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*Research Article*

# Abundant Symmetry-Breaking Solutions of the Nonlocal Alice–Bob Benjamin–Ono System

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The Benjamin–Ono equation is a useful model to describe the long internal gravity waves in deep stratified fluids. In this paper, the nonlocal Alice–Bob Benjamin–Ono system is induced via the parity and time-reversal symmetry reduction. By introducing an extended Bäcklund transformation, the symmetry-breaking soliton, breather, and lump solutions for this system are obtained through the derived Hirota bilinear form. By taking suitable constants in the involved ansatz functions, abundant fascinating symmetry-breaking structures of the related explicit solutions are shown.

## 1. Introduction

In the recent years, studying the local excitations in the nonlinear evolution equations (NEEs) has become great significance since the complex nonlinear phenomena related to the NEEs involve in fluid dynamics, plasma physics, superconducting physics, condensed matter physics, and optical problems [1–6]. In fact, researchers have discovered many powerful methods for studying these aspects, such as the Hirota bilinear method [7–9], the inverse scattering method [10, 11], the Painlevé analysis approach [12–14], the Bäcklund transformation [15], and the Darboux transformation [16–18]. Furthermore, the investigation of the solitary waves and solitons taking one or more of the above approaches for the NEEs has become more and more important and attractive.

Meanwhile, one of the proposed two-place nonlocal models, the nonlinear Schrödinger (NLS) equation

$$iA_t + A_{xx} \pm A^2 B = 0, B = \widehat{f} A = \widehat{P}\widehat{C}A = A^*(-x, t) \quad (1)$$

(where $\widehat{P}$ and $\widehat{C}$ are the usual parity and charge conjugation operators) had been investigated [19]. Recently, Lou proposed the Alice–Bob (AB) systems to describe two-place physical problems [20, 21]. The parity, time reversal, charge conjugation, and their suitable combinations were conserved for most of the above problems [20–30]. However, these AB symmetries exist in various physical models, although they are not directly used to solve the nonlinear physical systems, especially the $\widehat{P}$-$\widehat{C}$-$\widehat{T}$ symmetries [22]. Using the Bäcklund transformation, some types of $\widehat{P}\widehat{T}$ symmetry-breaking solutions including soliton and rogue wave solutions were explicitly obtained. In addition to nonlocal nonlinear Schrödinger equation (1), there are many other types of two-place nonlocal models, such as the nonlocal modified KdV systems [20] and the nonlocal Boussinesq-KdV systems [23].

In this work, we consider the $(1+1)$-dimensional Benjamin–Ono (BO) equation

$$u_{tt} + \beta\left(u^2\right)_{xx} + \gamma u_{xxxx} = 0, \quad (2)$$

where $\beta$ is the nonlinear term coefficient and $\gamma$ is a dispersion coefficient. The BO equation is one of the most important nonlinear equations that describes one-dimensional internal waves in deep water [31]. Ono had developed the Benjamin theory to obtain a species of the NEEs [32]. The two/four-place nonlocal Benjamin–Ono equation was explicitly solved

for special types of multiple soliton solutions via $\widehat{P}$-$\widehat{C}$-$\widehat{T}$ symmetric-antisymmetric separation approach [22]. Many authors also studied different properties for equation (2), such as the Bäcklund transformation, the existence of quasi-periodic solutions and the nontrivial time-periodic solutions, infinitely many conservation laws, and other integrable properties [33–38].

The outline of this paper is as follows: in Section 2, the AB-BO system and its Lax pair are introduced, and its bilinear form is written through an extended Bäcklund transformation. In Section 3, the symmetry-breaking soliton, breather, and lump solutions are presented through the derived Hirota bilinear form. According to the taken constants in the involved ansatz functions, some sets of the fascinating symmetry-breaking structures of the related explicit solutions are shown, correspondingly. Summary and conclusions are given in the last section.

