Finite-Time Stability of Fractional-Order Time-Varying Delays Gene Regulatory Networks with Structured Uncertainties and Controllers

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1. Introduction

Genetic regulatory networks (GRNs), which describe the interaction functions in gene expressions between DNAs, RNAs, proteins, and small molecules in an organism, are fundamental and important biological networks. The analysis and control of GRNs involve two aspects: first, understanding the widespread phenomena in living organisms and providing potential routes to prolong life span, cure cancer and diabetes, and so on; second, potential application of GRNs in the development of related disciplines, such as synthetic biology, network medicine, and personalized medicine [1–5].

With the development of sequencing technology, more and more genes and their regulatory sites are discovered. The structures and functions of a large number of genes are confirmed through experimental techniques, and even the regulatory mechanisms of the gene expressions controlled by some proteins are also identified [2]. Some qualitative models were proposed for distinguishing these regulatory mechanisms, such as directed graphs, Boolean networks, generalized logical networks, and rule-based formalisms [2]. However, it is difficult to know the biochemical reaction mechanisms’ underlying regulatory interactions when qualitatively handling great quantities of experimental data. Therefore, studying GRNs needs more accurately quantitative model. Based on experimental data, some simple genetic networks have been constructed, for example, genetic repressilator network [5], negative feedback GRNs [6], and genetic switch network [7]. The results in these experiments show that quantitative mathematical modeling approaches on dynamical systems have been great tools in...
providing insights on the mechanisms underlying the structure and the behaviors of GRNs [8–12].

Using integer-order differential equation to model GRNs is a classical method. However, the fractional-order differential equations are more suitable for modeling the gene regulatory mechanism. Ji et al. [13] applied the particle swarm optimization technique in modeling the fractional-order GRNs with eight real target genes. The experimental results confirmed that the fractional-order model has achieved much lower fitting error on test data than integer-order model. Other studies also revealed that the fractional-order systems have excellent performance in describing the memory and hereditary properties of various processes in GRNs, which could be far better than the integer-order ones [8–11].

Due to slow biochemical processes such as gene transcription, translation, and transportation (the synthesis of mRNAs and proteins at nucleus and cytoplasm, respectively, in eukaryotic cells), time delays are omnipresent in GRNs [14]. Many nonlinear differential equations with time delays have been proposed to model general GRNs, and the important role of time delays in dynamics of GRNs is now widely accepted [15–18]. Actually, time delays often degrade the system performance or destabilize the system [1, 19, 20]; even GRN models without time delay may generate wrong predictions [21]. As time delays often change with time and their precise measurement is difficult in real GRNs, the dynamics of fractional-order linear and nonlinear systems with time-varying delays has attracted increasing interest, and the results show that it is naturally of better practical significance than those with constant delays [21–25].

In addition, in order to avoid undesirable states associated with disease, the control of GRNs is often regarded as developing therapeutic intervention strategies for some diseases [26, 27]. And many literatures focus on the research of control in the dynamic system [28–33]. In [32], the authors obtained some stabilization results for neural networks with leakage delay by designing state-feedback controller. Ebihara et al. [33] discovered that exact robust control is indeed attained for discrete-time linear systems by designing periodically time-varying memory state-feedback controller. Therefore, it is necessary to consider the controller for the DFGRNs.

Since the modeling of GRNs is underlined with the real-world gene expression time-series data, some limitations of the current experimental techniques in GRNs make the modeling errors and parameter fluctuations unavoidable. Moreover, some point out that the system parameters identified with the experimental data may construct an unknown but bounded time-varying function, and this unknown nature is referred to as the structural uncertainty or the parametric uncertainty, also known as variation or fluctuation [34]. As is known, the structural uncertainties in GRNs may lead to the poor performance or even instability in real genetic networks [28, 34–37]. In [28], the authors studied the robust stabilization and state-feedback controller design for a class of integer-order GRNs with time-varying delays (DGRNs) and structured uncertainties and established some delay-dependent stability results by using some matrix techniques. Therefore, taking into account the structural uncertainties while investigating the dynamical behaviors of DFGRNs is essential.

Since the expression of gene and mRNA-translated protein is accomplished in a much relatively short period, in recent decades, some scholars have paid more attention to the finite-time stability of GRNs [4, 38]. For example, Wu et al. [4] investigated the finite-time stability associated with a class of integer-order GRNs by designing adaptive controllers. Wang et al. [38] established some new sufficient conditions of the finite-time stability for a class of integer-order uncertain GRNs with time-varying delays. Lazarević [22] investigated the finite-time stability for fractional-order nonlinear differential equation with time-varying delays by using generalized Gronwall inequality and the classical Bellman–Gronwall inequality, respectively. Phat and Thanh [23] established some new sufficient conditions of robust finite-time stability for a class of nonlinear fractional-order differential systems with time-varying delays. Wang et al. [39] considered a class of nonlinear fractional-order systems with constant delays and studied the existence and uniqueness of the solution for this kind of systems by using relevant properties of the fractional derivative.

However, the discussions on the existence and uniqueness of the solutions and the finite-time stability results for the fractional-order uncertain GRNs with time-varying delays and controllers seem rare.

From above discussions, we focus on the existence and uniqueness of the solution and the finite-time stability for a class of DFGRNs with structured uncertainties and controllers. The remainder of this paper is organized as follows. In Section 2, we give the model description, some definitions, and related properties on fractional calculus. In Section 3, we discuss the existence and uniqueness of the solution and give some sufficient criteria on the finite-time stability for the DFGRNs. In Section 4, we perform some numerical simulations, which support our findings. In Section 5, we briefly review and summarize the main results.

2. Problem Description and Preliminaries

For any vector \( x(t) \in \mathbb{R}^n \) and matrix \( \overline{A} = (\overline{a}_{ij})_{n \times n} \in \mathbb{R}^{n \times n} \), denote

\[
\|x(t)\| = \sum_{i=1}^{n} |x_i(t)|, \\
\|\overline{A}\| = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |\overline{a}_{ij}|.
\]

Let \( \sigma(\cdot) \) be the largest singular value of matrix,
\[ \eta_1 = \max[\|A\| + \mu_1, \|C\| + \mu_4], \]
\[ \eta_2 = \max[\|W\| + \mu_2L_1, \|D\| + \mu_3], \]
\[ \eta_3 = \max[\|K\| + \mu_2L_2, \|H\| + \mu_3], \]
\[ \eta_4 = \max[\sigma(A) + \mu_1 + \sigma(D) + \mu_2, (\sigma(W) + \mu_2)L_1 + \sigma(C) + \mu_4], \]
\[ \eta_5 = \max[\sigma(A) + \mu_1 + (\sigma(Q_1) + \mu_7)\sigma(y_1) + \sigma(D) + \mu_2, (\sigma(W) + \mu_2)L_1 + \sigma(C) + \mu_4 + (\sigma(Q_2) + \mu_9)\sigma(y_2)], \]
\[ \eta_6 = \max[\sigma(Q_1) + \mu_7, \sigma(Q_2) + \mu_8], \]
\[ \zeta_1 = \eta_4 + (\sigma(K) + \mu_2)L_2 + \sigma(H) + \mu_6, \]
\[ \zeta_2 = \|B\| + (\sigma(W) + \mu_2)\|F(0)\| + (\sigma(K) + \mu_3)\|G(0)\|, \]
\[ \zeta_3 = \eta_5 + (\sigma(K) + \mu_2)L_2 + (\sigma(Q_1) + \mu_7)\sigma(y_3) + (\sigma(H) + \mu_6) + (\sigma(Q_2) + \mu_9)\sigma(y_4), \]

where \(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9, L_1, L_2\) are positive constants that satisfy the later assumptions (I) and (II), respectively.

