z_j)\end{pmatrix}\right)\\&=\sum_{s=1}^{k}s\alpha_s\lambda_j^{s-1}\left(u_{n+1}Q_n^2(z_j)-u_nQ_{n-1}^2(z_j)-\sqrt{u_nu_{n+1}}\,Q_{n-1}(z_j)Q_{n+1}(z_j)+\sqrt{u_{n-1}u_n}\,Q_{n-2}(z_j)Q_n(z_j)\right).\end{aligned}\tag{50}$$

Substituting equation (50) into equation (48), then the summation of the left hand reaches zero.

In view of $Q_n(z_j)\longrightarrow c_j(t)z_j^n$ as $n\longrightarrow\infty$ and equations (21), (48), and (49), from equation (44), we then have

$$\begin{aligned}&c_{j,t}(t)+nc_j(t)z_{j,t}z_j^{-1}+\frac{1}{2}\sum_{l=0}^{k}\alpha_l\lambda_j^lc_j(t)z_j^{-1}\\&-\frac{1}{2}\sum_{l=0}^{k}\alpha_l\lambda_j^lc_j(t)z_j=0,\end{aligned}\tag{51}$$

which gives

$$z_{j,t}=0,\quad c_{j,t}=\frac{1}{2}\left[\sum_{l=0}^{k}\alpha_l\left(z_j+\frac{1}{z_j}\right)^l\right]\left(z_j-\frac{1}{z_j}\right)c_j(t).\tag{52}$$

Solving equation (52), we can easily arrive at equation (31). □

Theorem 3. *Based on the scattering data (29)–(31), exact and explicit semidiscrete N-soliton solutions of the system (1) can be obtained:*

$$u_n = \frac{\det\left(E + D_n(t)\right)\det\left(E + D_{n-2}(t)\right)}{\det\left(E + D_{n-1}(t)\right)^2}, \tag{53}$$

$$\begin{aligned} v_n = {} & \operatorname{tr}\left[\left(E + D_{n-1}(t)\right)^{-1} q_{n-1}(t) q_n(t)^T\right] \\ & - \operatorname{tr}\left[\left(E + D_n(t)\right)^{-1} q_n(t) q_{n+1}(t)^T\right], \end{aligned} \tag{54}$$

where E is the N-th-order identity matrix, $\operatorname{tr}(\cdot)$ denotes the trace of a matrix, and

$$\begin{aligned} & D_n(t) = \left(c_j(t) c_k(t) \frac{z_j^{n+1} z_k^{n+1}}{1 - z_j z_k}\right)_{N\times N}, \\ & c_j^2(t) = c_j^2(0) \exp\left\{\left[\sum_{l=0}^{k}\left(z_j + \frac{1}{z_j}\right)^l \int_0^t \alpha_l(\tau)\mathrm{d}\tau\right]\left(z_j - \frac{1}{z_j}\right)\right\}, \\ & q_n(t) = \left(c_1(t) z_1^n, c_2(t) z_2^n \cdots c_N(t) z_N^n\right)^T, z_j(t) = z_j(0). \end{aligned} \tag{55}$$

Proof. Supposing

$$\begin{aligned} \Psi_n(z) &= \sum_{j=n}^{\infty} H_{n,j} z^j, \\ \overline{\Psi}_n(z) &= \sum_{j=n}^{\infty} H_{n,j} z^{-j}, \end{aligned} \tag{56}$$

and substituting them into equation (27), then we can determine $H_{n,j}$ and hence recover u_n and v_n:

$$\begin{aligned} u_n &= \frac{H_{n,n}}{H_{n-1,n-1}}, \\ v_n &= \frac{H_{n,n+1}}{H_{n,n}} - \frac{H_{n-1,n}}{H_{n-1,n-1}}. \end{aligned} \tag{57}$$

We note here that $\widetilde{H}_{n,m} = H_{n,m}/H_{n,n}$ satisfies the following Gel'fand–Levitan–Marchenko integral equation of discrete form [47]:

$$\widetilde{H}_{n,m} + F_{n+m} + \sum_{s=n+1}^{\infty} \widetilde{H}_{n,s} F_{s+m} = 0, \quad m > n, \tag{58}$$

$$H_{n,n}^{-2} = 1 + F_{2n} + \sum_{s=n+1}^{\infty} \widetilde{H}_{n,s} F_{s+m} = 0, \quad m = n, \tag{59}$$

where

$$F_m = \sum_{j=1}^{N} c_j^2(t) z_j^m + \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} R(t,z) z^{m-1} \mathrm{d}z. \tag{60}$$

To construct N-soliton solutions of system (1), we set the reflection coefficient $R(t,z)$ to zero. In this case, from (60), we have

$$F_m = \sum_{j=1}^{N} c_j^2(t) z_j^m. \tag{61}$$

Further taking

$$\widetilde{H}_{n.m}(t) = \sum_{j=1}^{N} c_j(t) z_j^m p_{n,j}(t), \tag{62}$$

and substituting equations (61) and (62) into equation (58), we have

$$p_{n,j}(t) + c_j(t) z_j^n + \sum_{k=1}^{N} c_j(t) c_k(t) \frac{z_j^{n+1} z_k^{n+1}}{1 - z_j z_k} p_{n,k}(t) = 0, \quad j = 1, 2, \ldots, N, \tag{63}$$

which can be rewritten as

$$\left(E + D_n(t) p_n(t) = -q_n(t), \right. \tag{64}$$

by means of the vector $p_n(t) = (p_{n,1}(t), p_{n,2}(t), \ldots, p_{n,N}(t))^T$.

