Research Article

Robust Time-Varying Output Formation Control for Swarm Systems with Nonlinear Uncertainties

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Time-varying output formation control problems for high-order time-invariant swarm systems are studied with nonlinear uncertainties and directed network topology in this paper. A robust controller which consists of a nominal controller and a robust compensator is applied to achieve formation control. The nominal controller based on the output feedback is designed to achieve desired time-varying formation properties for the nominal system. And the robust compensator based on the robust signal compensator technology is constructed to restrain nonlinear uncertainties. The time-varying formation problem is transformed into the stability problem. And the formation errors can be arbitrarily small with expected convergence rate. Numerical examples are provided to illustrate the effectiveness of the proposed strategy.

1. Introduction

The past decades have witnessed an increasing attention in distributed cooperative control of large-scale swarm systems due to their broad applications in various fields, such as physics, biology, mobile robots, sensor networks, and unmanned aerial vehicles [1–5]. Compared with the individual agent, swarm systems have great benefits in high efficiency, low cost, robust, and easy maintenance. The research involves a variety of branches, including consensus, flocking, synchronization, formation, and containment [6–11]. As one of the critical problems in distributed cooperative control systems, the formation problem is to find control laws that drive states or outputs of all agents to reach a predefined configuration. Many approaches have been proposed to achieve formation control [12], to name a few, leader-follower [13], virtual structure [14], and behavior-based approaches [15].

Consensus is one of the fundamental problems of swarm systems [16, 17]. Inspired by the development of consensus control theory, more and more researchers are interested in realizing the predefined formation via consensus approach without a central controller [18]. A consensus protocol was applied to achieve distributed formation control by Ren [16]. Olfati-Saber, Fax, and Murray [19] proposed a consensus-based theoretical framework for swarm systems with fixed or dynamic network topology. Xie and Wang [20] presented sufficient conditions for second-order swarm systems to realize time-invariant formation via local neighboring information interaction. Lafferriere et al. [21] studied the time-invariant formation stability problem for integrator-type high-order multiagent systems.

The formation mentioned above in [19–21] is time-invariant. In practical applications, the formation may change with the environment and tasks. Dong et al. [22] presented necessary and sufficient conditions for linear swarm systems with time delay to achieve time-varying formation. Xiao et al. [23] proposed a finite-time formation control framework which can realize a variety of complex formations and greatly reduce the data exchange. Generally speaking, only the output information of each agent can be obtained rather than state information. As a result, it is significant to study the formation control problem only using output information. Fax and Murry [23] studied
output formation stability problems of high-order linear swarm systems. In [24], a dynamic output approach was applied to solve a fully distributed time-varying formation-tracking problem for linear swarm systems. Zuo et al. [25] studied the adaptive output formation-tracking problem of linear heterogeneous swarm systems whose followers only received relative output information from their neighbors, and leaders’ dynamics were only known to their neighboring followers. Dong et al. [26] proposed a consensus-based approach for swarm systems with directed interaction topologies to achieve time-varying output formations. All these results listed above were for linear swarm systems. Practically, there are many cases where each agent involves uncertainties such as unknown time-varying parameters, unknown functions, and bounded external disturbances. Considering uncertain leader dynamics and uncertain local dynamics, Peng et al. [27] presented a neural network-based adaptive formation control approach for swarm systems. Li et al. [28] provided sufficient and necessary conditions to achieve the leader-following formation for second-order swarm systems with time-varying delay and nonlinear dynamics. Liu et al. [29] proposed an iterative learning-based method to deal with the formation control problem of swarm systems with unknown dynamics. Lu [30] proposed a robust controller consisting of a nominal controller and a robust compensator which is linear, time-invariant, and easy to implement. Because of great advantages, robust control can well deal with nonsmooth and discontinuous uncertainties compared to other approaches.

Compared with the approaches on formation control mentioned above, the main contributions of this paper are as follows. First, the dynamics of each agent is high-order with nonlinear uncertainties, and an output feedback control approach is proposed to deal with the time-varying formation problem. Second, the output formation error can be as small as desired with arbitrarily specified convergence rate under the proposed controller. Meanwhile, the proposed controller can be easily applied to practical situations.

The rest of the paper are arranged as follows. In Section 2, some basic concepts on graph theory and the problem description are introduced, respectively. Meanwhile, some assumptions, definitions, and useful lemmas are presented. In Section 3, sufficient and necessary conditions for swarm systems to achieve time-varying formation are given, and the robust compensator is introduced to suppress the nonlinear part. In Section 4, simulation examples are shown to verify the analytical results. Finally, Section 5 concludes the whole work.

**Notation 1.** Let $0_N$ and $1$ denote the matrix with all elements being 0 and column vectors of ones with dimension $N$, respectively. Use the superscripts $T$ and $H$ to represent the transpose and Hermitian adjoint of a matrix, respectively. $\| \cdot \|$ and $\otimes$ denote the Euclidean norm and Kronecker product, respectively. $\mathcal{L}$ and $\mathcal{L}^{-1}$ represent Laplace transform and inverse transformation, respectively. And $\ast$ represents the convolution operator.

### 2. Problem Formulation

#### 2.1. Basic Concept on Graph Theory

A directed graph of order $N$ can be denoted by $G = (V, E, W)$, where $V = \{v_1, v_2, \ldots, v_N\}$ is the set of nodes, $E \subseteq \{(v_i, v_j), v_i, v_j \in V\}$ is the set of edges, and $W = [w_{ij}] \in \mathbb{R}^{N \times N}$ is the adjacency matrix with nonnegative weights $w_{ij}$. For an edge $v_i = (v_i, v_j)$, $v_i$ is the parent node, $v_j$ is the child node, and $v_i$ is a neighbour of $v_j$. The set of neighbours of node $v_i$ is denoted by $N_i = \{v_j \in V: (v_i, v_j) \in E\}$. For the adjacency matrix $W$, $w_{ij} > 0$ if $(v_i, v_j) \in E$ and $w_{ij} = 0$, otherwise. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ is defined as $l_{ii} = \sum_{j=1,i \neq j}^{N} w_{ij}$ and $l_{ij} = -w_{ij}$ for all $i, j \in \{1, 2, \ldots, N\}$. If at least one node has a directed path to all the other nodes, the graph $G$ is said to have a spanning tree. More details on graph theory can be found in [31].