We will focus on a class of DFGRNs with structured uncertainties and controllers, which is established as follows:

\[
\begin{align*}
C^D p(t) &= -(A + \Delta A(t))m(t) + (W + \Delta W(t))F(p(t)) \\
&\quad + (K + \Delta K(t))G(p(t - \tau_1(t))) + B + (Q_1 + \Delta Q_1(t))u_1(t), \\
C^D m(t) &= -(C + \Delta C(t))p(t) + (D + \Delta D(t))m(t) \\
&\quad + (H + \Delta H(t))m(t - \tau_2(t)) + (Q_2 + \Delta Q_2(t))u_2(t),
\end{align*}
\]

where

\[
\begin{align*}
m(t) &= [m_1(t), m_2(t), \ldots, m_n(t)]^T, \\
p(t) &= [p_1(t), p_2(t), \ldots, p_n(t)]^T, \\
F(p(t)) &= [f_1(p_1(t)), f_2(p_2(t)), \ldots, f_n(p_n(t))]^T, \\
B &= [B_1, B_2, \ldots, B_n]^T, \\
A &= \text{diag}[a_1, a_2, \ldots, a_n], \\
C &= \text{diag}[c_1, c_2, \ldots, c_n], \\
D &= \text{diag}[d_1, d_2, \ldots, d_n], \\
H &= \text{diag}[e_1, e_2, \ldots, e_n], \\
G(p(t - \tau_1(t))) &= [g_1(p_1(t - \tau_1(t))), g_2(p_2(t - \tau_1(t))), \ldots, g_n(p_n(t - \tau_1(t)))]^T,
\end{align*}
\]

in which \(C^D\) represents Caputo’s fractional derivative and \(q \in (0, 1)\), \(m_i(t), p_i(t) \in \mathbb{R}\) are the concentrations of mRNA and protein of the \(i\)th node, respectively. The parameters \(a_i > 0\) and \(c_i > 0\) are the decay rates of mRNA and protein, respectively; \(d_i > 0\) are the translation rates; \(e_i \geq 0\) are the translation rates. Both \(f_j(p_j(t))\) and \(g_j(p_j(t - \tau_1(t)))\) represent the feedback regulation of the protein on the transcription. Generally, each one of the two functions is a nonlinear function but has a form of monotonicity with its variable. As a monotonic increasing or decreasing regulatory function, \(f_j\) and \(g_j\) are usually of the Michaelis–Menten or Hill forms \([21]\). \(B_i = \sum_{j \in I_i} b_{ij} + \sum_{j \in I_{\overline{i}}} \overline{b}_{ij}\), where \(b_{ij}\) and \(\overline{b}_{ij}\) are bounded constants which are, respectively, the dimensionless transcriptional rates of transcription factor \(j\) to \(i\) at time \(t\) and \(t - \tau_1(t)\), and \(I_i, I_{\overline{i}}\), respectively, are the set of all the \(j\) where the transcription factor \(j\) is a repressor of gene \(i\) at time \(t\) and \(t - \tau_1(t)\). \(W = (w_{ij}) \in \mathbb{R}^{n \times n}, K = (k_{ij}) \in \mathbb{R}^{n \times n}\) are the coupling matrices of the gene network, which are defined as follows:
Complexity

The transcriptional delay $\tau_1(t)$ and translational delay $\tau_2(t)$ are bounded continuous functions on $R$ with $0 \leq \tau_i(t) \leq \tau^*$ $(i = 1, 2)$; here $\tau^*$ is a positive constant. $u_1(t) = (u_{11}(t), u_{12}(t), \ldots, u_{1n}(t))^T$, $u_2(t) = (u_{21}(t), u_{22}(t), \ldots, u_{2n}(t))^T$ are controller vectors, $Q_i(i = 1, 2)$ are $n$ dimensions coefficient matrices. $\Delta A(t)$, $\Delta D(t)$, $\Delta C(t)$, $\Delta W(t)$, $\Delta K(t)$, $\Delta H(t)$, $\Delta Q_1(t)$, $\Delta Q_2(t)$ are norm-bounded unknown matrices with time-varying structured uncertainties.

The initial conditions for DFGRN (3) are as follows:

\[
\begin{cases}
   m(\theta) = \phi_1(\theta), & \theta \in [-\tau^*, 0], \\
   p(\theta) = \phi_2(\theta), & \theta \in [-\tau^*, 0],
\end{cases}
\]

where $\phi_1(t) \in C([-\tau^*, 0], R^n)$ $(i = 1, 2)$ is the given initial function with $\|\phi_1\|_c = \sup_{-\tau^* \leq \theta \leq 0} \|\phi_1(\theta)\|$ $(i = 1, 2)$ and $\phi_2 = \|\phi_1\|_c + \|\phi_2\|_c$.

(i) Assumption (I): the norm-bounded unknown matrices satisfy the following inequalities:

\[
\begin{align*}
\|\Delta A(t)\| & \leq \mu_1, \\
\|\Delta W(t)\| & \leq \mu_2, \\
\|\Delta K(t)\| & \leq \mu_3, \\
\|\Delta C(t)\| & \leq \mu_4, \\
\|\Delta D(t)\| & \leq \mu_5, \\
\|\Delta H(t)\| & \leq \mu_6, \\
\|\Delta Q_1(t)\| & \leq \mu_7, \\
\|\Delta Q_2(t)\| & \leq \mu_8,
\end{align*}
\]

where $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8$ are positive constants.

(ii) Assumption (II): the functions $F, G$ satisfy the following inequalities:

\[
\begin{align*}
\|F(x) - F(y)\| & \leq L_1\|x - y\|, \\
\|G(x) - G(y)\| & \leq L_2\|x - y\|, & x, y \in R^n,
\end{align*}
\]

where $L_1, L_2$ are positive constants.

Next, we give some definitions and lemmas.

**Definition 1** (see [40]). The fractional integral of order $q$ for a function $f(t)$ is defined as

\[
a_t^q I_a^q f(t) = \frac{1}{\Gamma(q)} \int_a^t (t - r)^{q-1} f(r)dr,
\]

where $t \geq a, a \in R, q > 0$. The gamma function $\Gamma(q)$ is defined by the integral $\Gamma(q) = \int_0^\infty t^{q-1} e^{-t}dt$.

**Definition 2** (see [40]). Caputo’s fractional derivative of order $q$ for a function $f$ is defined by

\[
cD^q_a f(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{1}{(t-s)^{n-q}} f^{(n)}(s)ds,
\]

where $t \geq a$ and $n$ is a positive integer such that $n - 1 < q < n$.

**Definition 3** (see [40]). The Riemann–Liouville fractional derivative of order $q$ for a function $f$ is defined as

\[
rD^q_a f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{ds^n} \int_a^t (t-s)^{n-q-1} f(s)ds,
\]

where $t \geq a$ and $n$ is a positive integer such that $n - 1 < q < n$.

For convenience, we choose the notation $I_q^a = qI_q^a$, $C\!D^q_a = C\!D^q_a$, $R\!L\!D^q_a = R\!L\!D^q_a$.

**Definition 4.** A mild solution of DFGRN (3) with initial condition (6) is a vector $(m(t), p(t))^T$ composed of continuous functions

\[
m(t), p(t): [-\tau^*, \alpha) \longrightarrow R^n,
\]

satisfying

\[
\begin{align*}
m(t) &= m(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(-\Delta A(s) m(s) + (\Delta W(s) F(p(s)) + (K + \Delta K(s)) G(p(s - \tau_1(s))) + B + \Delta Q_1(s) u_1(s) \right) ds, \\
p(t) &= p(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(-\Delta C(s) p(s) + (D + \Delta D(s)) m(s) + (H + \Delta H(s)) m(s - \tau_2(s)) + (Q_2 + \Delta Q_2(s) u_2(s) \right) ds, \\
m(\theta) &= \phi_1(\theta), & \theta \in [-\tau^*, 0], \\
p(\theta) &= \phi_2(\theta), & \theta \in [-\tau^*, 0].
\end{align*}
\]
Definition 5 (see [22]). The system given by (3) (when $Q_i = 0, \Delta Q_i = 0, i = 1, 2$) satisfying the initial condition (6) is finite-time stable with respect to $\{\delta, \varepsilon, t_0, J\}$, $\delta < \varepsilon$ if and only if $\phi_0 < \delta$ imply $\|m(t)\| + \|p(t)\| < \varepsilon$, $\forall t \in J, J \subset R$.

Definition 6 (see [22]). The system given by (3) satisfying the initial condition (6) is finite-time stable with respect to $\{\delta, \varepsilon, \alpha_1, t_0, J\}$, $\delta < \varepsilon$ if and only if $\phi_0 < \delta$ and $\|u_1(t)\| + \|u_2(t)\| < \alpha_1$, $\forall t \in J$. Simply $\|m(t)\| + \|p(t)\| < \varepsilon$, $\forall t \in J, J \subset R$, where $\alpha_1$ is a positive constant.