## 2. The AB-BO System and Its Lax Pair, Bäcklund Transformation, and Bilinear Form

Based on the principle of the AB system [20, 21], after substituting $u = (A+B)/2$ into equation (2), the nonlocal AB-BO system is derived as follows:

$$\begin{aligned} &A_{tt} + B_{tt} + \beta\left(A_x + B_x\right)^2 + \beta(A+B)\left(A_{xx} + B_{xx}\right) \\ &\quad + \gamma\left(A_{xxxx} + B_{xxxx}\right) = 0. \end{aligned} \tag{3}$$

Equation (3) can be split into the coupled equations

$$A_{tt} + \frac{\beta}{2}\left(A_x + B_x\right)^2 + \beta A\left(A_{xx} + B_{xx}\right) + \gamma A_{xxxx} + G(A,B) = 0, \tag{4a}$$

$$B_{tt} + \frac{\beta}{2}\left(A_x + B_x\right)^2 + \beta B\left(A_{xx} + B_{xx}\right) + \gamma B_{xxxx} - G(A,B) = 0, \tag{4b}$$

where

$$\begin{aligned} B &= \widehat{f}A = A^{\widehat{f}}, \\ \widehat{f} &\in \Theta \equiv \left\{1, \widehat{P}_s, \widehat{T}_d, \widehat{P}_s\widehat{T}_d\right\}, \\ A^{\widehat{P}_s} &= A(-x + x_0, t), \\ A^{\widehat{T}_d} &= A(x, -t + t_0), \end{aligned} \tag{5}$$

with $x_0$ and $t_0$ are two arbitrary constants, and $G(A,B)$ is an arbitrary function of $A$ and $B$, with $\widehat{f}$ invariant. That is, $G(A,B) = \widehat{f}G(A,B) = G(A,B)^{\widehat{f}}$. Although there are infinite functions satisfying this $G(A,B)$ for equations (4a) and (4b), we can construct the function $G(A,B)$ as

$$G(A,B) = \beta\left(A_x^2 - B_x^2\right) + \frac{\beta}{2}(A+B)\left(A_{xx} - B_{xx}\right), \tag{6}$$

and equations (4a) and (4b) are reduced to the following AB-BO system:

$$\begin{aligned} &A_{tt} + \frac{\beta}{2}\left(A_x + B_x\right)\left(3A_x - B_x\right) + \frac{\beta}{2}(3A+B)A_{xx} \\ &\quad + \frac{\beta}{2}(A-B)B_{xx} + \gamma A_{xxxx} = 0, \end{aligned} \tag{7a}$$

$$\begin{aligned} &B_{tt} - \frac{\beta}{2}\left(A_x + B_x\right)\left(A_x - 3B_x\right) + \frac{\beta}{2}(3B+A)B_{xx} \\ &\quad - \frac{\beta}{2}(A-B)A_{xx} + \gamma B_{xxxx} = 0. \end{aligned} \tag{7b}$$

For $\widehat{f} = 1\,(B = A)$, equations (7a) and (7b) are just the usual local BO equations. For $\widehat{f} \neq 1\,(B \neq A)$, equations (7a) and (7b) express three types of nonlocal AB-BO systems with three different nonlocalities, the parity nonlocal AB-BO (PNAB-BO) system $(B = A^{\widehat{P}_s})$, the time-reversal nonlocal AB-BO (TNAB-BO) system $(B = A^{\widehat{T}_d})$, and the parity and time-reversal nonlocal AB-BO (PTNAB-BO) $(B = A^{\widehat{P}_s\widehat{T}_d})$ system.

Obviously, systems (7a) and (7b) are integrable, and their Lax pair can be written as

$$\psi_{xxx} = \begin{pmatrix} M & 0 \\ Z & M \end{pmatrix}\psi, \tag{8a}$$

$$\psi_t = \sqrt{3\gamma}\psi_{xx} + \frac{\sqrt{3}\beta}{6\sqrt{\gamma}}\begin{pmatrix} A+B & 0 \\ A-B & A+B \end{pmatrix}\psi, \tag{8b}$$

with

$$\begin{aligned} M &= -\frac{\beta}{4\gamma}(A+B)\partial_x - \frac{\sqrt{3}\beta}{24\gamma^{3/2}}\int\left(A_t + B_t\right)\mathrm{d}x \\ &\quad - \frac{\beta}{8\gamma}\left(A_x + B_x\right) + \lambda_1, \end{aligned} \tag{9a}$$

$$\begin{aligned} Z &= -\frac{\beta}{4\gamma}(A-B)\partial_x - \frac{\sqrt{3}\beta}{24\gamma^{3/2}}\int\left(A_t - B_t\right)\mathrm{d}x \\ &\quad - \frac{\beta}{8\gamma}\left(A_x - B_x\right) + \lambda_2, \end{aligned} \tag{9b}$$

$\lambda_1$ and $\lambda_2$ being arbitrary constants.