#### 2.2. Problem Description

A swarm system consists of $N$ agents with the directed graph $G$ which is used to describe the interaction topology. The dynamics of each agent $i (i \in \{1, 2, \ldots, N\})$ is described by

$$
\begin{align*}
\dot{x}_i(t) &= Ax_i(t) + Bu_i(t) + B_\omega \omega_i(\Delta_i, x_i), \\
y_i(t) &= Cx_i(t),
\end{align*}
$$

where $x_i \in \mathbb{R}^m$ is the state of agent $i$, $x(t) = [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T$, $u_i(t) \in \mathbb{R}^n$, and $y_i(t) \in \mathbb{R}^q$ denote the control input and output of agent $i$ with $u(t) = [u_1^T(t), u_2^T(t), \ldots, u_N^T(t)]^T$ and $y(t) = [y_1^T(t), y_2^T(t), \ldots, y_N^T(t)]^T$. $\Delta_i$ is the external disturbance, and $\omega_i(\Delta_i, x_i)$ is the nonlinear uncertainty. Simply, the nonlinear uncertainty $\omega_i(\Delta_i, x_i)$ is represented as $\omega_i(t) \in \mathbb{R}^p$ with $\omega(t) = [\omega_1^T(t), \omega_2^T(t), \ldots, \omega_N^T(t)]^T$.

**Assumption 1.** The matrix $B_1$ is of full-column rank, i.e., $\text{rank}(B_1) = m$. The matrix $C$ is of full-row rank, i.e., $\text{rank}(C) = q$. Moreover, the output dimension $q$ and input dimension $m$ satisfy $q \geq m$.

**Assumption 2.** There are positive constants $\xi_{x1}, \xi_{u1}$, and $\xi_1$ such that $\|\omega(t)\|_\infty \leq \xi_{x1} \|x(t)\|_{\infty}^\eta_1 + \xi_{u1} \|u(t)\|_\infty + \xi_1$, where $\eta_1 \geq 1$.

**Assumption 3.** There are positive constants $\xi_{x2}, \xi_{u2}$, and $\xi_2$ such that $\|N_0^{-1}(s)L(s)\omega(t)\|_\infty \leq \xi_{x2} \|x(t)\|_{\infty}^{\eta_2} + \xi_{u2} \|u(t)\|_\infty + \xi_2$, where $\eta_2 \geq 1$ and $\xi_{u2} < 1$.

**Remark 1.** $D_0^{-1}(s)N_0(s)$ and $D_0^{-1}(s)L(s)$ can be seen in the following in equation (4). $N_0^{-1}(s)L(s) = (D_0^{-1}(s)N_0 (s))^{-1} - D_0^{-1}(s)L(s)$. The nonlinear uncertainties associated with the $i$th agent are assumed to be bounded, and the assumption is widely used in literature studies.

**Definition 1.** (see [26]). The swarm system consisting of (1) is said to achieve time-varying output formation $h(t)$ if for
any given bounded initial states, there exists a vector-valued function \( r(t) \in \mathbb{R}^7 \) satisfying
\[
\lim_{t \to \infty} (y_i(t) - h_i(t) - r(t)) = 0 \quad (i = 1, 2, \ldots, N),
\]
where \( r(t) \) is called an output formation reference function.

**Definition 2.** (see [26]). If the swarm system can achieve time-varying output formation \( h(t) \) under control input \( u_i(t) = [h_1^T(t), h_2^T(t), \ldots, h_N^T(t)]^T \in \mathbb{R}^{7N} \), then \( h(t) \) is said to be feasible.

**Definition 3.** (see [26]). Swarm system (1) is said to achieve output consensus if there exists a function \( c(t) \in \mathbb{R}^7 \) satisfying
\[
\lim_{t \to \infty} (y_i(t) - c(t)) = 0 \quad (i = 1, 2, \ldots, N),
\]
where \( c(t) \) is called an output consensus function.

**Remark 2.** From Definitions 1 and 3, one sees that the output formation reference function is equal to the output consensus function if \( h(t) \equiv 0 \), and output formation problems are converted into state formation problems if \( C = I \).

**Lemma 1** (see [32]). \( L \in \mathbb{R}^{N \times N} \) is the Laplacian matrix of a directed graph \( G \); then,
\[(i) L \) has at least one zero eigenvalue, and 1 is the associated eigenvector, that is, \( L1 = 0_N \)
\[(ii) If G has a spanning tree, then 0 is a simple eigenvalue of \( L \), and all the other \( N - 1 \) eigenvalues have positive real parts.
\]

Consider the following system:
\[
\begin{cases}
\dot{x}(t) = A\bar{x}(t) \\
y(t) = Cx(t),
\end{cases}
\]
where \( A = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \) and \( C = \begin{bmatrix} 1 & 0 \end{bmatrix} \).

**Lemma 2** (see [33]). If \( (\bar{A}_{22}, \bar{A}_{12}) \) is completely observable, then system (2) is asymptotically stable with respect to \( y(t) \) if and only if \( \bar{A} \) is Hurwitz.

3. Main Results

In this section, a robust control approach is introduced to deal with time-varying output formation problems with nonlinear uncertainties in directed networks. The robust controller consists of a nominal controller and a robust compensator. Section 3.1 designs the robust compensator, and Section 3.2 is to design the nominal controller and provide time-varying output formation analysis.

3.1. Robust Compensator Design. The robust controller is constructed with two parts:
\[
u_i = u_i^{\mathrm{nom}}(t) + u_i^{\mathrm{rob}}(t),
\]
where \( u_i^{\mathrm{nom}}(t) \) stands for the nominal controller for the nominal model to obtain desired time-varying formation and \( u_i^{\mathrm{rob}}(t) \) stands for the robust compensator to restrain the influence of nonlinear uncertainty \( \omega_i(t) \).