Lemma 1 (see [40]). If $f(t) \in C^\infty([0, \infty))$ and $n - 1 < \alpha < n \in \mathbb{Z}^+$, then

(i) $I_t^{\alpha}[C^1 D_t^\alpha f(t)] = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0)(t^k/k!)$.
(ii) $R^L D_t^\alpha[I_t^{\alpha} f(t)] = f(t)$.
(iii) $R^L D_t^\alpha I_t^{\alpha} f(t) = C D_t^\alpha f(t) + \sum_{k=0}^{n-1} \frac{(t^k-k)!}{\Gamma(k+1-q)} f^{(k)}(0)$.

Lemma 2 (see [41]). Suppose $\beta > 0$; if $0 \leq t < T$ (some $T \leq +\infty$), $a(t)$ is a locally integrable nonnegative function, $v(t)$ is a nonnegative and nondecreasing continuous function, $v(t) \leq M$ (constant), and $u(t)$ is a nonnegative and locally integrable function with $u(t) \leq a(t) + v(t) \int_0^t (t-s)^{\beta-1} a(s)ds$, then

$$u(t) \leq a(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(v(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n-1} a(s) \, ds, \quad \forall 0 \leq t \leq T.$$  

(14)

In addition, if $a(t)$ is a nondecreasing function, then

$$u(t) \leq a(t) E_{\beta}(v(t)\Gamma(\beta)^{\beta}), \quad \text{where } E_{\beta}(z) = \sum_{k=0}^{\infty} (z^{k}/\Gamma(k\beta + 1)).$$

3. Main Results

3.1. The Existence and Uniqueness of the Mild Solution of DFGRN

Theorem 1. Continuously differentiable functions $m(t), p(t) : [-r^*, T] \rightarrow \mathbb{R}^n (T < +\infty)$ form a mild solution $(m(t), p(t))^T$ to DFGRN (3) with initial condition (6) if and only if

$$\begin{align*}
m(t) &= \phi_1(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (-A + \Delta A(s)m(s)) + (W + \Delta W(s))F(p(s)) + (K + \Delta K(s)G(p(s-t_1(s))) + B + (Q_1 + \Delta Q_1(s)u_1(s))ds, \quad \text{for } t \in [0, T],
p(t) &= \phi_2(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (-C + \Delta C(s)p(s)) + (D + \Delta D(s)m(s)) + (H + \Delta H(s)m(s-t_2(s))) + (Q_2 + \Delta Q_2(s)u_2(s))ds, \quad \text{for } t \in [0, T],
m(t) &= \phi_1(t), \quad \text{for } t \in [-r^*, 0],
p(t) &= \phi_2(t), \quad \text{for } t \in [-r^*, 0].
\end{align*}$$

(15)

Proof. We firstly give the sufficient condition of the existence of the mild solution to DFGRN (3).

When $-r^* \leq t \leq 0$, $(m(t), p(t))^T = (\phi_1(t), \phi_2(t))^T$ is obvious. For $0 \leq t \leq T$, according to (15), applying $R^L D_t^\alpha$ and property (ii) of Lemma 1, we obtain

$$\begin{align*}
R^L D_t^{\alpha} m(t) &= \phi_1(0) \frac{t^{-q}}{\Gamma(1-q)} - (A + \Delta A(t))m(t) + (W + \Delta W(t))F(p(t))
+ (K + \Delta K(t))G(p(t-t_1(t))) + B + (Q_1 + \Delta Q_1(t))u_1(t),
\quad \text{for } t \in [0, T],
R^L D_t^{\alpha} p(t) &= \phi_2(0) \frac{t^{-q}}{\Gamma(1-q)} - (C + \Delta C(t))p(t) + (D + \Delta D(t)m(t)
+ (H + \Delta H(t)m(t-t_2(t))) + (Q_2 + \Delta Q_2(t)u_2(t).)
\end{align*}$$

(16)
According to the property (iii) of Lemma 1 and $0 < q < 1$, we get

\[
\begin{align*}
RL_D_t^q m(t) &= C D_t^q m(t) + \phi_1(0) \frac{t^{-q}}{\Gamma(1-q)}, \\
RL_D_t^q p(t) &= C D_t^q p(t) + \phi_2(0) \frac{t^{-q}}{\Gamma(1-q)}.
\end{align*}
\]

(17)

From (16) and (17), we have

\[
\begin{align*}
C D_t^q m(t) &= -(A + \Delta A(t))m(t) + (W + \Delta W(t))F(p(t)) \\
&+ (K + \Delta K(t))G(p(t - \tau_1(t))) + B + (Q_1 + \Delta Q_1(t))u_1(t), \\
C D_t^q p(t) &= -(C + \Delta C(t))p(t) + (D + \Delta D(t))m(t) \\
&+ (H + \Delta H(t))m(t - \tau_2(t)) + (Q_2 + \Delta Q_2(t))u_2(t).
\end{align*}
\]

(18)

We secondly give the necessary condition of the existence of the mild solution to DFGRN (3).

When $t \in [-r^*, 0]$, the solution of DFGRN (3) is

\[
m(t) = \phi_1(t), \\
p(t) = \phi_2(t),
\]

(19)

If $0 \leq t \leq T$, from DFGRN (3), we have

\[
\begin{align*}
I_t^\alpha [C D_t^q m(t)] &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left((A + \Delta A(s))m(s) + (W + \Delta W(s))F(p(s)) + (K + \Delta K(s))G(p(s - \tau_1(s))) + B + (Q_1 + \Delta Q_1(s))u_1(s)\right) ds, \\
I_t^\alpha [C D_t^q p(t)] &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(-(C + \Delta C(s))p(s) + (D + \Delta D(s))m(s) + (H + \Delta H(s))m(s - \tau_2(s)) + (Q_2 + \Delta Q_2(s))u_2(s)\right) ds.
\end{align*}
\]

(20)

In the case of $0 < q < 1$, from property (i) of Lemma 1, we can obtain

\[
\begin{align*}
m(t) &= \phi_1(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(-(A + \Delta A(s))m(s) + (W + \Delta W(s))F(p(s)) + (K + \Delta K(s))G(p(s - \tau_1(s))) + B + (Q_1 + \Delta Q_1(s))u_1(s)\right) ds, \\
p(t) &= \phi_2(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(-(C + \Delta C(s))p(s) + (D + \Delta D(s))m(s) + (H + \Delta H(s))m(s - \tau_2(s)) + (Q_2 + \Delta Q_2(s))u_2(s)\right) ds.
\end{align*}
\]

(21)

The proof is completed.

\[
\Box
\]

**Theorem 2.** If assumptions (I) and (II) hold, then DFGRN (3) with initial condition (6) has a unique mild solution.

**Proof.** Let $(m(t), p(t))^T$ and $(\tilde{m}(t), \tilde{p}(t))^T$ be any two different solutions to DFGRN (3) with initial condition (6);

\[
\begin{align*}
x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(-(A + \Delta A(s))x(s) + (W + \Delta W(s))(F(p(s)) - F(\tilde{p}(s))) + (K + \Delta K(s))(G(p(s - \tau_1(s))) - G(\tilde{p}(s - \tau_1(s))))\right) ds, \\
y(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(-(C + \Delta C(s))y(s) + (D + \Delta D(s))x(s) + (H + \Delta H(s))x(s - \tau_2(s))\right) ds.
\end{align*}
\]

(22)
From (22), by using the norm \(\|\cdot\|\) and assumptions (I) and (II), we can obtain

\[
\begin{align*}
\|x(t)\| & \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[ (\|A\| + \mu_1) \cdot \|x(s)\| + (\|W\| + \mu_2) \cdot L_1 \cdot \|y(s)\| + (\|K\| + \mu_3) \cdot L_2 \cdot \|y(s - \tau_1(s))\| \right] ds, \quad t \in [0, T], \\
\|y(t)\| & \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[ (\|C\| + \mu_4) \cdot \|y(s)\| + (\|D\| + \mu_5) \cdot \|x(s)\| + (\|H\| + \mu_6) \cdot \|x(s - \tau_2(s))\| \right] ds, \quad t \in [0, T].
\end{align*}
\]

(23)

In accordance with (24), we can obtain

\[
\begin{align*}
\|x(t)\| & \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[ (\|A\| + \mu_1) \cdot \|x(s)\| + (\|W\| + \mu_2) \cdot L_1 \cdot \|y(s)\| \right] ds, \\
\|y(t)\| & \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[ (\|C\| + \mu_4) \cdot \|y(s)\| + (\|D\| + \mu_5) \cdot \|x(s)\| \right] ds.
\end{align*}
\]