Considering the positive definiteness of $E + D_n(t)$ when the scattering data z_j and $c_j(t)$ are all real numbers, we have from equation (62),

$$\widetilde{H}_{n.m}(t) = -\operatorname{tr}\left(E + D_n(t)\right)^{-1} q_n(t) q_m(t)^T\Big), \quad m > n, \tag{65}$$

When $m = n$, $H_{n,n}(t)$ can be determined as follows [47]:

$$H_{n,n}(t) = \frac{\det\left(E + D_n(t)\right)}{\det\left(E + D_{n-1}(t)\right)}. \tag{66}$$

Finally, from equations (57), (65), and (66) we obtain equations (53) and (54). □

4. Nonlinear Soliton Dynamics

We study in this section the nonlinear soliton dynamics of the semidiscrete soliton solutions (53) and (54) in the three cases when $N = 1, 2, 3$.

When $N = 1$, we have

$$\det\left(E + D_n(t)\right) = 1 + \frac{c_1^2(0) z_1^{2n+2} e^{2\xi_1}}{1 - z_1^2}, \tag{67}$$

where

$$\xi_1 = \frac{1}{2}\left[\sum_{l=0}^{k}\left(z_1 + \frac{1}{z_1}\right)^l \int_0^t \alpha_l(\tau)\mathrm{d}\tau\right]\left(z_1 - \frac{1}{z_1}\right). \tag{68}$$

Then, the semidiscrete one-soliton solutions of system (1) are obtained:

$$u_n = \frac{\left(1 - z_1^2 + c_1^2(0) z_1^{2n+2} e^{2\xi_1}\right)\left(1 - z_1^2 + c_1^2(0) z_1^{2n-2} e^{2\xi_1}\right)}{\left(1 - z_1^2 + c_1^2(0) z_1^{2n} e^{2\xi_1}\right)^2}, \tag{69}$$

$$v_n = \frac{c_1^2(0) z_1^{2n-1} e^{2\xi_1}\left(1 - z_1^2\right)}{1 - z_1^2 + c_1^2(0) z_1^{2n} e^{2\xi_1}} - \frac{c_1^2(0) z_1^{2n+1} e^{2\xi_1}\left(1 - z_1^2\right)}{1 - z_1^2 + c_1^2(0) z_1^{2n+2} e^{2\xi_1}}. \tag{70}$$

We simulate the spatiotemporal structures of the semidiscrete one-soliton solutions (69) and (70) in Figures 1 and 2, where $\alpha_0 = 1$, $\alpha_1 = t$, $\alpha_2 = \cos t$, $\alpha_3 = \operatorname{sech} t$, $c_1(0) = 1$,

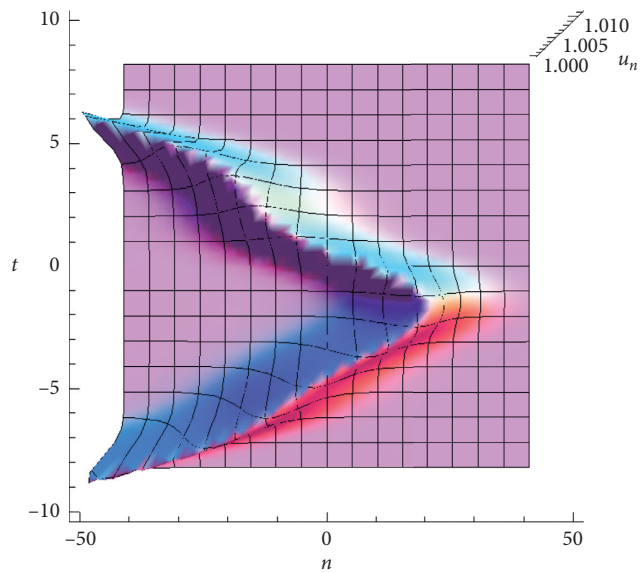

Figure 1: Spatiotemporal structure of semidiscrete one-soliton solution (69) with the parameters $\alpha_0 = 1$, $\alpha_1 = t$, $\alpha_2 = \cos t$, $\alpha_3 = \operatorname{sech} t$, $c_1(0) = 1$, $k = 3$, and $z_1 = 0.9$.