Based on Laplace transformation, the following frequency domain equation can be found from (1):
\[
y_i(s) = D_0^{-1}(s)N_0(s)u_i(s) + D_0^{-1}(s)L(s)\omega_i(s),
\]
where \( D_0^{-1}(s)N_0(s) = C(sI - A)^{-1}B_1 \) and \( D_0^{-1}(s)L(s) = C(sI - A)^{-1}B_2 \).

To restrain uncertainties, a robust compensator is designed as
\[
u_i^{\mathrm{rob}}(s) = -N_0^{-1}(s)L(s)\omega_i(s).
\]
Substituting (5) and (7) into (6), we can get
\[
y_i(s) = D_0^{-1}(s)N_0(s)u_i^{\mathrm{nom}}(s),
\]
which means robust compensator (7) can restrain the nonlinear uncertainties perfectly.

As a matter of fact that nonlinear uncertainty \( \omega_i(s) \) cannot be measured directly, substituting \( \omega_i(s) \) into (7), one can further express the robust compensator by
\[
u_i^{\mathrm{rob}}(s) = -(N_0^{-1}(s)D_0(s)y_i(s) + u_i(s)).
\]
To avoid the high-order derivative terms of the output, a robust filter is considered which can be described as
\[
u_i^{\mathrm{rob}}(s) = F(s)u_i^{\mathrm{nom}}(s) = -F(s)[N_0^{-1}(s)D_0(s)y_i(s) - u_i(s)],
\]
where the robust filter is \( F(s) = (f/(s + f))^d \); \( f \) is a positive constant and \( d \) is a positive integer greater than or equal to the relative degree of the elements in \( N_0^{-1}(s)D_0(s) \).

Substituting (5) into (10), the robust compensator \( u_i^{\mathrm{rob}}(s) \) is finally represented as
\[
u_i^{\mathrm{rob}}(s) = -(1 - F(s))^{-1}F(s)[N_0^{-1}(s)D_0(s)y_i(s) - u_i^{\mathrm{nom}}(s)].
\]

Thus, one can obtain that
\[
u_i^{\mathrm{rob}}(t) = -\mathcal{L}^{-1}[(1 - F(s))^{-1}F(s)[N_0^{-1}(s)D_0(s)\mathcal{L}(y_i(t)) - \mathcal{L}(u_i^{\mathrm{nom}})(t))].
\]

3.2. Nominal Controller Design and Feasibility Analysis.
The nominal system without uncertainties is shown as
\[
\begin{cases}
\dot{x}_m(t) = Ax_m(t) + B_1u_i^{\mathrm{nom}}(t), \\
y_m(t) = Cx_m(t).
\end{cases}
\]
To achieve time-varying output formation, the nominal controller can be designed as
\[
\begin{align*}
    u^\text{nom}(t) &= K_1 y^1_n(t) + K_2 (y_n(t) - h_1(t)) \\
    &+ K_3 \sum_{j \in N_i} u_j \left( (y_m(t) - h_1(t)) - (y_{nj}(t) - h_j(t)) \right).
\end{align*}
\]

(14)

Under protocol (14), swarm system (13) can be written as follows:
\[
\begin{align*}
    \dot{x}_n(t) &= (I_N \otimes (A + B_1 K_1 C + B_1 K_2 C) + L \otimes B_1 K_3 C) x_n(t) \\
    &- (I_N \otimes B_1 K_2 + L \otimes B_1 K_3) h(t), \\
    y_n(t) &= (I_N \otimes C)x_n(t),
\end{align*}
\]

(15)

where \(x_n(t) = [x^T_n(t), x^2_n(t), \ldots, x^N_n(t)]^T\) and \(y_n(t) = [y^T_n(t), y^2_n(t), \ldots, y^N_n(t)]^T\).

According to Assumption 1, one can find \(C \in \mathbb{R}^{(n-q) \times n}\) such that \(T = [CT, CT]^T\) is nonsingular. Let \(\gamma_{ni} = C x^2_{ni}(t)(i = 1, 2, \ldots, N), \quad \gamma_n(t) = [\gamma^T_{n1}(t), \gamma^T_{n2}(t), \ldots, \gamma^T_{nN}(t)]^T\) and
\[
\begin{align*}
    TAT^{-1} &= \begin{bmatrix}
    \bar{A}_{11} & \bar{A}_{12} \\
    \bar{A}_{21} & \bar{A}_{22}
    \end{bmatrix}, \\
    T B_1 &= \begin{bmatrix}
    \bar{B}_{11} \\
    \bar{B}_{12}
    \end{bmatrix}, \\
    T B_{2u} &= \begin{bmatrix}
    \bar{B}_{u1} \\
    \bar{B}_{u2}
    \end{bmatrix}.
\end{align*}
\]

(16)

Using nonsingular transformation \(I_N \otimes T\), swarm system (15) can be rewritten as
\[
\begin{align*}
    \dot{y}_n(t) &= (I_N \otimes (\bar{A}_{11} + \bar{B}_{11} K_1 + \bar{B}_{11} K_2) + L \otimes \bar{B}_{11} K_3) y_n(t) \\
    &+ (I_N \otimes \bar{A}_{12}) \gamma_n(t) - (I_N \otimes \bar{B}_{11} K_2 + L \otimes \bar{B}_{11} K_3) h(t), \\
    \bar{y}_n(t) &= (I_N \otimes (\bar{A}_{21} + \bar{B}_{12} K_1 + \bar{B}_{12} K_2) + L \otimes \bar{B}_{12} K_3) y_n(t) \\
    &+ (I_N \otimes \bar{A}_{22}) \gamma_n(t) - (I_N \otimes \bar{B}_{12} K_2 + L \otimes \bar{B}_{12} K_3) h(t).
\end{align*}
\]

(17)