(24)

Second, when \(t \in (0, \tau^*]\), from (23), we have

\[
\begin{align*}
\|x(t)\| & \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[ (\|A\| + \mu_1) \cdot \|x(s)\| + (\|W\| + \mu_2) \cdot L_1 \cdot \|y(s)\| \right] ds, \\
\|y(t)\| & \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[ (\|C\| + \mu_4) \cdot \|y(s)\| + (\|D\| + \mu_5) \cdot \|x(s)\| \right] ds.
\end{align*}
\]

(25)

In accordance with (24), we can obtain

\[
\begin{align*}
z(t) = & \|x(t)\| + \|y(t)\| \\
\leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[ \eta_1 \|z(s)\| + \eta_2 \|z(s)\| \right] ds \\
= & \frac{\eta_1 + \eta_2}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|z(s)\| ds.
\end{align*}
\]

(26)

Thus, \(z(t) \leq 0, t \in (0, \tau^*].\) That is to say, \(\|x(t)\| + \|y(t)\| \leq 0, t \in (0, \tau^*].\) So, \(m(t) = \tilde{m}(t), p(t) = \tilde{p}(t)\) for \(t \in (0, \tau^*].\)

Third, when \(t \in (\tau^*, T],\) according to (23), we have

\[
\begin{align*}
z(t) & = \|x(t)\| + \|y(t)\| \\
& \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[ \eta_1 \|z(s)\| + \eta_2 \|z(s)\| + \eta_3 \|x(s - \tau_2(s))\| \right] ds + \|y(s - \tau_1(s))\| ds, \quad t \in (\tau^*, T].
\end{align*}
\]

(27)

Summarizing the above three cases, we can obtain that \(m(t) = \tilde{m}(t), p(t) = \tilde{p}(t)\) for \(t \in [-\tau^*, T].\) Due to the arbitrary nature of the solution \((m(t), \rho(t))^T\) and \((\tilde{m}(t), \tilde{p}(t))^T\) of DFGRN (3) and in accordance with Definition 4, we can conclude that DFGRN (3) has a unique mild solution. The proof is completed.

3.2 Finite-Time Stability of DFGRNs with Structured Uncertainties

**Theorem 3.** If assumptions (I) and (II) and \([1 + (\zeta_1 + \zeta_2)^T \Gamma (q + 1)] \cdot E_q (\zeta_1 t^q) \leq (\varepsilon/\delta), \forall t \in I_0 = [0, T] \) hold, then the uncertain DFGRNs with controllers given by (3) with initial condition (6) are finite-time stable with respect to \(\{\delta, \varepsilon, \alpha_1, I_0\}, \delta < \varepsilon, \) where \(\zeta_4 := ((\eta_6 \alpha_1 + \zeta_2)/\delta).\)
Proof. According to Theorem 1 and Theorem 2, we can know that DFGRN (3) has a mild solution and the solution satisfies the following integral equation:

\[
\begin{align*}
  m(t) &= m(0) + \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} \left( \left( A + \Delta A(s) \right) m(s) + \left( W + \Delta W(s) \right) F(p(s)) + \left( K + \Delta K(s) \right) G(p(s - \tau(s))) + B(s) + \left( Q_1 + \Delta Q_1(s) \right) u_1(s) \right) ds,
  \\
  p(t) &= p(0) + \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} \left( \left( C + \Delta C(s) \right) p(s) + \left( D + \Delta D(s) \right) m(s) + \left( H + \Delta H(s) \right) m(s - \tau(s)) + \left( Q_2 + \Delta Q_2(s) \right) u_2(s) \right) ds.
\end{align*}
\]

(30)

Using the norm \( \| (\cdot) \| \), we can obtain the solution estimate of system (30):

\[
\begin{align*}
  \| m(t) \| &\leq \| m(0) \| + \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} \left( \left\| A + \Delta A(s) \right\| \| m(s) \| + \left\| W + \Delta W(s) \right\| \| F(p(s)) \| + \left\| K + \Delta K(s) \right\| \| G(p(s - \tau(s))) \| + \| B(s) \| + \left\| Q_1 + \Delta Q_1(s) \right\| \| u_1(s) \| \right) ds, \\
  \| p(t) \| &\leq \| p(0) \| + \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} \left( \left\| C + \Delta C(s) \right\| \| p(s) \| + \left\| D + \Delta D(s) \right\| \| m(s) \| + \left\| H + \Delta H(s) \right\| \| m(s - \tau(s)) \| + \left\| Q_2 + \Delta Q_2(s) \right\| \| u_2(s) \| \right) ds.
\end{align*}
\]

(31)

By applying norm \( \| (\cdot) \| \) to DFGRN (3) and combining assumptions (I) and (II), we can get

\[
\begin{align*}
  \left\| D^q_\tau m(t) \right\| &\leq \left( \sigma(A) + \mu_1 \right) \| m(t) \| + \left( \sigma(W) + \mu_2 \right) \| L_1 \| \| p(t) \| + \| F(0) \| \\
  &\quad + \left( \sigma(K) + \mu_3 \right) \| L_2 \| \| p(t - \tau_1(t)) \| + \| G(0) \| + \| B(t) \| + \left( \sigma(Q_1) + \mu_2 \right) \| u_1(t) \|, \\
  \left\| D^q_\tau p(t) \right\| &\leq \left( \sigma(C) + \mu_4 \right) \| p(t) \| + \left( \sigma(D) + \mu_5 \right) \| m(t) \| + \left( \sigma(H) + \mu_6 \right) \| m(t - \tau_2(t)) \| + \left( \sigma(Q_2) + \mu_6 \right) \| u_2(t) \|.
\end{align*}
\]

(32)

Let \( x(t) = \| m(t) \| + \| p(t) \| \). According to (31), (3), and (32), if \( \| u_1(t) \| + \| u_2(t) \| \leq \eta_1 \), we have

\[
\begin{align*}
  x(t) &\leq \| m(0) \| + \| p(0) \| + \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} \left( \left\| D^q_\tau m(s) \right\| + \left\| D^q_\tau p(s) \right\| \right) ds \\
  &\leq \| m(0) \| + \| p(0) \| + \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} \left( \left\| (\sigma(A) + \mu_1 + \sigma(D) + \mu_5) \| m(s) \| \right\| + \left\| \left( \sigma(W) + \mu_2 \right) L_1 + \sigma(C) + \mu_4 \right\| \| p(s) \| \\
  &\quad + \left( \sigma(K) + \mu_3 \right) L_2 \| p(s - \tau_1(s)) \| + \left( \sigma(H) + \mu_6 \right) \| m(s - \tau_2(s)) \| + \| B \| + \left( \sigma(W) + \mu_2 \right) \| F(0) \| + \left( \sigma(K) + \mu_2 \right) \| G(0) \| \\
  &\quad + \left( \sigma(Q_1) + \mu_2 \right) \| u_1(s) \| + \left( \sigma(Q_2) + \mu_6 \right) \| u_2(s) \| \right) ds
\end{align*}
\]
\[ \leq x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\left( (\eta_4x(s) + (\sigma(K) + \mu_3)L_2x(s - \tau_1(s)) + (\sigma(H) + \mu_6)x(s - \tau_2(s)) \\
+ \zeta_2 + (\sigma(Q_1) + \mu_7)\|u_1(s)\| + (\sigma(Q_2) + \mu_8)\|u_2(s)\|) \right)ds, \]
\[ \leq x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}(\eta_4x(s) + (\sigma(K) + \mu_3)L_2x(s - \tau_1(s)) + (\sigma(H) + \mu_6)(\sup_{s - \tau \leq t \leq s} x(t^*) + \phi_0) \\
+ \zeta_2 + (\sigma(Q_1) + \mu_7)\|u_1(s)\| + (\sigma(Q_2) + \mu_8)\|u_2(s)\|)ds \]
\[ \leq x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\left( \zeta_1\left(\sup_{s - \tau \leq t \leq s} x(t^*) + \phi_0\right) + \zeta_2 + \eta_6\left(\|u_1(s)\| + \|u_2(s)\|\right) \right)ds \]
\[ \leq \phi_0 + \frac{1}{\Gamma(q)} \zeta_1 \int_0^t (t-s)^{q-1} \sup_{s - \tau \leq t \leq s} x(t^*)ds \frac{\zeta_1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \phi_0ds \\
+ \frac{\eta_6}{\Gamma(q)} \int_0^t (t-s)^{q-1}\left(\|u_1(s)\| + \|u_2(s)\|\right)ds \frac{\zeta_2}{\Gamma(q)} \int_0^t (t-s)^{q-1} \phi_0ds \]
\[ \leq \phi_0 + \frac{1}{\Gamma(q)} \zeta_1 \int_0^t (t-s)^{q-1} \sup_{s - \tau \leq t \leq s} x(t^*)ds \frac{\zeta_1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \phi_0ds \\
+ \frac{\eta_6}{\Gamma(q)} \int_0^t (t-s)^{q-1}\left(\|u_1(s)\| + \|u_2(s)\|\right)ds \frac{\zeta_2}{\Gamma(q)} \int_0^t (t-s)^{q-1} \phi_0ds \]
\[ \leq \phi_0 \left[ 1 + \frac{\zeta_1 t^q}{\Gamma(q + 1)} \right] + \frac{\eta_6}{\Gamma(q + 1)} \frac{\zeta_1 + \zeta_2}{t^q}, \quad t > 0. \]  