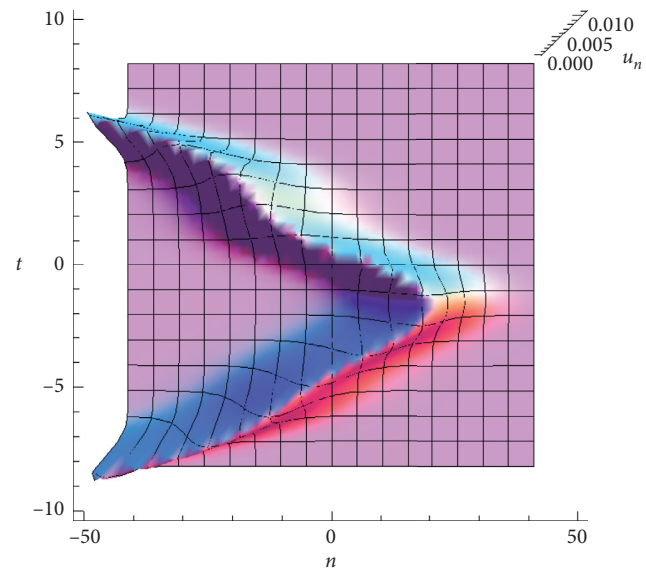

Figure 2: Spatiotemporal structure of semidiscrete one-soliton solution (70) with the parameters $\alpha_0 = 1$, $\alpha_1 = t$, $\alpha_2 = \cos t$, $\alpha_3 = \operatorname{sech} t$, $c_1(0) = 1$, $k = 3$, and $z_1 = 0.9$.

$k = 3$, and $z_1 = 0.9$. It can be seen that the bell-solitons determined by solutions (69) and (70) have very similar structures. In fact, the computer simulation hints $|u_n - 1 - v_n| \le 0.0005$. If we rewrite (69) and (70) to have the same numerator:

$$u_n = 1 + \frac{c_1^2(0)\left(1 - z_1^2\right)^3 z_1^{2n} e^{2\xi_1}}{z_1^2\left(1 - z_1^2 + c_1^2(0) z_1^{2n} e^{2\xi_1}\right)^2},$$
$$v_n = \frac{c_1^2(0)\left(1 - z_1^2\right)^3 z_1^{2n} e^{2\xi_1}}{z_1\left[\left(1 - z_1^2\right)^2 + c_1^2(0)\left(1 + z_1^2\right)\left(1 - z_1^2\right) z_1^{2n} e^{2\xi_1} + z_1^{4n+2} e^{4\xi_1}\right]}, \tag{71}$$

then the subtraction of the denominators of $u_n - 1$ and v_n is calculated as

$$0.81 \times 0.9^{4n} e^{2\xi_1} - 0.00171 \times 0.9^{2n} e^{\xi_1} - 0.003249, \tag{72}$$

which is very small when the parameters are selected as those of Figures 1 and 2. In Figure 3, we show the dynamical evolutions of bell-soliton solution (69) at three different times. We can see from Figures 1 and 3 that the bell-soliton solution first propagates in the positive direction of n-axis and then in the reverse direction. It is the soliton propagation in different directions that forms the V-shaped trajectory in Figure 1. However, if we let $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and all the other parameters unchanged, then the trajectory is a straight line (see Figure 4). Since the dynamical evolutions of solution (70) are very similar to the ones of solution (69), we omit them here for simplicity.

When $N = 2$, we have

$$\det = \begin{vmatrix} 1 + \dfrac{c_1^2(0) z_1^{2n+2} e^{2\xi_1}}{1 - z_1^2} & \dfrac{c_1(0) c_2(0)\left(z_1 z_2\right)^{n+1} e^{\xi_1 + \xi_2}}{1 - z_1 z_2} \\ \dfrac{c_2(0) c_1(0)\left(z_2 z_1\right)^{n+1} e^{\xi_1 + \xi_2}}{1 - z_2 z_1} & 1 + \dfrac{c_2^2(0) z_2^{2n+2} e^{2\xi_2}}{1 - z_2^2} \end{vmatrix}, \tag{73}$$