Let \(\theta_n(t) = y_n(t) - h(t)(i = 1, 2, \ldots, N)\) and \(\theta(t) = [\theta^T_1(t), \theta^T_2(t), \ldots, \theta^T_N(t)]^T\); it is easy to find
\[
\begin{align*}
    \dot{\theta}(t) &= (I_N \otimes (\bar{A}_{11} + \bar{B}_{11} K_1 + \bar{B}_{11} K_2) + L \otimes \bar{B}_{11} K_3) \theta(t) \\
    &+ (I_N \otimes \bar{A}_{12}) \gamma_n(t) - (I_N \otimes \bar{B}_{11} K_2 + L \otimes \bar{B}_{11} K_3) h(t), \\
    \bar{\gamma}_n(t) &= (I_N \otimes (\bar{A}_{21} + \bar{B}_{12} K_1 + \bar{B}_{12} K_2) + L \otimes \bar{B}_{12} K_3) \theta(t) \\
    &+ (I_N \otimes \bar{A}_{22}) \gamma_n(t) - (I_N \otimes \bar{B}_{12} K_2 + L \otimes \bar{B}_{12} K_3) h(t).
\end{align*}
\]

(18)

According to Definition 1, time-varying output formation means swarm system (18) achieves output consensus. Let \(\lambda_i(i = 1, 2, \ldots, N)\) be the eigenvalues of the Laplacian matrix \(L\) and \(\lambda_1 = 0\) with the associated eigenvector \(\bar{u}_i = 1\). \(L\) is the Jordan canonical form of \(L\), where \(U = [\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N]\) and \(U^{-1} = [\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N]^H\) such that \(U^{-1} L U = J\). Let \(c_k \in \mathbb{R}^q(k = 1, 2, \ldots, q)\) be linearly independent vectors and \(p_j = \bar{u}_i \otimes c_k(j = (i-1)q + k; i = 1, 2, \ldots, N; k = 1, 2, \ldots, q)\). The subspace \(C(U)\) spanned by \(p_k = \bar{u}_i \otimes c_k = 1 \otimes c_k\), \(k = 1, 2, \ldots, q\) is an output consensus subspace, and the subspace \(\overline{C}(U)\) spanned by \(p_{(p-1)}, p_{(p+1)}, \ldots, p_{(qN)}\) is a complement output consensus subspace. Because \(p_j(j = 1, 2, \ldots, qN)\) are linearly independent, Lemma 3 can be obtained as follows.

**Lemma 3** (see [33]). \(C(U) \oplus \overline{C}(U) = C^{jN}\).

\(J\) can be denoted as \(J = \text{diag}(0, J)\) by Lemma 1, and \(J\) consists of Jordan blocks corresponding to \(\lambda_i(i = 2, 3, \ldots, N)\). Let \(\bar{U} = [\bar{u}_2, \bar{u}_3, \ldots, \bar{u}_N]^H\), \(\bar{\xi}(t) = (\bar{u}_1^H \otimes I) \theta(t), \quad \bar{\xi}(t) = (\bar{u}_1^H \otimes I) \gamma_n(t), \quad \bar{\xi}(t) = (\bar{u}_1^H \otimes I) \bar{\gamma}_n(t), \quad \bar{\xi}(t) = (\bar{u}_1^H \otimes I) \bar{y}_n(t)\); then, swarm system (18) can be converted into
\[
\begin{align*}
    \hat{\xi}(t) &= (\bar{A}_{11} + \bar{B}_{11} K_1 + \bar{B}_{11} K_2) \hat{\xi}(t) + (\bar{A}_{12}) \bar{\xi}(t) + (\bar{u}_1^H \otimes (\bar{A}_{11} + \bar{B}_{11} K_1)) h(t) - (\bar{u}_1^H \otimes I) \hat{h}(t), \\
    \bar{\xi}(t) &= (\bar{A}_{21} + \bar{B}_{12} K_1 + \bar{B}_{12} K_2) \bar{\xi}(t) + (\bar{A}_{22}) \bar{\xi}(t) + (\bar{u}_1^H \otimes (\bar{A}_{21} + \bar{B}_{12} K_1)) h(t), \\
    \hat{\xi}(t) &= (I_{N-1} \otimes (\bar{A}_{11} + \bar{B}_{11} K_1 + \bar{B}_{11} K_2) + J \otimes \bar{B}_{11} K_3) \hat{\xi}(t) + (I_{N-1} \otimes \bar{A}_{12}) \bar{\xi}(t) + (\bar{u} \otimes (\bar{A}_{11} + \bar{B}_{11} K_1)) h(t) - (\bar{u} \otimes I) \hat{h}(t), \\
    \bar{\xi}(t) &= (I_{N-1} \otimes (\bar{A}_{21} + \bar{B}_{12} K_1 + \bar{B}_{12} K_2) + J \otimes \bar{B}_{12} K_3) \bar{\xi}(t) + (I_{N-1} \otimes \bar{A}_{22}) \bar{\xi}(t) + (\bar{u} \otimes (\bar{A}_{21} + \bar{B}_{12} K_1)) h(t).
\end{align*}
\]

(19)
Lemma 4 (see [26]). Swarm system (15) achieves time-varying output formation \( h(t) \) if and only if

\[
\lim_{t \to \infty} \zeta(t) = 0. \tag{21}
\]