Now, we introduce an extended Bäcklund transformation

$$\begin{aligned} A &= \frac{6\gamma}{\beta}(\ln F)_{xx} + b_1(\ln F)_{xxx} + b_2(\ln F)_{xxt} + \alpha, \\ B &= \frac{6\gamma}{\beta}(\ln F)_{xx} - b_1(\ln F)_{xxx} - b_2(\ln F)_{xxt} + \alpha, \end{aligned} \tag{10}$$

with $b_1, b_2, \alpha$ being arbitrary constants, and $F \equiv F(x,t)$ is an undetermined real function of variables $x, t$ and satisfies

$$F = \widehat{f}F = F^{\widehat{f}}. \tag{11}$$

When $b_1 = 0$ and $b_2 = 0$, equation (10) becomes one normal Bäcklund transformation of equation (2). Substituting equation (10) into equations (7a) and (7b), the bilinear form can be written as follows:

$$\left(D_t^2 + 2\alpha\beta D_x^2 + \gamma D_x^4\right)(F \cdot F) = 0, \tag{12}$$

where $D_t^2, D_x^2$, and $D_x^4$ are the bilinear derivative operators defined by [8, 9].

$$D_x^m D_t^n (F \cdot G) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \times F(x,t)G(x',t')\big|_{x'=x,t'=t}. \tag{13}$$

According to the properties of bilinear operator $D$, equation (12) is equal to

$$FF_{tt} - F_t^2 + 2\alpha\beta\left(FF_{xx} - F_x^2\right) + \gamma\left(FF_{xxxx} - 4F_xF_{xxx} + 3F_{xx}^2\right) = 0, \tag{14}$$

which is a bilinear form of equation (2).

## 3. Symmetry-Breaking Soliton, Breather, and Lump Solutions to the AB-BO System

In this section, we turn our attention to the Hirota bilinear form (12) of AB-BO systems (7a) and (7b) to derive the symmetry-breaking soliton, symmetry-breaking breather, and symmetry-breaking lump solutions.

*3.1. Symmetry-Breaking Soliton and Breather Solutions to the AB-BO System.* Based on the bilinear form (12), we can first determine the symmetry-breaking soliton and breather solutions through the Bäcklund transformation (10) of AB-BO systems (7a) and (7b) with the function $F$ be written as a summation of some special functions [20, 21, 23]:

$$\begin{aligned} F &= F_N = \sum_{\{\nu\}} K_{\{\nu\}} \cosh\left(\frac{1}{2}\sum_{i=1}^{N} \nu_i \xi_i\right), \\ \xi_i &= k_i\left(x - \frac{x_0}{2}\right) + \omega_i\left(t - \frac{t_0}{2}\right) + \eta_{i0}, \\ \omega_i &= \delta_i\sqrt{-2\alpha\beta k_i^2 - \gamma k_i^4}, \\ \delta_i^2 &= 1, \end{aligned} \tag{15}$$

where the summation of $\{\nu\} = \{\nu_1, \nu_2, \ldots, \nu_N\}$, $k_i, \eta_{i0}\,(i = 1, 2, \ldots, N)$ are arbitrary constants, while

$$K_{\{\nu\}} = \prod_{i<j}^{N} \sqrt{k_i k_j\left(2\gamma k_i^2 + 2\gamma k_2^2 + 2\alpha\beta - 3\nu_i\nu_j\gamma k_i k_j\right) + \omega_i\omega_j}. \tag{16}$$

We notice that when $\eta_{i0} = 0$, the invariant condition $F(x,t) = F(-x + x_0, -t + t_0)$ of the function $F$ (15) is satisfied.

For $N = 1$, equation (15) possesses the form

$$\begin{aligned} F &= F_1 = \cosh\left(\frac{\xi_1}{2}\right), \\ \xi_1 &= k_1\left(x - \frac{x_0}{2}\right) + \omega_1\left(t - \frac{t_0}{2}\right). \end{aligned} \tag{17}$$

We have the single soliton solution

$$\begin{aligned} A &= \frac{k_1^2}{4}\left[\frac{6\gamma}{\beta} - (b_1k_1 + b_2\omega_1)\tanh\left(\frac{\xi_1}{2}\right)\right]\operatorname{sech}^2\left(\frac{\xi_1}{2}\right) + \alpha, \\ B &= \widehat{f}A = A(-x + x_0, -t + t_0). \end{aligned} \tag{18}$$

Figure 1 shows the profile of the single soliton solution to the AB-BO system. The velocity of this solitary wave is equal to −3 after our choice of the free parameters. At the same time, we also know that the amplitude of the solitary wave increases with the increase of parameters $b_1$ and $b_2$.