Let
\[ \rho(t) = \phi_0 \left[ 1 + \frac{\zeta_1 t^q}{\Gamma(q + 1)} \right] + \frac{\eta_6}{\Gamma(q + 1)} \frac{\zeta_1 + \zeta_2}{t^q}, \quad t > 0. \]  

Then, we know that \( \rho(t) \) is a nonnegative and nondecreasing function. By using Lemma 2 (the generalized Gronwall inequality), we have
\[ x(t) \leq \sup_{t - \tau \leq t \leq t} x(t^*) \leq \rho(t)E_q\left( \frac{\zeta_1}{\Gamma(q)} t^q \right). \]  

If \( \phi_0 < \delta \), we have
\[ x(t) \leq \delta \left[ 1 + \frac{\zeta_1 t^q}{\Gamma(q + 1)} + \frac{\eta_6}{\Gamma(q + 1)} \frac{\zeta_1 + \zeta_2}{\delta} \right] E_q(\zeta_1 t^q). \]

Because \( [1 + ((\zeta_1 + \zeta_3) t^q/\Gamma(q + 1))]E_q(\zeta_1 t^q) \leq (e/\delta) \) and \( \zeta_3 = ((\eta_6 + \zeta_2)/\delta) \), then
\[ x(t) < \varepsilon, \quad \forall t \in J_0. \]

Hence,
\[ \|m(t)\| + \|\rho(t)\| < \varepsilon, \quad \forall t \in J_0. \]

The proof is completed. \( \square \)

**Remark 1.** If we adopt \( u_1(t) \equiv 0, u_2(t) \equiv 0, \forall t \in J_0 \) in DFGRN (3), we can obtain the following conclusion. The uncertain DFGRN (3) satisfying the initial condition (6) is finite-time stable with respect to \( \{\delta, \varepsilon, J_0\}, \delta < \varepsilon \), if assumptions (I) and (II) hold and the following condition is satisfied:
\[ 1 + \frac{\zeta_1 + \zeta_5}{\Gamma(q + 1)} \leq \frac{\varepsilon}{\delta} \quad \forall t \in J_0 = [0, T]. \]

**Remark 2.** In the proof of Theorem 3, if we use the “classical” Bellman–Gronwall inequality instead of the generalized Grónwall inequality, we can get the following result.

The uncertain DFGRN with controllers given by (3) satisfying the initial condition (6) is finite-time stable with respect to \( \{\delta, \varepsilon, J_0\}, \delta < \varepsilon \), if assumptions (I) and (II) hold and the following condition is satisfied:
\[ \left[ 1 + \frac{\zeta_1 + \zeta_5}{\Gamma(q + 1)} \right] \left( t - t_0 \right)^q \exp\left( \frac{\zeta_1}{\Gamma(q + 1)} \right) < \frac{\varepsilon}{\delta} \]

**Remark 3.** If we take \( u_1(t) \equiv 0, u_2(t) \equiv 0, \forall t \in J_0 \) in system (3), the above results turn into the following conclusion.

The uncertain DFGRN (3) satisfying the initial condition (6) is finite-time stable with respect to \( \{\delta, \varepsilon, J_0\}, \delta < \varepsilon \). If assumptions (I) and (II) hold, the following condition is satisfied:
\[ \left[ 1 + \frac{\zeta_1 + \zeta_5}{\Gamma(q + 1)} \right] \left( t - t_0 \right)^q \exp\left( \frac{\zeta_1}{\Gamma(q + 1)} \right) < \frac{\varepsilon}{\delta} \]
3.3 Finite-Time Stability of DFGRNs with Memory State-Feedback Controllers. We consider the following memory state-feedback controllers on DFGRN (3):

\[
\begin{align*}
  u_1(t) &= \gamma_1 m(t) + \gamma_3 p(t - \tau_1(t)), \\
  u_2(t) &= \gamma_2 p(t) + \gamma_4 m(t - \tau_2(t)),
\end{align*}
\]

where \(\gamma_i, i = 1, 2, 3, 4\) are the gain matrices of \(u_i(t), 0 \leq \tau_1(t) \leq \tau', 0 \leq \tau_2(t) \leq \tau'\). Then, DFGRN (3) can be changed into

\[
\begin{align*}
  C^D q m(t) &= -(A + \Delta A(t)) m(t) + (W + \Delta W(t)) F(p(t)) + (K + \Delta K(t)) G(p(t - \tau_1(t))) + B \\
  &+ (Q_1 + \Delta Q_1(t)) (\gamma_1 m(t) + \gamma_3 p(t - \tau_1(t))), \\
  C^D q p(t) &= -(C + \Delta C(t)) p(t) + (D + \Delta D(t)) m(t) + (H + \Delta H(t)) m(t - \tau_2(t)) \\
  &+ (Q_2 + \Delta Q_2(t)) (\gamma_2 p(t) + \gamma_4 m(t - \tau_2(t))).
\end{align*}
\]

Theorem 4. If assumptions (I) and (II) and

\[
\left[ 1 + \frac{\xi_3 + \xi_5}{\Gamma (q + 1)} \right]^q E_q (\xi_i t^q) \leq \frac{\xi}{\delta},
\]

hold, then the uncertain DFGRN (3) with memory state-feedback controllers given by (43) satisfying the initial condition (6) is finite-time stable with respect to \([\delta, \varepsilon, I_0], \delta < \varepsilon\).

Proof. Similar to Theorem 1 and Theorem 2, it is easy to prove that DFGRN (43) has a mild solution satisfying the following integral equation:

\[
\begin{align*}
  m(t) &= m(0) + \frac{1}{\Gamma (q)} \int_0^t (t - s)^{q - 1} \left( -(A + \Delta A(s)) m(s) + (W + \Delta W(s)) F(p(s)) + (K + \Delta K(s)) G(p(s - \tau_1(s))) + B \right. \\
  &\left. + (Q_1 + \Delta Q_1(s)) (\gamma_1 m(s) + \gamma_3 p(s - \tau_1(s))) \right) ds, \\
  p(t) &= p(0) + \frac{1}{\Gamma (q)} \int_0^t (t - s)^{q - 1} \left( -(C + \Delta C(s)) p(s) + (D + \Delta D(s)) m(s) + (H + \Delta H(s)) m(s - \tau_2(s)) \right. \\
  &\left. + (Q_2 + \Delta Q_2(s)) (\gamma_2 p(s) + \gamma_4 m(s - \tau_2(s))) \right) ds.
\end{align*}
\]