Lemma 4 shows that the two subsystems with states \( \xi(t) \) and \( \zeta(t) \) determine the output consensus and complement output consensus parts of system (18), respectively. From Lemma 2, it is easy to find that only observable components of \((\mathcal{A}_{22}, \mathcal{A}_{12})\) affect the subsystem with state \( \zeta(t) \). Let \( \mathcal{T} \) be a nonsingular matrix, and the observability decomposition of \((\mathcal{A}_{22}, \mathcal{A}_{12})\) is given as follows:

\[
\begin{array}{c}
\mathcal{F} = \begin{bmatrix}
\mathcal{A}_{11} + \mathcal{B}_{11} K_1 & \mathcal{B}_{11} K_2 & \mathcal{F}_1 \\
\mathcal{A}_{11} + \mathcal{B}_{11} K_3 & \mathcal{F}_2 & \mathcal{F}_3 \\
\mathcal{A}_{11} + \mathcal{B}_{11} K_4 & \mathcal{F}_4 & \mathcal{F}_5 \\
\end{bmatrix}
\end{array}
\begin{array}{c}
\mathcal{H} = \begin{bmatrix}
\mathcal{B}_{12} K_2 & \mathcal{B}_{12} K_3 & \mathcal{B}_{12} K_4 \\
\mathcal{B}_{12} K_5 & \mathcal{B}_{12} K_6 & \mathcal{B}_{12} K_7 \\
\mathcal{B}_{12} K_8 & \mathcal{B}_{12} K_9 & \mathcal{B}_{12} K_{10} \\
\end{bmatrix}
\end{array}
\begin{array}{c}
\mathcal{G} = \begin{bmatrix}
\mathcal{A}_1 \\
\mathcal{A}_2 \\
\mathcal{A}_3 \\
\end{bmatrix}
\end{array}
\]

For \( \mathcal{B}_{11} \) and \( \mathcal{B}_{121} \), there exist nonsingular matrices \( \mathcal{T} = \begin{bmatrix} T_1 & T_2 \\ T_2 & T_3 \end{bmatrix} \) and \( \mathcal{T} \) with \( \mathcal{T}^{-1} = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \) such that

\[
\begin{bmatrix}
T_1 T_2 \\
T_2 T_3 \\
\end{bmatrix}
\begin{bmatrix}
\mathcal{B}_{11} \\
\mathcal{B}_{121} \\
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
0 & 0 \\
\end{bmatrix}
\]

Let

\[
x_e(t) = x(t) - x_n(t),
\]

\[
y(t) = y(t) - y_n(t),
\]

\[
q(t) = \mathcal{D}^{-1} \left[ \frac{1 - F(s)}{s + f} \right] \ast \omega(t),
\]

\[
z(t) = x_e(t) - (I_N \otimes B_n) q(t).
\]

Theorem 1. Suppose that the system described by (1), satisfying Assumptions 1–3. If the controller given by (5), (12), and (14) is applied, for any given constant \( \epsilon \) and any initial conditions, one can find \( T \geq T_0 \) that \( \| e(t) \| \leq \epsilon, \ t \geq T \) if and only if the following conditions hold simultaneously:

(i) For \( \forall i \in \{1, 2, \ldots, N\} \) and \( j \in N_i \),

\[
\lim_{t \to \infty} \begin{bmatrix}
\mathcal{A}_{11} + \mathcal{B}_{11} K_1 & \mathcal{F}_1 + \mathcal{B}_{121} K_2 \\
\mathcal{A}_{11} + \mathcal{B}_{11} K_3 & \mathcal{F}_2 \\
\mathcal{A}_{11} + \mathcal{B}_{11} K_4 & \mathcal{F}_3 \\
\end{bmatrix}
\begin{bmatrix}
\hat{h}_i(t) - \hat{h}_j(t) \\
\hat{h}_i(t) - \hat{h}_j(t) \\
\end{bmatrix}
= 0.
\]

(ii) The following \( N - 1 \) matrices are Hurwitz:

\[
\begin{bmatrix}
\tilde{D}_1 & 0 \\
\tilde{D}_2 & \tilde{D}_3 \\
\end{bmatrix}
\begin{bmatrix}
\tilde{E}_1 \\
0 \\
\end{bmatrix}
\]

where \((\tilde{D}_1, \tilde{E}_1)\) is completely observable. Denote \( \tilde{\xi}_i(t) = \tilde{\xi}_i(t) = \begin{bmatrix} \tilde{\xi}_{1o}^T(t), \tilde{\xi}_{2o}^T(t), \ldots, \tilde{\xi}_{N_N}^T(t) \end{bmatrix}^T \), \( \tilde{\zeta}_i(t) = \begin{bmatrix} \tilde{\zeta}_{1o}^T(t), \tilde{\zeta}_{2o}^T(t), \ldots, \tilde{\zeta}_{N_N}^T(t) \end{bmatrix}^T \), \( \tilde{T}_1^{-1} \mathcal{A}_{21} = \begin{bmatrix} \mathcal{F}_1^T, \mathcal{F}_2^T \end{bmatrix} \), \( \tilde{T}^T \mathcal{B}_{12} = \begin{bmatrix} \mathcal{B}_{121}^T, \mathcal{B}_{122}^T \end{bmatrix}, \) and \( \tilde{T}^{-1} \mathcal{B}_{a2} = \begin{bmatrix} \mathcal{B}_{a21}^T, \mathcal{B}_{a22}^T \end{bmatrix} \). Then, system (20) can be transformed into

\[
\begin{bmatrix}
\tilde{T}_1^{-1} \mathcal{A}_{21} & \tilde{A}_{12} \tilde{T} \\
\end{bmatrix}
= \begin{bmatrix}
\tilde{D}_1 & 0 \\
\tilde{D}_2 & \tilde{D}_3 \\
\end{bmatrix}
\begin{bmatrix}
\tilde{E}_1 \\
0 \\
\end{bmatrix}
\]

Proof. From Corollary 1 in [26], one can see that (28) and (29) are sufficient and necessity conditions for nominal system (13) with nominal controller (14) to achieve time-varying output formation \( h(t) \).

Then, \( u_{\text{nom}}(t) = [u_{\text{nom}}^1(t)^T, u_{\text{nom}}^2(t)^T, \ldots, u_{\text{nom}}^N(t)^T]^T \).