For $N = 2$, equation (15) becomes

$$\begin{aligned} F &\equiv F_2 = K_{\{\}}\cosh\left(\frac{(\xi_1 + \xi_2)}{2}\right) + K_{\{1\}}\cosh\left(\frac{(\xi_1 - \xi_2)}{2}\right), \\ \xi_i &= k_i\left(x - \frac{x_0}{2}\right) + \omega_i\left(t - \frac{t_0}{2}\right), \quad i = 1, 2, \end{aligned} \tag{19}$$

where

$$\begin{aligned} K_{\{\}} &= \sqrt{k_1k_2\left(2\gamma k_1^2 + 2\gamma k_2^2 + 2\alpha\beta - 3\gamma k_1k_2\right) + \omega_1\omega_2}, \\ K_{\{1\}} &= \sqrt{k_1k_2\left(2\gamma k_1^2 + 2\gamma k_2^2 + 2\alpha\beta + 3\gamma k_1k_2\right) + \omega_1\omega_2}. \end{aligned} \tag{20}$$

The two-soliton solution is obtained by substituting equations (19) and (20) into equation (10):

$$\begin{aligned} A &= \frac{6\gamma}{\beta}(\ln F_2)_{xx} + b_1(\ln F_2)_{xxx} + b_2(\ln F_2)_{xxt} + \alpha, \\ B &= \widehat{f}A = A(-x + x_0, -t + t_0). \end{aligned} \tag{21}$$

After taking $\alpha = 4, \beta = \gamma = -1, \delta_1 = \delta_2 = k_1 = 1$, $k_2 = -(4/5), x_0 = t_0 = 0$, the two-soliton is generated by equation (21). The corresponding structures are plotted in Figure 2. Figure 2(a) shows the wave shape, wave velocity, and amplitude are unchanged after two solitons' head-on collision.

In earlier works [33], by some constraints to the parameters on the two solitons, a family of analytical breather solutions can be obtained. Inspired by this technique, we give the breather solution to equation (19) by setting

$$\begin{aligned} \alpha &= -4, \\ \beta &= \gamma = k_2 = -1, \\ \delta_1 &= \delta_2 = k_1 = 1, \\ x_0 &= t_0 = 0. \end{aligned} \tag{22}$$

Then, $F$ can be written as

$$F = 2\cos(\sqrt{7}\,t) + \sqrt{7}\cosh(x), \tag{23}$$

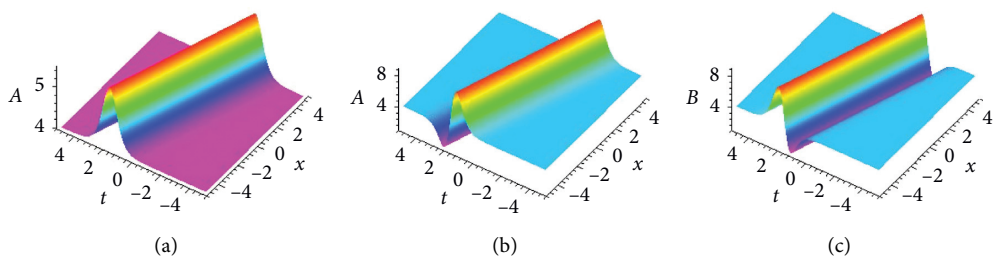



Figure 1: The single soliton solution (18) with the parameters $\alpha = 4, \beta = \gamma = -1, \delta_1 = k_1 = 1$, and $x_0 = t_0 = 0$: (a) $b_1 = b_2 = 0$; (b) and (c) $b_1 = b_2 = 10$.

and the corresponding $t$-breather solution is obtained which is shown in Figure 3.