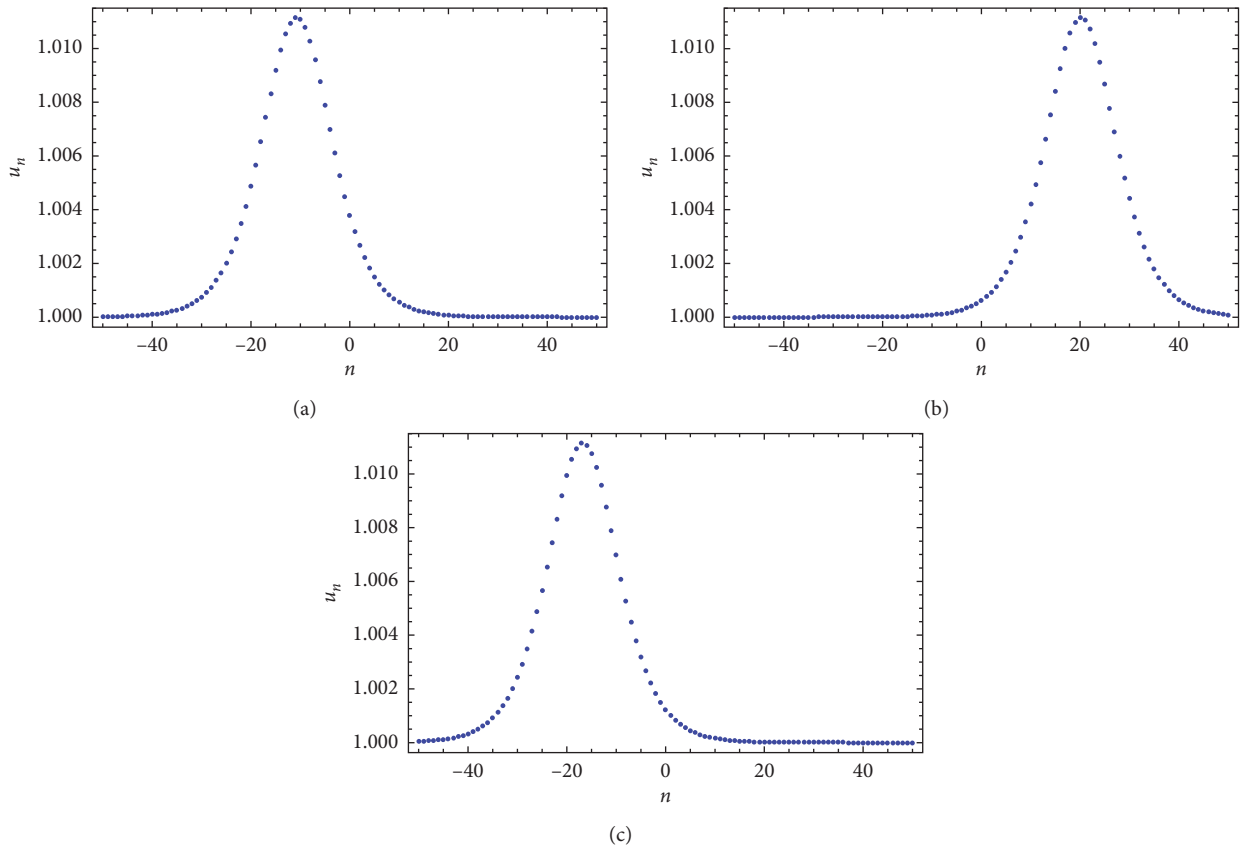

FIGURE 3: Dynamical evolution of semidiscrete one-soliton solution (69) with the parameters $\alpha_0 = 1$, $\alpha_1 = t$, $\alpha_2 = \cos t$, $\alpha_3 = \operatorname{sech} t$, $c_1(0) = 1$, $k = 3$, and $z_1 = 0.9$. (a) $t = -6$. (b) $t = 2$. (c) $t = 3$.

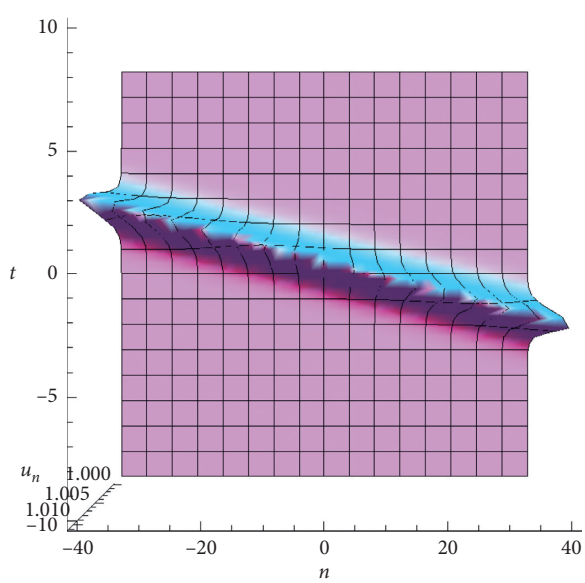

FIGURE 4: Spatiotemporal structure of semidiscrete one-soliton solution (69) with the parameters $\alpha_0 = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $c_1(0) = 1$, $k = 3$, and $z_1 = 0.9$.

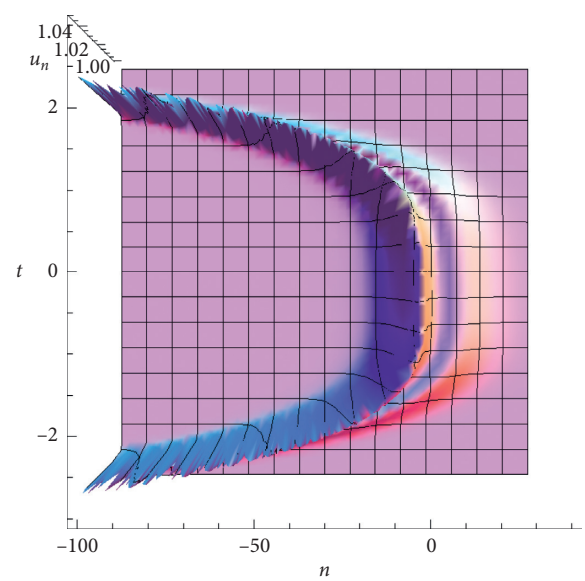

FIGURE 5: Spatiotemporal structure of the semidiscrete two-soliton solution determined by solution (53) with the parameters $\alpha_0 = 1$, $\alpha_1 = t, \alpha_2 = t^2, \alpha_3 = t^3, c_1(0) = 1, c_2(0) = 5.5, z_1 = 0.8$, and $z_2 = 0.9$.