And \( u_{\text{nom}}(t) \) can be described as

\[
u_{\text{nom}}(t) = D_j y(t) + D_i h(t).
\]

From (1), (5), and (10), one has

\[
\begin{bmatrix}
\dot{x}(t) = (I_N \otimes (A + B_1 K_1 C + B_2 K_2 C) \otimes L \otimes B_1 K_3 C) x(t) \\
-I_N \otimes B_1 K_2 \otimes (I_N \otimes B_3) h(t) + (I_N \otimes B_2) \omega(t) \ast \mathcal{D}^{-1} (1 - F(s)) \\
\end{bmatrix} \\
y(t) = (I_N \otimes C) x(t).
\]

From (15), (24), (26), and (31), one can obtain the error system given by

\[
\begin{bmatrix}
\dot{x}_e(t) = A_n x_e(t) + (I_N \otimes B_n) \omega(t) \ast \mathcal{D}^{-1} (1 - F(s)) \\
e(t) = C x_e(t)
\end{bmatrix}
\]

where \( A_n = I_N \otimes (A + B_1 K_1 C + B_2 K_2 C) \otimes L \otimes B_1 K_3 C \).

Combining (26), (27), and (32), one can get that
\[ \dot{z}(t) = \dot{x}_c(t) - B_w q(t) \]

\[ = A_w x_c + (I_N \otimes B_w) \omega(t) * \mathcal{L}^{-1} (1 - F(s)) - (I_N \otimes B_w) \mathcal{L}^{-1} (sq(s)) \]

\[ = A_w z(t) + A_n (I_N \otimes B_w) q(t) + (I_N \otimes B_w) \mathcal{L}^{-1} (\omega(s) (1 - F(s))) - (I_N \otimes B_w) \mathcal{L}^{-1} \left( \frac{s}{s + f} \omega(s) (1 - F(s)) \right) \]

\[ = A_w z(t) + A_n (I_N \otimes B_w) q(t) + \mathcal{L}^{-1} \left( \frac{f}{s + f} \omega(s) (1 - F(s)) \right) \]

\[ = A_w z(t) + A_n (I_N \otimes B_w) q(t) + f (I_N \otimes B_w) q(t) \]

\[ = A_w z(t) + (A_n + f I_{nn}) (I_N \otimes B_w) q(t). \]

Moreover, from (5), (10), (31), and Assumption 3, one gets

\[ \| \mu(t) \|_\infty = \| \mu_{\text{nom}}(t) + \mu_{\text{rob}}(t) \|_\infty \]

\[ \leq \| \mu_{\text{nom}}(t) \|_\infty + \| F(s) \| N_0 \| (s)L(s) \omega(t) \|_\infty \]

\[ \leq \xi_x \| x(t) \|_\infty + \xi_{x_2} \| x(t) \|_{\infty}^q + \xi_{u_2} \| u(t) \|_\infty + \xi_{u} \| u(t) \|_\infty \]

\[ \leq \xi_x \| x(t) \|_\infty + \xi_{x_2} \| x(t) \|_{\infty}^q + \xi_{u_2} \| u(t) \|_\infty + \xi_{u} \| u(t) \|_\infty \]

\[ \tag{34} \]

Combining (34) and Assumption 3, one can obtain that

\[ \| \mu(t) \|_\infty \leq \zeta_x \| x(t) \|_{\infty}^q + \zeta_u, \]

where \( \zeta_x \) and \( \zeta_u \) are positive constants.

From Assumption 2, (24), and (35), positive constants \( \zeta \) and \( \mu \) can be found satisfying

\[ \| \omega(t) \|_\infty \leq \zeta \| x_c(t) \|_{\infty}^q + \mu, \]

where \( \eta = \max(\eta_1, \eta_2) \).

From (26), (27), and (35), positive constants \( \zeta_q \) and \( \mu_q \) can be found such that

\[ \| q(t) \|_\infty \leq \| q(t) \|_{\infty} \]

\[ \leq \frac{1}{\beta + f} \left( \| \omega(t) \|_{\infty} \right) \]

\[ \leq \frac{\zeta_x}{f} (\| x(t) \|_{\infty} + \| q(t) \|_{\infty} + \mu_q), \]

\[ \tag{37} \]

where \( \| q(t) \|_{\infty} = \max_{1 \leq i \leq r} \| q_i(t) \| \),

\[ \| q(t) \|_{\infty} = \max_{1 \leq i \leq r} \sup_{t \in t_{i-1}} \| q_i(t) \| \]

\[ \tag{38} \]

If \( f \) is large enough and satisfies \( \zeta_q \| q \|_{\infty} \leq (f/2) \), from (37), one can get

\[ \| q(t) \|_{\infty} \leq \| q(t) \|_{\infty} \leq \frac{\zeta_x}{f} (\| z(t) \|_{\infty} + \mu_q). \]

\[ \tag{39} \]

Choose the Lyapunov function candidate

\[ V(t) = z^T(t) P z(t), \]

\[ \tag{40} \]

where \( P \) is a symmetric positive definite constant matrix and satisfies

\[ PA_n + A_n^T P = -Q, \quad Q = Q^T > 0. \]

Then, the derivative of \( V \) is as follows:

\[ \dot{V}(t) = z^T(t) P \dot{z}(t) + z^T(t) P \dot{z}(t) \]

\[ = -z^T(t) Q z(t) + 2 z^T(t) P (A_n + f I_{nn}) (I_N \otimes B_w) q(t) \]

\[ \leq -z^T(t) Q z(t) + \zeta_z \| z(t) \|_2 \| q(t) \|_2 \]

\[ \leq -\lambda_q \| z(t) \|_2^2 + \zeta_z \| z(t) \|_2 \frac{2 \zeta_q \sqrt{n}}{f} \left( \| z(t) \|_{\infty} + \mu_q \right) \]

\[ \leq -\lambda_q \| z(t) \|_2 \left( \| z(t) \|_2 - \frac{\zeta_q \sqrt{n}}{\lambda_q} \right) \left( \| z(t) \|_{\infty} + \mu_q \right). \]

\[ \tag{43} \]

If \( f \) is large enough to satisfy

\[ \| z(t) \|_{\infty} \leq \frac{\lambda_q \sqrt{f}}{2 \sqrt{\zeta_q}} - \mu_q, \]

then one can obtain

\[ \dot{V} \leq -\lambda_q \| z(t) \|_2 \left( \| z(t) \|_2 - \frac{1}{\sqrt{f}} \right) \]

\[ \leq -\lambda_q \| z(t) \|_2 \sqrt{\frac{V}{\lambda_{\text{max}}(P)} - \frac{1}{\sqrt{f}}}. \]