Using the norm \(\| (\cdot) \|\), we have

\[
\begin{align*}
  \|m(t)\| &\leq \|m(0)\| + \frac{1}{\Gamma (q)} \int_0^t (t - s)^{q - 1} \left\| -(A + \Delta A(s)) m(s) + (W + \Delta W(s)) F(p(s)) + (K + \Delta K(s)) G(p(s - \tau_1(s))) + B \right. \\
  &\left. + (Q_1 + \Delta Q_1(s)) (\gamma_1 m(s) + \gamma_3 p(s - \tau_1(s))) \right\| ds, \\
  \|p(t)\| &\leq \|p(0)\| + \frac{1}{\Gamma (q)} \int_0^t (t - s)^{q - 1} \left\| -(C + \Delta C(s)) p(s) + (D + \Delta D(s)) m(s) + (H + \Delta H(s)) m(s - \tau_2(s)) \right. \\
  &\left. + (Q_2 + \Delta Q_2(s)) (\gamma_2 p(s) + \gamma_4 m(s - \tau_2(s))) \right\| ds.
\end{align*}
\]
From (43), by using assumptions (I) and (II), we have

\[
\begin{aligned}
\|C D_t^q m(t)\| &\leq (\sigma(A) + \mu_1)\|m(t)\| + (\sigma(W + \mu_2) (L_1 + \|F(0)\|) + (\sigma(K) + \mu_3) (L_2 + \|p(t-t_1)\|) + \|G(0)\|) \\
+ \|B\| + (\sigma(Q_1) + \mu_2) (\|m(t)\| + \|\sigma(y_1)\| (p(t-t_1)))} \\
\|C D_t^q p(t)\| &\leq (\sigma(C) + \mu_4)\|p(t)\| + (\sigma(D) + \mu_5)\|m(t)\| + (\sigma(H) + \mu_6)\|m(t-t_2)\| \\
+ (\sigma(Q_2) + \mu_5) (\|p(t)\| + \|\sigma(y_2)\| (m(t-t_2))}. 
\end{aligned}
\]  

(47)

Let \( x(t) = \|m(t)\| + \|p(t)\|. \) From (46) and (47), we obtain

\[
\begin{aligned}
x(t) &\leq \|m(0)\| + \|p(0)\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ((\sigma(A) + \mu_1) (\sigma(Q_1) + \mu_2) (\sigma(y_1) + \sigma(D) + \mu_3) \|m(s)\| + \|B\| \\
&\quad + (\sigma(W) + \mu_2) \|F(0)\| + (\sigma(K) + \mu_3) \|G(0)\| + (\sigma(C) + \mu_4) (\sigma(Q_2) + \mu_5) (\sigma(y_2) + (\sigma(H) + \mu_6) \|m(s-t_2)\|) \\
&\quad + (\sigma(Q_2) + \mu_5) (\|m(s-t_2)\|) } ds.
\end{aligned}
\]  

(48)

Hence,

\[
\begin{aligned}
x(t) &\leq x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (\mu_1 (\sigma(y_1) + \sigma(D) + \mu_3) \|m(s)\| + \|B\| \\
&\quad + (\sigma(W) + \mu_2) \|F(0)\| + (\sigma(K) + \mu_3) \|G(0)\| + (\sigma(C) + \mu_4) (\sigma(Q_2) + \mu_5) (\sigma(y_2) + (\sigma(H) + \mu_6) \|m(s-t_2)\|) \\
&\quad + (\sigma(Q_2) + \mu_5) (\|m(s-t_2)\|) + (\sigma(Q_1) + \mu_2) (\sigma(y_1) + \sigma(D) + \mu_3) \|m(s-t_1)\| \\
&\quad + \sigma(H) + \mu_6) \|m(s-t_2)\| ) ds.
\end{aligned}
\]  

(49)

Let

\[
\rho(t) = \phi_0 \left[ 1 + \frac{\zeta_3 t^q}{\Gamma(q+1)} \right] + \frac{\zeta_3 t^q}{\Gamma(q+1)} \quad t > 0.
\]  

(50)

Then, we know \( \rho(t) \) is a nonnegative and nondecreasing function. From Lemma 2, we have

\[
x(t) \leq \sup_{s-t_1 \leq t_2 \leq t} x(t') \leq \rho(t) E_q \left( \frac{\zeta_3}{\Gamma(q)} t^q \right).
\]  

(51)

If \( \phi_0 < \delta \), we obtain

\[
x(t) \leq \delta \left[ 1 + \frac{\zeta_3 (\delta)}{\Gamma(q+1)} t^q \right] E_q \left( \frac{\zeta_3}{\Gamma(q)} t^q \right).
\]  

(52)

From the condition of \( [1 + ((\zeta_3 + \zeta_5) t^q)/\Gamma(q+1)]) E_q(\zeta_3 t^q) \leq (e/\delta) \) and \( \zeta_5 = (\zeta_2/\delta) \), we can get

\[
x(t) < \epsilon, \quad \forall t \in J_0.
\]  

(53)

Therefore,

\[
\|m(t)\| + \|p(t)\| < \epsilon, \quad \forall t \in J_0.
\]  

(54)

The proof is completed. \( \square \)

**Remark 4.** Similar to Remark 2, we can get the following result.
The uncertain DFGRN (3) with memory state-feedback controller given by (43) satisfying the initial condition (6) is finite-time stable with respect to \([\delta, \epsilon, f_0], \delta < \epsilon\), if assumptions (I) and (II) hold and the following condition is satisfied:

\[
\left[1 + \frac{\zeta_3 + \zeta_5}{\Gamma(q + 1)}(t - t_0)^q \right] \exp \left[\frac{\zeta_3 (t - t_0)^q}{\Gamma(q + 1)}\right] < \frac{\epsilon}{\delta} \tag{55}
\]

**Remark 5.** We can obtain the same conclusion as Theorem 3 and Theorem 4 if the inequalities in assumption (II) are satisfied:

\[
\|F(x)\| \leq L_1\|x\|, \\
\|G(x)\| \leq L_2\|x\|. \tag{56}
\]

**Remark 6.** All the results in Remarks 1–4 are still new.

### 4. Numerical Examples

In this section, some numerical examples are given to illustrate the effectiveness of above theoretical results. In the following examples, the functions \(f_j\) and \(g_j\) are taken as the Hill form. And in the Adams–Bashforth–Moulton predictor-corrector scheme [42], the step length is \(h = 0.1\).

**Example 1.** Consider the following DFGRNs of three mRNA and protein nodes with structured uncertainties and memory state-feedback controllers:

\[
\begin{aligned}
\dot{C}_D q(t) &= -(A + \Delta A(t))m(t) + (W + \Delta W(t))F(p(t)) \\
&+ (K + \Delta K(t))G(p(t - \tau_1(t))) + B + (Q_1 + \Delta Q_1(t))u_1(t), \\
\dot{C}_D q p(t) &= -(C + \Delta C(t))p(t) + (D + \Delta D(t))m(t) \\
&+ (H + \Delta H(t))m(t - \tau_2(t)) + (Q_2 + \Delta Q_2(t))u_2(t).
\end{aligned} \tag{57}
\]

Let

\[
A = \begin{pmatrix}
3 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 3
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
2.5 & 0 & 0 \\
0 & 2.5 & 0 \\
0 & 0 & 2.5
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
H = \begin{pmatrix}
0.3 & 0 & 0 \\
0 & 0.3 & 0 \\
0 & 0 & 0.3
\end{pmatrix},
\]

\[
W = \begin{pmatrix}
0.8147 & -0.9134 & 0.2785 \\
0.9058 & 0.6324 & -0.5469 \\
-0.1270 & 0.0975 & 0.9575
\end{pmatrix},
\]

\[
K = \begin{pmatrix}
0.28947 & 0.28716 & -0.04257 \\
0.04728 & -0.14562 & 0.12654 \\
-0.29118 & 0.24009 & 0.27471
\end{pmatrix},
\]

\[
\Delta A(t) = \begin{pmatrix}
0.1 \cos(t) - 0.07 \sin(t) & 0.02 \cos(t) - 0.05 \sin(t) & 0.04 \cos(t) - 0.06 \sin(t) \\
0.1 \cos(t) + 0.01 \sin(t) & 0.02 \cos(t) & 0.04 \cos(t) + 0.03 \sin(t) \\
0.05 \cos(t) + 0.03 \sin(t) & 0.01 \cos(t) + 0.01 \sin(t) & 0.02 \cos(t) + 0.06 \sin(t)
\end{pmatrix},
\]

\[
f_j(t) = (0.1 + 0.1 \cos(t)) \sin(t), \\
g_j(t) = (0.1 + 0.02 \cos(t)) \sin(t).
\]
Complexity