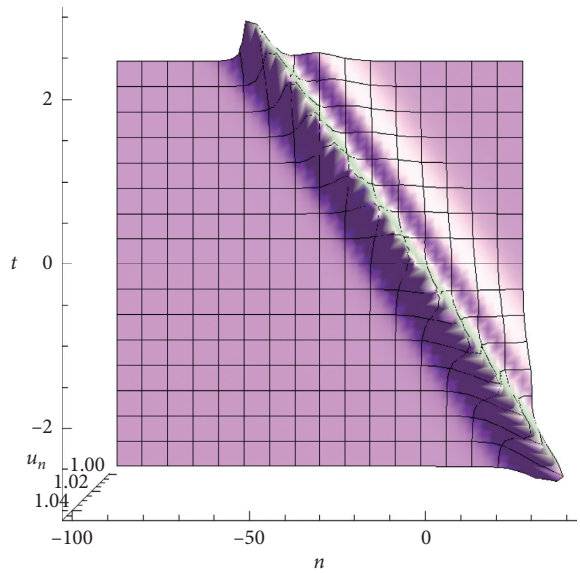

Figure 6: Spatiotemporal structure of the semidiscrete two-soliton solution determined by solution (53) with the parameters $\alpha_0 = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $c_1(0) = 1$, $c_2(0) = 5.5$, $z_1 = 0.8$, and $z_2 = 0.9$.

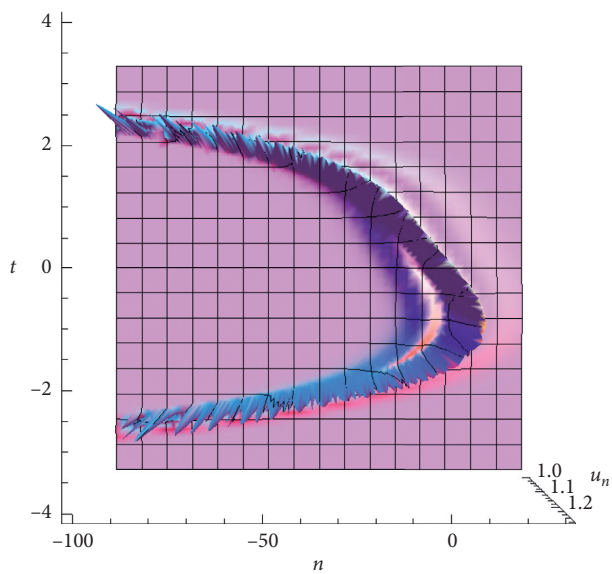

Figure 7: Spatiotemporal structure of the semidiscrete three-soliton solution determined by solution (53) with the parameters $\alpha_0 = 1$, $\alpha_1 = t$, $\alpha_2 = t^3$, $\alpha_3 = \operatorname{sech}^2 t$, $c_1(0) = 1$, $c_2(0) = 2$, $c_3(0) = -3$, $z_1 = 0.8$, $z_2 = 0.9$, and $z_3 = 0.6$.

where

$$\xi_2 = \frac{1}{2}\left[\sum_{l=0}^{k}\left(z_2 + \frac{1}{z_2}\right)^l \int_0^t \alpha_l(\tau)\mathrm{d}\tau\right]\left(z_2 - \frac{1}{z_2}\right). \tag{74}$$

With the help of equations (73) and (74), we simulate in Figure 5, the spatiotemporal structure of the semidiscrete two-soliton solution determined by equation (53). Here, the parameters are selected as $\alpha_0 = 1$, $\alpha_1 = t$, $\alpha_2 = t^2$, $\alpha_3 = t^3$, $c_1(0) = 1$, $c_2(0) = 5.5$, $k = 3$, $z_1 = 0.8$, and $z_2 = 0.9$. In Figure 6, we simulate a catching up process between the bell-shaped two-soliton solutions. In the process of chasing each other, the high soliton gradually surpasses the low soliton after turning and therefore forms a U-shaped trajectory, in which there is no complete collision. If we let $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and all the other parameters unchanged, then the two-soliton solution shown in Figure 6 propagates in a straight line and such process of chasing each other has not happened within the same time period.