After setting ($I$ is the imagine unit, $I^2 = 1$),

$$\begin{aligned} \alpha &= -4, \\ \beta &= \gamma = -1, \\ \delta_1 &= \delta_2 = 1, \\ k_1 &= \frac{3I}{2}, \\ k_2 &= -\frac{3I}{2}, \\ x_0 &= t_0 = 0. \end{aligned} \tag{24}$$

The function $F$ can be written as

$$F = \sqrt{41}\cos\left(\frac{3x}{2}\right) + 2\sqrt{17}\cosh\left(\frac{3\sqrt{41}t}{4}\right), \tag{25}$$

and the corresponding $x$-breather solution is obtained which is shown in Figure 4.

For $N = 3$, the function $F$ of equation (15) is described as

$$\begin{aligned} F \equiv F_3 = {} & K_{\{\}}\cosh\left\{\frac{1}{2}(\xi_1 + \xi_2 + \xi_3)\right\} + K_{\{1\}}\cosh\left\{\frac{1}{2}(\xi_1 - \xi_2 - \xi_3)\right\} \\ & + K_{\{2\}}\cosh\left\{\frac{1}{2}(\xi_1 - \xi_2 + \xi_3)\right\} + K_{\{3\}}\cosh\left\{\frac{1}{2}(\xi_1 + \xi_2 - \xi_3)\right\}, \end{aligned} \tag{26}$$

where

$$\begin{aligned} K_{\{\}} &= a_{12}^{-}a_{13}^{-}a_{23}^{-}, \\ K_{\{1\}} &= a_{12}^{+}a_{13}^{+}a_{23}^{-}, \\ K_{\{2\}} &= a_{12}^{+}a_{13}^{-}a_{23}^{+}, \\ K_{\{3\}} &= a_{12}^{-}a_{13}^{+}a_{23}^{+}, \end{aligned} \tag{27a}$$

$$\begin{aligned} \xi_i &= k_i\left(x - \frac{x_0}{2}\right) + \delta_i\sqrt{-2\alpha\beta k_i^2 - \gamma k_i^4}\left(t - \frac{t_0}{2}\right), \\ a_{ij}^{\pm} &= \sqrt{k_i k_j\left(2\gamma k_i^2 + 2\gamma k_2^2 + 2\alpha\beta \pm 3k_i k_j\right) + \omega_i\omega_j} \\ & \qquad (i, j = 1, 2, 3). \end{aligned} \tag{27b}$$

The three-soliton solution is obtained by substituting equations (26) and (27a) and (27b) into equation (10):

$$\begin{aligned} A &= \frac{6\gamma}{\beta}(\ln F_3)_{xx} + b_1(\ln F_3)_{xxx} + b_2(\ln F_3)_{xxt} + \alpha, \\ B &= \hat{f}A = A(-x + x_0, -t + t_0). \end{aligned} \tag{28}$$

After setting $\alpha = 4, \beta = \gamma = -1, \delta_1 = \delta_2 = \delta_3 = k_1 = 1$, $k_2 = -(4/5), k_3 = (6/5), x_0 = t_0 = 0$, the three solitons can be constructed through equation (28), and these related structures are plotted in Figure 5.

For $N = 4$, the function $F$ of equation (15) can be rewritten regularly as

$$\begin{aligned} F \equiv F_4 = {} & K_{\{\}}\cosh\left\{\frac{1}{2}(\xi_1 + \xi_2 + \xi_3 + \xi_4)\right\} + K_{\{1\}}\cosh\left\{\frac{1}{2}(-\xi_1 + \xi_2 + \xi_3 + \xi_4)\right\} \\ & + K_{\{2\}}\cosh\left\{\frac{1}{2}(\xi_1 - \xi_2 + \xi_3 + \xi_4)\right\} + K_{\{3\}}\cosh\left\{\frac{1}{2}(\xi_1 + \xi_2 - \xi_3 + \xi_4)\right\} \\ & + K_{\{4\}}\cosh\left\{\frac{1}{2}(\xi_1 + \xi_2 + \xi_3 - \xi_4)\right\} + K_{\{23\}}\cosh\left\{\frac{1}{2}(\xi_1 - \xi_2 - \xi_3 + \xi_4)\right\} \\ & + K_{\{24\}}\cosh\left\{\frac{1}{2}(\xi_1 - \xi_2 + \xi_3 - \xi_4)\right\} + K_{\{34\}}\cosh\left\{\frac{1}{2}(\xi_1 + \xi_2 - \xi_3 - \xi_4)\right\}, \end{aligned} \tag{29}$$