\[ \Delta C(t) = \begin{bmatrix}
0.04 \cos(t) - 0.08 \sin(t) & 0.04 \cos(t) - 0.03 \sin(t) & 0.02 \cos(t) - 0.01 \sin(t) \\
0.04 \cos(t) + 0.04 \sin(t) & 0.04 \cos(t) + 0.04 \sin(t) & 0.02 \cos(t) + 0.03 \sin(t) \\
0.02 \cos(t) + 0.08 \sin(t) & 0.02 \cos(t) + 0.07 \sin(t) & 0.01 \cos(t) + 0.05 \sin(t)
\end{bmatrix}, \]

\[ \Delta D(t) = \begin{bmatrix}
0.01 \sin(t) & 0.04 \cos(t) & -0.09 \sin(t) \\
0.02 \sin(t) & 0.04 \cos(t) & -0.03 \sin(t) \\
0.03 \sin(t) & 0.02 \cos(t) & -0.03 \sin(t)
\end{bmatrix}, \]

\[ \Delta H(t) = \begin{bmatrix}
0.06 \cos(t) + 0.01 \sin(t) & 0.02 \cos(t) - 0.04 \sin(t) & 0.04 \cos(t) - 0.02 \sin(t) \\
0.06 \cos(t) + 0.02 \sin(t) & 0.02 \cos(t) + 0.02 \sin(t) & 0.04 \cos(t) + 0.01 \sin(t) \\
0.03 \cos(t) + 0.03 \sin(t) & 0.01 \cos(t) + 0.04 \sin(t) & 0.02 \cos(t) + 0.02 \sin(t)
\end{bmatrix}, \]

\[ \Delta K(t) = \begin{bmatrix}
0.02 \cos(t) - 0.01 \sin(t) & 0.06 \cos(t) - 0.05 \sin(t) & 0.04 \cos(t) - 0.02 \sin(t) \\
0.02 \cos(t) + 0.03 \sin(t) & 0.06 \cos(t) & 0.04 \cos(t) + 0.01 \sin(t) \\
0.01 \cos(t) + 0.05 \sin(t) & 0.03 \cos(t) + 0.01 \sin(t) & 0.02 \cos(t) + 0.02 \sin(t)
\end{bmatrix}, \]

\[ \Delta W(t) = \begin{bmatrix}
0.04 \cos(t) - 0.1 \sin(t) & 0.06 \cos(t) - 0.02 \sin(t) & 0.08 \cos(t) + 0.01 \sin(t) \\
0.04 \cos(t) & 0.06 \cos(t) + 0.01 \sin(t) & 0.08 \cos(t) + 0.01 \sin(t) \\
0.02 \cos(t) + 0.02 \sin(t) & 0.03 \cos(t) + 0.02 \sin(t) & 0.04 \cos(t) + 0.03 \sin(t)
\end{bmatrix}, \]

\[ \Delta Q_1(t) = \begin{bmatrix}
0.02 \cos(t) - 0.02 \sin(t) & 0.04 \cos(t) - 0.05 \sin(t) & 0.04 \cos(t) - 0.08 \sin(t) \\
0.02 \cos(t) + 0.01 \sin(t) & 0.04 \cos(t) & 0.04 \cos(t) - 0.01 \sin(t) \\
0.01 \cos(t) + 0.02 \sin(t) & 0.02 \cos(t) + 0.01 \sin(t) & 0.02 \cos(t)
\end{bmatrix}, \]

\[ \Delta Q_2(t) = \begin{bmatrix}
0.02 \cos(t) + 0.02 \sin(t) & 0.02 \cos(t) - 0.04 \sin(t) & 0.04 \cos(t) - 0.08 \sin(t) \\
0.02 \cos(t) + 0.04 \sin(t) & 0.02 \cos(t) + 0.02 \sin(t) & 0.04 \cos(t) - 0.01 \sin(t) \\
0.01 \cos(t) + 0.06 \sin(t) & 0.01 \cos(t) + 0.04 \sin(t) & 0.02 \cos(t)
\end{bmatrix}. \]

(58)

The memory state-feedback controllers are defined as follows:

\[ u_1(t) = \gamma_1 m(t) + \gamma_2 p(t - \tau_1(t)), \]

\[ u_2(t) = \gamma_3 p(t) + \gamma_4 m(t - \tau_2(t)), \]

where

\[ \gamma_1 = \begin{bmatrix}
0.0465 & 0.0457 & -0.0358 \\
-0.0342 & -0.0015 & -0.0078 \\
0.0471 & 0.0300 & 0.0416
\end{bmatrix}, \]

\[ \gamma_2 = \begin{bmatrix}
0.0195 & -0.0466 & 0.0266 \\
-0.0183 & -0.0061 & 0.0295 \\
0.0450 & -0.0118 & -0.0313
\end{bmatrix}, \]

\[ \gamma_3 = \begin{bmatrix}
-0.0010 & 0.0209 & 0.0180 \\
-0.0054 & 0.0255 & 0.0155 \\
0.0146 & -0.0224 & -0.0337
\end{bmatrix}, \]

\[ \gamma_4 = \begin{bmatrix}
-0.0381 & -0.0160 & 0.0251 \\
-0.0002 & 0.0085 & -0.0245 \\
0.0460 & -0.0276 & 0.0006
\end{bmatrix}. \]

(59)

(60)

Let \( Q_1 = Q_2 = \text{diag}(2,2,2), q = 0.95, \delta = 1, \varepsilon = 50, \tau_1(t) = \tau_2(t) = ((\cos(t) + 1)/4), \tau^* = (1/2), (\phi_1(t), \phi_2(t))^T = (0.1392, 0.2734, 0.4788, 0.4824, 0.0788, 0.4853)^T (-r^* \leq t \leq 0), L_1 = L_2 = 1, F(x) = G(x) = x^2/(1 + x^2). \) According to the
notations in Section 2, we obtain $\phi_0 = 0.9641 < 1$, $\sigma(A) = 3$, $\sigma(D) = 1$, $\sigma(W) = 1.3710$, $\sigma(C) = 2.5$, $\sigma(H) = 0.3$, $\sigma(K) = 0.4793$, $\sigma(Q_i) = 2$, $\eta_5 = 4.5081$, $\zeta_3 = 5.8292$, $\zeta_5 = 7.0183$. When $t < 0.3339$, simple computation reveals that

$$
\left[ 1 + \frac{(\zeta_3 + \zeta_5) t^\eta}{\Gamma(q + 1)} \right] E_q(\zeta_3 t^\eta) < \left[ 1 + \frac{(5.8292 + 0.3339) \times 0.3339^{0.95}}{\Gamma(0.95 + 1)} \right] E_q(5.8292 \times 0.3339^{0.95}) < \frac{e}{\delta} = \frac{50}{1}.
$$

From Theorem 4, system (57) is finite-time stable with respect to $[1, 50, [0, 0.3339]]$. Denote $T_x = 0.3339$ as the “estimated time” of finite-time stability. The transient states of the variable $m_i(t)$ and $p_i(t) (i = 1, 2, 3)$ of DFGRN (57) with $q = 0.95$ and $q = 0.6$ are shown in Figures 1(a) and 1(b), respectively.

$$
\begin{align*}
C D^\eta m(t) &= -(A + \Delta A(t)) m(t) + (W + \Delta W(t)) F(p(t)) + (K + \Delta K(t)) G(p(t - \tau_1(t))) + B, \\
C D^\eta p(t) &= -(C + \Delta C(t)) p(t) + (D + \Delta D(t)) m(t) + (H + \Delta H(t)) m(t - \tau_2(t)).
\end{align*}
$$

Example 2. Consider the following DFGRNs of three mRNA and protein nodes with structured uncertainties and without controller:

$$
\left[ 1 + \frac{(\zeta_3 + \zeta_5) t^\eta}{\Gamma(q + 1)} \right] E_q(\zeta_3 t^\eta) < \left[ 1 + \frac{(5.8292 + 0.3339) \times 0.3339^{0.95}}{\Gamma(0.95 + 1)} \right] E_q(5.8292 \times 0.3339^{0.95}) < \frac{e}{\delta} = \frac{50}{1}.
$$

From Remark 1, system (62) is finite-time stable with respect to $[1, 50, [0, 0.33585]]$. The transient states of the variables $m_i(t)$ and $p_i(t) (i = 1, 2, 3)$ of DFGRN (62) with $q = 0.95$ and $q = 0.6$ are shown in Figures 2(a) and 2(b), respectively.