For the nonlinear soliton dynamics of the semidiscrete three-soliton solutions of system (1), we set $N = 3$ and have

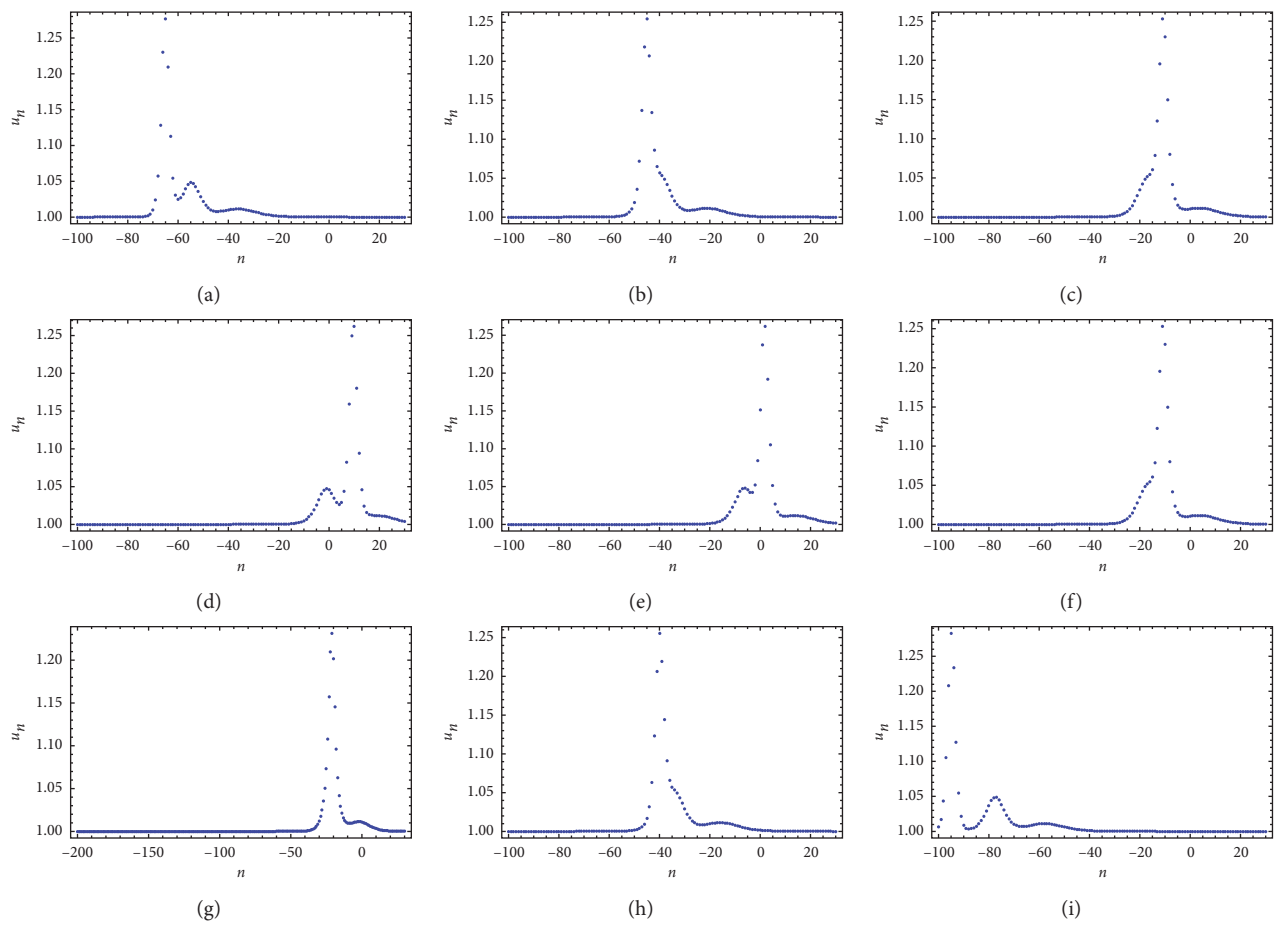

Figure 8: Dynamical evolution of the semidiscrete three-soliton solution determined by solution (53) with the parameters $\alpha_0 = 1$, $\alpha_1 = t$, $\alpha_2 = t^3$, $\alpha_3 = \operatorname{sech}^2 t$, $c_1(0) = 1$, $c_2(0) = 2$, $c_3(0) = -3$, $z_1 = 0.8$, $z_2 = 0.9$, and $z_3 = 0.6$. (a) $t = -2.7$. (b) $t = -2.5$. (c) $t = -2$. (d) $t = -1$. (e) $t = 0$. (f) $t = 1.5$. (g) $t = -1.5$. (h) $t = 0$. (i) $t = 2$.

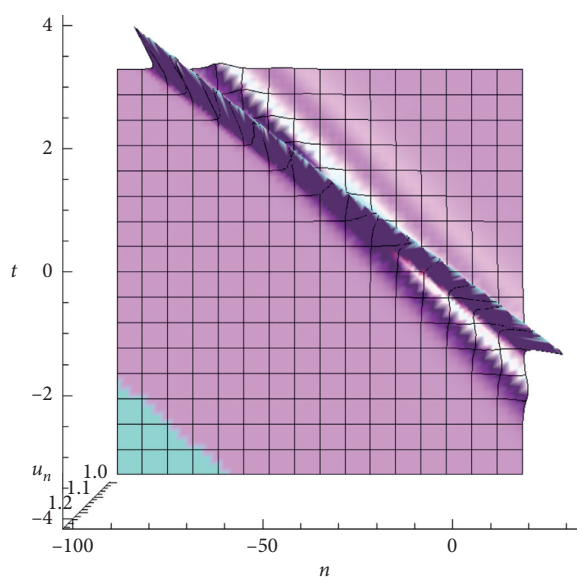

Figure 9: Spatiotemporal structure of the semidiscrete three-soliton solution determined by solution (53) with the parameters $\alpha_0 = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $c_1(0) = 1$, $c_2(0) = 2$, $c_3(0) = -3$, $z_1 = 0.8$, $z_2 = 0.9$, and $z_3 = 0.6$.