where

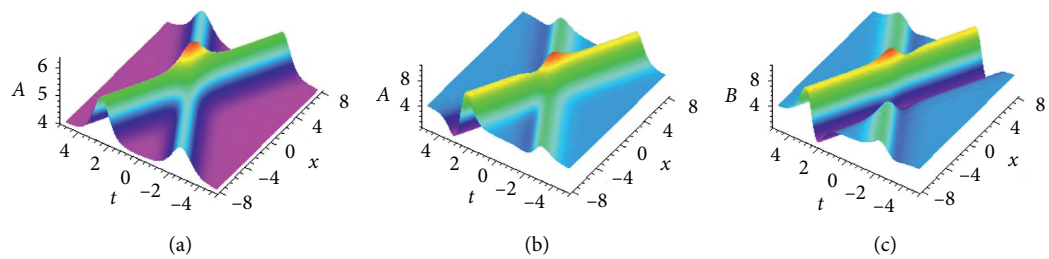



FIGURE 2: The two-soliton solution (21) with (a) $b_1 = b_2 = 0$; (b) and (c) $b_1 = b_2 = 10$.

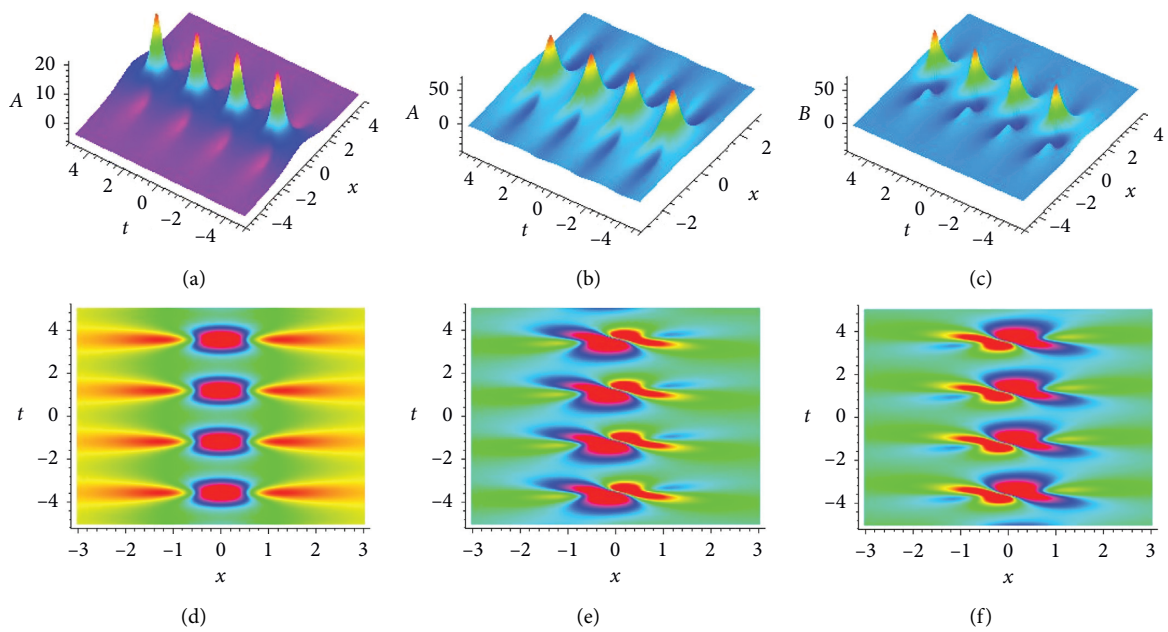



FIGURE 3: The $t$-breather solution of equation (23) with (a) $b_1 = b_2 = 0$; (b) and (c) $b_1 = b_2 = 5$. (d)–(f) The corresponding density plots of (a)–(c), respectively.