From (45), one can get that if \( V(z(0)) \leq (\lambda_{\text{max}}(P) / f) \), then \( V(z(t)) \leq (\lambda_{\text{max}}(P) / f) \); hence, \( \| z(t) \|_2 \leq \sqrt{(\lambda_{\text{max}}(P)) / (\lambda_{\text{min}}(P))} \). Else, if \( V(z(0)) > (\lambda_{\text{max}}(P) / f) \), then \( V(z(t)) \leq V(z(0)) \) so that \( \| z(t) \|_2 \leq \sqrt{(V(z(0))) / (\lambda_{\text{min}}(P))} \). That means \( z(t) \) is bounded.
Because (39) and $z(t)$ are bounded, it follows that if $f$ is sufficiently large, then
\[ \|q(t)\|_{\infty} \leq \frac{\lambda}{f} \]  \hspace{1cm} (46)

where $\lambda$ is a positive constant.

As a result, $x_e(t)$ is bounded and converges for any initial conditions. And for any given positive constant $\varepsilon$,
\[ \|e(t)\|_2 \leq \varepsilon, \forall t \geq T, \]  \hspace{1cm} (47)

where $T$ is a positive constant depending on initial conditions. Therefore, the time-varying output formation is achieved. \hfill \square

Remark 3. From equation (28), we can see that the condition is very conservative and restrictive. Only a few formations $h(t)$ can satisfy the condition. The contribution of this paper is to find sufficient and necessary conditions to achieve time-varying formation for general high-order linear systems. Especially, when the formation $h(t)$ can be described as sin function, cos function, or exponential function, it is very likely to satisfy condition (28). Moreover, the trial-and-error method can be used to find $K_1$, $C$ and $T$ can be adjusted to find proper $K_1$.

Remark 4. From controllers (11) and (14), one can see that only gain matrices $K_1$, $K_2$, and $K_3$ and positive constants $f$ and $d$ are required. $K_1$ and $K_2$ can be found based on the pole place method. According to He and Wang [34], iterative linear matrix inequality algorithm can be applied to find $K_3$. $d$ is the relative degree of the elements in $N_0^1(s)D_0(s)$. The value of $f$ can be chosen by using the trial-and-error method. In general, the proposed controller can be easily applied to practical situations.

4. Numerical Simulations

In this section, a numerical simulation is presented to illustrate the effectiveness of the proposed control method. The designing process of nominal controller (12) is presented for swarm system (13) to achieve time-varying output formation.

Step 1: solving feasible condition (28) for $K_1$

Step 2: choosing $K_2$ to assign the eigenvalues of $A + BK_1C + BK_2C$ at desired locations in the complex plane

Step 3: designing $K_3$ to make condition (29) satisfied

Let
\[ A_0 = \begin{bmatrix} \overline{A}_{11} + \overline{B}_{11}K_1 + \overline{B}_{12}K_2 & \overline{B}_1 \\ \overline{B}_1 + \overline{B}_{121}K_1 + \overline{B}_{122}K_2 & \overline{B}_2 \end{bmatrix}, \]
\[ B_0 = \begin{bmatrix} \overline{B}_{11} \\ \overline{B}_{121} \end{bmatrix}, \]
\[ C_0 = \begin{bmatrix} I \\ 0 \end{bmatrix}. \]  \hspace{1cm} (48)

Then, $\Gamma_i (i = 2, 3, \ldots, N)$ in condition (ii) can be re-written as
\[ \Gamma_i = A_0 + \lambda_iB_0K_iC_0 \]  \hspace{1cm} (49)

From (49), one can see that if and only if $K_3$ can stabilize subsystems $(A_0, \lambda_iB_0, C_0)$ through the static output feedback (SOF), then $\Gamma_i$ are Hurwitz. To find the gain matrix $K_3$, an improved iterative linear matrix inequality (ILMI) algorithm [34] is employed which can solve the SOF problem without introducing any additional variables. And the dimensions of the LMIs need not be increased.

Time-varying output formation control for a fifth-order system with eight agents is considered. The directed interaction topology is shown in Figure 1, and its adjacency matrix is assumed to be 0-1 matrix. The dynamics of each agent described by (1) are

\[ x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \\ x_{i4} \\ x_{i5} \end{bmatrix}, \]
\[ A = \begin{bmatrix} 2 & -2 & 0 & -4 & 1 \\ 0 & 0 & -2 & 0 \\ -2 & 1 & -1 & 2 & -1 \end{bmatrix}, \]
\[ B_1 = \begin{bmatrix} -3 \\ 4 \\ 0 \\ 3 \end{bmatrix}, \]
\[ B_w = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1.2 & 0.5 & 0.8 & -0.3 \\ 3.25 & -2.5 & 2 & 1.75 \end{bmatrix}, \]
\[ C = \begin{bmatrix} 2 & -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}. \]

Choose
\[ \overline{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \]  \hspace{1cm} (51)
Then, we can get

\[
T = \begin{bmatrix}
2 & -1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
x_1(0) = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

\[
x_2(0) = \begin{bmatrix}
0.2 \\
1.1 \\
0.5 \\
0 \n\end{bmatrix}
\]

\[
x_4(0) = \begin{bmatrix}
0.9 \\
-3 \\
2 \\
\end{bmatrix}
\]

\[
A_{11} = \begin{bmatrix}
-1 & 1 & -1 \\
1 & 0 & 2 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

\[
A_{12} = \begin{bmatrix}
2 & 0 \\
1 & 0 \\
\end{bmatrix}
\]

\[
A_{21} = \begin{bmatrix}
0 & 0 & 0 \\
-1 & 0 & -1 \\
\end{bmatrix}
\]

\[
A_{22} = \begin{bmatrix}
-2 & 0 \\
2 & -1 \\
\end{bmatrix}
\]

\[
B_{11} = \begin{bmatrix}
1 \\
1 \\
0 \\
\end{bmatrix}
\]

\[
B_{12} = \begin{bmatrix}
0 \\
3 \\
\end{bmatrix}
\]