In Example 2, when $t \to +\infty$, the case of infinite time, DFGRN (62) with structured uncertainties is unstable. The numerical simulations of the variables $m_i(t)$ and $p_i(t) (i = 1, 2, 3)$ of DFGRN (62) with $q = 0.95$ and $q = 0.6$ are shown in Figures 3(a) and 3(b), respectively.

$$
\begin{align*}
C D^\eta m(t) &= -A(m(t) + WF(p(t)) + KG(p(t - \tau_1(t))) + B + Q_1(y_1 m(t) + y_3 p(t - \tau_1(t))), \\
C D^\eta p(t) &= -C p(t) + D m(t) + H m(t - \tau_1(t)) + Q_2(y_2 p(t) + y_4 m(t - \tau_1(t))).
\end{align*}
$$

Using the same parameters in Example 1, we similarly obtain $\eta_5 = 4.1799$, $\zeta_3 = 5.2009$, $\zeta_5 = 7.0183$. When $t < 0.3697$, we can get

$$
\left[ 1 + \frac{(\zeta_3 + \zeta_5) t^\eta}{\Gamma(q + 1)} \right] E_q(\zeta_3 t^\eta) < \left[ 1 + \frac{(5.2009 + 0.3697) \times 0.3697^{0.95}}{\Gamma(0.95 + 1)} \right] E_q(5.2009 \times 0.3697^{0.95}) < \frac{e}{\delta} = \frac{50}{1}.
$$

Remark 7. It is worthy to note that in a special case of DFGRN (62) without structured uncertainties, it is proved that in the sense of infinite stability, (62) is globally asymptotically stable [16].

Example 3. Consider the following DFGRNs of three mRNA and protein nodes with memory state-feedback controllers and without structured uncertainties:

$$
\begin{align*}
C D^\eta m(t) &= -A(m(t) + WF(p(t)) + KG(p(t - \tau_1(t))) + B + Q_1(y_1 m(t) + y_3 p(t - \tau_1(t))), \\
C D^\eta p(t) &= -C p(t) + D m(t) + H m(t - \tau_1(t)) + Q_2(y_2 p(t) + y_4 m(t - \tau_1(t))).
\end{align*}
$$

Using the same parameters in Example 1, we similarly obtain $\eta_5 = 4.1799$, $\zeta_3 = 5.2009$, $\zeta_5 = 7.0183$. When $t < 0.3697$, we can get

$$
\left[ 1 + \frac{(\zeta_3 + \zeta_5) t^\eta}{\Gamma(q + 1)} \right] E_q(\zeta_3 t^\eta) < \left[ 1 + \frac{(5.2009 + 0.3697) \times 0.3697^{0.95}}{\Gamma(0.95 + 1)} \right] E_q(5.2009 \times 0.3697^{0.95}) < \frac{e}{\delta} = \frac{50}{1}.
$$
From Theorem 4, system (64) is finite-time stable with respect to $T^*_s = 0.3697$. The transient states of the variables $m_i(t)$ and $p_i(t)$ $(i = 1, 2, 3)$ of DFGRN (64) with $q = 0.95$ and $q = 0.6$ are shown in Figures 4(a) and 4(b), respectively.

**Example 4.** Consider the following DFGRNs of three mRNA and protein nodes without structured uncertainties or controller:

$$
\begin{align*}
C D^p_t m(t) &= -A m(t) + W F (p(t)) + K G (p(t - \tau_1(t))) + B, \\
C D^p_t p(t) &= -C p(t) + D m(t) + H m(t - \tau_2(t)).
\end{align*}
$$

(66)

Using the same parameters in Example 1, we also obtain the “estimated time” of finite-time stability for system (66) as $T^*_s = 0.3984$. The transient states of the variables $m_i(t)$ and $p_i(t)$ $(i = 1, 2, 3)$ of DFGRN (66) with $q = 0.95$ and $q = 0.6$ are shown in Figures 5(a) and 5(b), respectively.

If we adopt constant time-delay $\tau_1(t) = \tau_2(t) = 2$ and $q = 0.4$ in DFGRN (66), then system (66) is finite-time stable, and the “estimated time” of finite-time stability is 0.0315. The transient states of the variables $m_i(t)$ and $p_i(t)$ $(i = 1, 2, 3)$ of DFGRN (66) with $q = 0.4$ are shown in Figure 6.

**Remark 8.** If $\tau_1(t) = \tau_2(t) = 2$ and $q = 0.4$ in DFGRN (66), then system (66) converges to system (4.1) in [16]. When
\[ t \to +\infty, \] it is proved that system (4.1) is unstable in the sense of infinite-time stability [16], which means that the finite-time stability is different from the infinite-time stability of DFGRNs.

If we take \( K = \Delta K(t) = H = \Delta H(t) = 0 \) and \( \gamma_3 = \gamma_4 = 0 \) in DFGRNs (57), (62), (64), and (66) systems (57), (62), (64), and (66) convert to the corresponding fractional-order gene regulatory networks without time delays (FGRNs).

In order to investigate the effects of structured uncertainties, controllers and time delays on the stability of the DFGRNs, we calculate the “estimated time” \( T_e \) of finite-time stability for above four examples and the corresponding FGRNs with different fractional-order \( q \); the results are shown in Tables 1 and 2, respectively.

From Table 1 or Table 2, we have the following conclusions:

(i) The effect of the controllers: comparing column 2 with 3 (or column 4 with 5), we can know that the controllers can shorten the “estimated time” of finite-time stability under the same conditions of fractional-order \( q \) and structured uncertainties.

(ii) The effect of the structured uncertainties: comparing column 3 with 5, we can know that the structured uncertainties can shorten the “estimated time” of finite-time stability under the same fractional-order \( q \).
The difference between the structured uncertainties and the controllers: comparing column 3 with 4, we can know that the size of “estimated time” of finite-time stability for DFGRN (62) with structured uncertainties is longer than DFGRN (64) with controllers under the same fractional-order \( q \).

The effect of the fractional-order \( q \): in the same column, we can know that decreasing the fractional-
order \( q \) will be useful to decrease the “estimated time” of finite-time stability for DFGRNs or FGRNs.

(v) The effect of time delays: comparing Table 1 with Table 2, we can know that the “estimated time” of finite-time stability is reduced under the same fractional-order \( q \) when considering time delays.

5. Concluding Remarks

This paper deals with the existence and uniqueness of the solution and the finite-time stability for a class of DFGRNs with structured uncertainties and controllers. In particular, we design the memory state-feedback controllers for DFGRNs with structured uncertainties and give the sufficient conditions for the system to achieve the finite-time stability.

It should be pointed out that the conditions of finite-time stability in the present paper are dependent on the fractional-order \( q \), which is more different from the previous stability results for the case of integer order, i.e., the finite-time stability is independent of the integer order.

In addition, from the numerical results, we find that all of the controllers, uncertain terms, fractional-order \( q \) and time delays can affect the “estimated time” of finite-time stability. Particularly, (i) the size of “estimated time” of finite-time stability with controllers is shorter than the case without controller but only with structured uncertainties, which means that the controllers are more beneficial for controlling the “estimated time” than the structured uncertainties; (ii) the size of “estimated time” of finite-time stability with time delays is shorter than the case without time delays, which means that time delays degrade the GRN performance.

If we take \( \Delta A (t) = \Delta W (t) = \Delta K (t) = \Delta C (t) = \Delta D (t) = \Delta H (t) = \Delta Q_1 (t) = \Delta Q_2 (t) = 0 \) and controllers terms \( u_1 (t) = u_2 (t) = 0 \), meanwhile, in the special case constant time delay, system (3) convert to (2.2) in [16], and we find that numerically: as \( t \rightarrow + \infty \), DFGRN (62) in this paper is unstable; however, DFGRN (4.1) in [16] is globally asymptotically stable, which means that the structured uncertainty can change the stability of DFGRNs. Furthermore, from Remark 8, we know that DFGRN (66) is finite-time stable, while the corresponding system (4.1) in [16] is infinite-time unstable, which means that an infinite-time unstable system can change to a finite-time stable one under extra conditions. The analytical study on above questions is desirable in the future.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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References


Table 2: The “estimated time” \( T_e \) of finite-time stability without time delays.

<table>
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<th>( q )</th>
<th>FGRN (57) ( T_e )</th>
<th>FGRN (62) ( T_e )</th>
<th>FGRN (64) ( T_e )</th>
<th>FGRN (66) ( T_e )</th>
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<td>0.95</td>
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