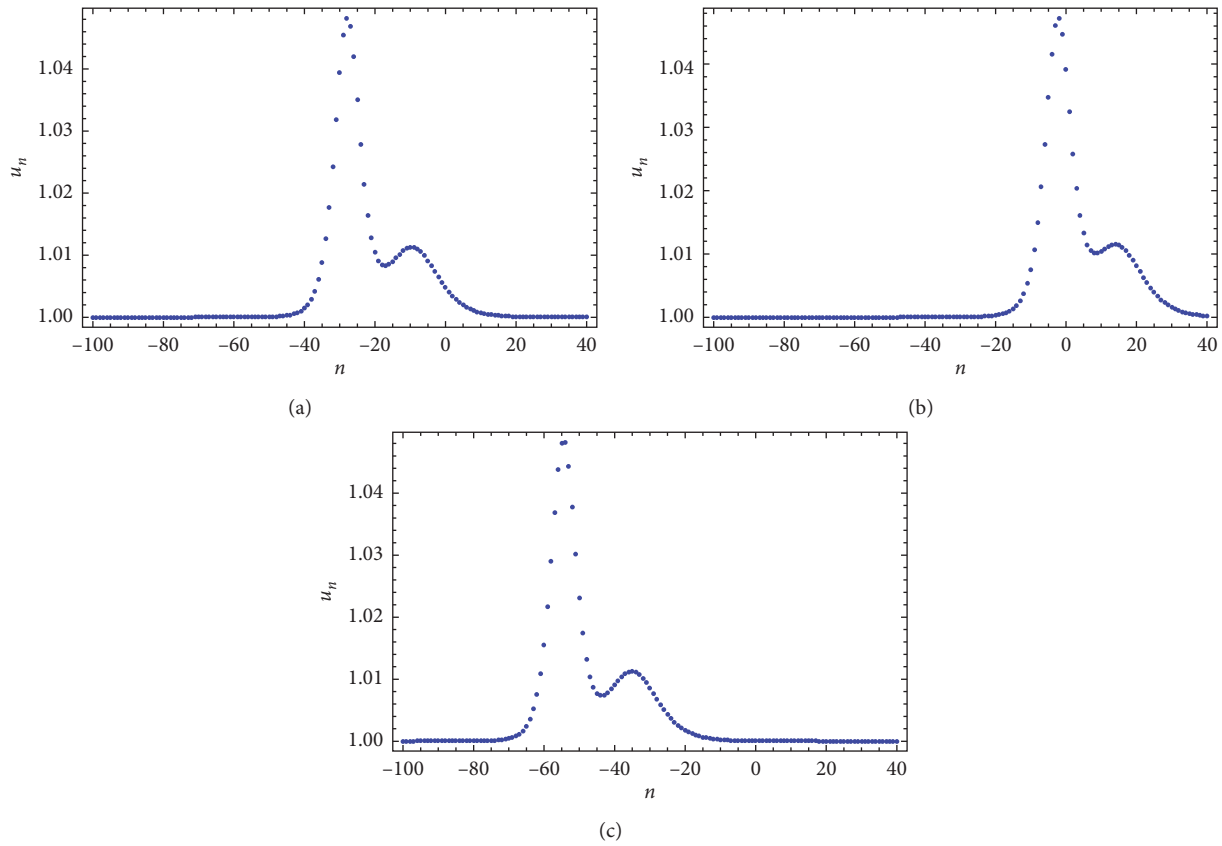

Figure 10: Dynamical evolution of the semidiscrete two-soliton solution determined by solution (53) with the parameters $\alpha_0 = 1$, $\alpha_1 = t$, $\alpha_2 = t^2$, $\alpha_3 = t^3$, $c_1(0) = 1$, $c_2(0) = 5.5$, $z_1 = 0.8$, and $z_2 = 0.9$. (a) $t = -2$. (b) $t = 0$. (c) $t = 2$.