(52)

\[
\begin{align*}
\bar{T} & = T^{-1} = I = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}. \\
\end{align*}
\]

(53)

\[
x_3(0) = \begin{bmatrix}
-1.5 \\
-1.5 \\
1 \\
\end{bmatrix}
\]

\[
x_5(0) = \begin{bmatrix}
3 \\
7 \\
-2.6 \\
\end{bmatrix}
\]

\[
\bar{D}_1 = -2, \bar{D}_2 = 2, \bar{D}_3 = -1, \bar{E}_1 = \begin{bmatrix}
2 \\
1 \\
1 \\
\end{bmatrix}
\]

\[
\bar{F}_1 = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

\[
\bar{F}_2 = \begin{bmatrix}
-1 \\
1 \\
1 \\
\end{bmatrix}
\]

\[
\bar{B}_{121} = 0, \bar{B}_{123} = 3.
\]

(54)

Obviously, \((A, B)\) is stabilizable, and \((\bar{A}_{22}, \bar{A}_{12})\) is not observable. The nonsingular matrix \(\bar{T}\) is selected as

\[
\bar{T} = T^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. 
\]

(55)

It can be obtained that

\[
\bar{D}_1 = -2, \bar{D}_2 = 2, \bar{D}_3 = -1, \bar{E}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},
\]

\[
\bar{F}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},
\]

\[
\bar{F}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},
\]

\[
\bar{B}_{121} = 0, \bar{B}_{123} = 3.
\]

According to the designing process of protocol (5), choose \(K_1\) to satisfy condition (i) in Theorem 1. \(K_1\) can be chosen as \(K_1 = [0, 0, 0]\). Choose \(K_2 = [-1.2, -3.3, 14.1]\) to assign the eigenvalue of \(A + B_1K_2C + B_2K_2C\) at \(-1, -2.0773, -1.7114 + 3.5863j, -1.7114 - 3.5863j\), and -2 with \(j^2 = -1\). Finally, the gain matrix \(K_3\) is chosen as \(K_3 = [-1.4457, -2.1366, 0.4930]\).

The initial state is

\[
\begin{bmatrix}
1 \\
1.1 \\
-1.3 \\
0.5 \\
-0.3 \\
\end{bmatrix}
\]
The eight agents need to achieve predefined time-varying output formations which are defined as $h_i(t) = [10\sin(t + ((i-1)\pi/4))10\cos(t + ((i-1)\pi/4)) - 10\sin(t + ((i-1)\pi/4))]^T$ $(i=1,2,\ldots,8)$. The nonlinear uncertainty is denoted as $w_i(t) = [0.3\sin(t + ((i-1)\pi/4))0.4\cos(t + ((i-1)\pi/4)) + 0.6x_i10x_i20x_i3]^T$ $(i=1,2,\ldots,8)$ which is related to external disturbances and agents’ states. The robust compensator parameters are $f = 60$ and $d = 2$.

Considering nonlinear uncertainty, Figures 2 and 3 show the output formation snapshot of eight agents with the
Figure 4: Output error of eight agents without the robust compensator.

Figure 5: Continued.
Figure 5: Output error of eight agents with the robust compensator.

Figure 6: Continued.
nominal controller and robust controller, respectively. The outputs of agents are denoted by the plus, triangle, diamond, square, asterisk, x-mark, circle, and point, and the output formation reference function is denoted by the pentagram. From the comparison of two figures, one can see that the robust controller has better control effect under nonlinear uncertainties. Figure 3 achieves the specified formation in ten seconds while Figure 2 does not.

Figures 4 and 5 show the output formation error of eight agents with the nominal controller and robust controller, respectively. From Figure 4, we can see that output errors converge. Because of the existence of nonlinear uncertainty $\omega_i(t)$, output errors can only converge to the small neighborhood of zero. Compared with Figure 4, output errors in Figure 5 can also converge to the small neighborhood of zero. However, the convergence speed is faster, and the convergence domain is smaller in Figure 5. From the comparison of two figures, one can see that the robust controller can restrain the nonlinear uncertainties well. Meanwhile, the formation errors can be made as small as desired with arbitrarily specified convergence rate.

Figure 6 shows the input signal of eight agents including $\|u_{i,\text{norm}}(t)\|_2$ and $\|u_{i,\text{rob}}(t)\|_2$. From Figure 6, we can see that two controllers tend to be stable. The robust compensator outputs change over time and are similar to nonlinear uncertainty $\omega_i(t)$ which also verifies the effectiveness of the proposed controller. Meanwhile, input constrain may be encountered considering actuator saturation in practical situations. As a result, it is meaningful to study the influence of input constrain on multiagent systems in future work.

### 5. Conclusion

Time-varying output formation control problems for multiagent systems with directed interaction topologies and nonlinear uncertainties are studied. The nonlinear uncertainties are related to external disturbances, parameter uncertainties, nonlinearities, and couplings. By extending the dimensions of the observation matrix, the formation control problem is transformed into a consensus problem and then into a stability problem of the subsystem. A novel robust controller is proposed which includes a nominal controller and a robust compensator. The nominal controller based on relative outputs of neighbour agents is designed to achieve expected performance for the nominal system. And the robust compensator is to restrain nonlinear uncertainties. Based on the proposed controller, the output formation error can be as small as desired. Future work will focus on swarm systems with switching topologies, input constrain, and time-delay group formation control problems.

Some related works can be found in [35]. Compared with previous works, different problems are addressed using the similar method. First, time-varying formation is studied in this paper. However, it is time-invariant formation that was studied. Second, most of the formulas are different. And more assumptions and remarks are added to compare with other works. Third, the proof of Theorem 1 is different. The process of proof in this paper is more rigorous and complete.

### Data Availability

The data used in this paper have been given in the simulation section.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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