$$\det\left(E + D_n(t)\right) = \begin{vmatrix} 1 + \dfrac{c_1^2(0) z_1^{2n+2} e^{2\xi_1}}{1 - z_1^2} & \dfrac{c_1(0) c_2(0) \left(z_1 z_2\right)^{n+1} e^{\xi_1+\xi_2}}{1 - z_1 z_2} & \dfrac{c_1(0) c_3(0) \left(z_1 z_3\right)^{n+1} e^{\xi_1+\xi_3}}{1 - z_1 z_3} \\ \dfrac{c_2(0) c_1(0) \left(z_2 z_1\right)^{n+1} e^{\xi_1+\xi_2}}{1 - z_2 z_1} & 1 + \dfrac{c_2^2(0) z_2^{2n+2} e^{2\xi_2}}{1 - z_2^2} & \dfrac{c_2(0) c_3(0) \left(z_2 z_3\right)^{n+1} e^{\xi_2+\xi_3}}{1 - z_2 z_3} \\ \dfrac{c_3(0) c_1(0) \left(z_3 z_1\right)^{n+1} e^{\xi_3+\xi_1}}{1 - z_3 z_1} & \dfrac{c_3(0) c_2(0) \left(z_3 z_2\right)^{n+1} e^{\xi_3+\xi_2}}{1 - z_3 z_2} & 1 + \dfrac{c_3^2(0) z_3^{2n+2} e^{2\xi_3}}{1 - z_3^2} \end{vmatrix}, \tag{75}$$

where

$$\xi_3 = \frac{1}{2}\left[\sum_{l=0}^{k}\left(z_3 + \frac{1}{z_3}\right)^l \int_0^t \alpha_l(\tau)\mathrm{d}\tau\right]\left(z_3 - \frac{1}{z_3}\right). \tag{76}$$

Based on equation (75), the spatiotemporal structures of the semidiscrete three-soliton solution determined by solution (53) are simulated in Figure 7 by selecting $\alpha_0 = 1$, $\alpha_1 = t$, $\alpha_2 = t^3$, $\alpha_3 = \operatorname{sech}^2 t$, $c_1(0) = 1$,$c_2(0) = 2$, $c_3(0) = -3$, $k = 3$, $z_1 = 0.95$, $z_2 = 0.99$, and $z_3 = 0.97$. In Figure 8, two catching up processes between the bell-shaped three-soliton solutions are simulated. We can see in the U-shaped trajectory that the high soliton surpasses the sub-high soliton twice and there happen two complete collisions between the high soliton and sub-high soliton. When $\alpha_1 = \alpha_2 = \alpha_3 = 1$, the corresponding three-soliton propagation in the negative direction of n-axis is shown in Figure 9. In this process of linear propagation, there is only a complete collision between the high soliton and sub-high soliton.

5. Conclusion

In summary, we have derived and solved the new and more general semidiscrete system (1) with time-varying coefficients. To the best of our knowledge, the obtained N-soliton solutions (53) and (54) are new, and they have not been reported in the literature. When $k = 1$, with the help of Mathematica 4.0, the validity of the one-soliton solutions (69) and (70) has been verified by inserting them back into original system (1). Owing to the embedded finite time-

varying coefficient functions, the obtained soliton solutions (53) and (54) possess rich spatiotemporal structures and more freedom to discuss the dynamical evolution of soliton solutions. In this paper, the bell-shaped one-, two-, and three-soliton solutions determined by solutions (53) and (54) are shown under the condition of $0.6 < z_1, z_2, z_3 < 1$. In this case, the one-, two-, and three-solitons keep soliton characteristics, and the spatiotemporal structures of u_n and v_n are very similar. But when we select small values of z_1, z_2, and z_3, the simulations always output the disordered spatiotemporal structures, and even some of them have many singularities, which have not been shown in this paper.

It is shown in Figures 1–10 that these embedded time-varying coefficient functions α_0, α_1, α_2, and α_3 affect the spatiotemporal structure and propagation velocity of semi-discrete solitons. The influences from the coefficient functions include not only the propagation speed but also the propagation direction. To some extent, this can also be verified from a mathematical point of view. For example, solutions (69) and (70) can be rewritten as

$$u_n = 1 + \frac{\left(1 - z_1^2\right)^2}{4z_1^2} \sec h^2\left(\xi_1 + n \ln z_1 + \frac{1}{2} \ln \frac{c_1^2(0)}{1 - z_1^2}\right), \tag{77}$$

$$v_n = \frac{1 - z_1^2}{2z_1} \tan h\left(\xi_1 + n \ln z_1 + \frac{1}{2} \ln \frac{c_1^2(0)}{1 - z_1^2}\right) - \frac{1 - z_1^2}{2z_1} \tan h\left[\xi_1 + (n+1) \ln z_1 + \frac{1}{2} \ln \frac{c_1^2(0)}{1 - z_1^2}\right]. \tag{78}$$

Then, the velocity v of the one soliton determined by equation (77) has the following expression:

$$v = \frac{\left(1 - z_1^2\right)}{2z_1 \ln z_1} \sum_{l=0}^{k} \left(z_1 + \frac{1}{z_1}\right)^l \alpha_l(t). \tag{79}$$

Obviously, velocity (79) depends on the coefficient functions $\alpha_l (l = 0, 1, 2, \ldots, k)$ and the constant $z_1 (0 < z_1 < 1)$ but remains unchanged for any fixed z_1 when these coefficient functions are constants. Since the velocity affects the propagation trajectory of the one soliton, the coefficient functions affect the spatiotemporal structure of the one soliton.

Data Availability

The data used to supported the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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