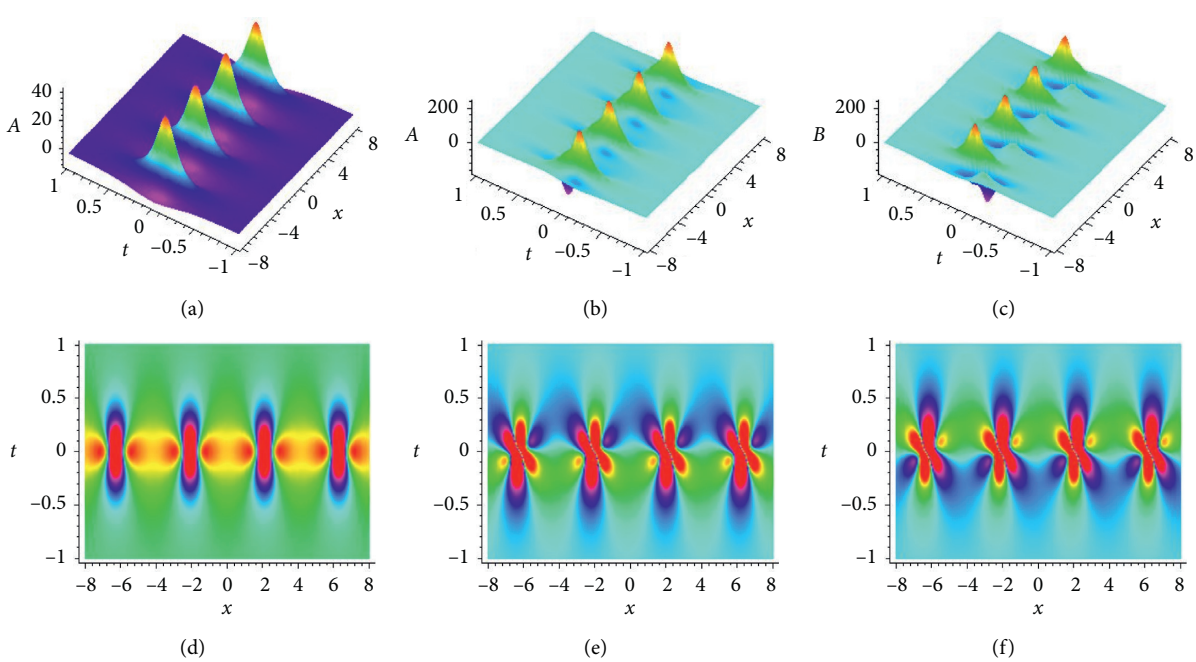
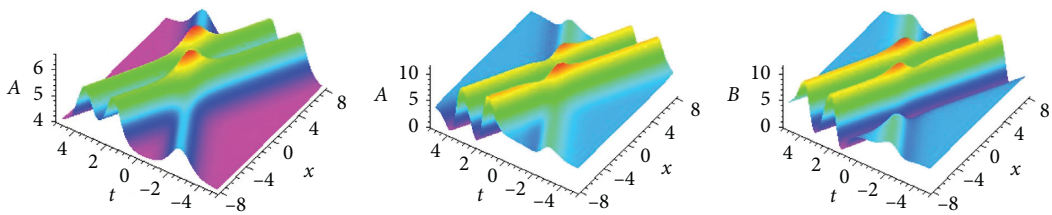



FIGURE 4: The $x$-breather solution of equation (25) with (a) $b_1 = b_2 = 0$; (b) and (c) $b_1 = b_2 = 5$. (d)–(f) The corresponding density plots of (a)–(c), respectively.




(a) (b) (c)

Figure 5: The three-soliton solution (28) with (a) $b_1 = b_2 = 0$; (b) and (c) $b_1 = b_2 = 10$.

$$K_{\{\}} = a_{12}^{-}a_{13}^{-}a_{14}^{-}a_{23}^{-}a_{24}^{-}a_{34}^{-}, \quad K_{\{1\}} = a_{12}^{+}a_{13}^{+}a_{14}^{+}a_{23}^{-}a_{24}^{-}a_{34}^{-},$$
$$K_{\{2\}} = a_{12}^{+}a_{13}^{-}a_{14}^{-}a_{23}^{+}a_{24}^{+}a_{34}^{-}, \quad K_{\{3\}} = a_{12}^{-}a_{13}^{+}a_{14}^{-}a_{23}^{+}a_{24}^{-}a_{34}^{+},$$
$$K_{\{4\}} = a_{12}^{-}a_{13}^{-}a_{14}^{+}a_{23}^{-}a_{24}^{+}a_{34}^{+}, \quad K_{\{23\}} = a_{12}^{+}a_{13}^{+}a_{14}^{-}a_{23}^{-}a_{24}^{+}a_{34}^{+},$$
$$K_{\{24\}} = a_{12}^{+}a_{13}^{-}a_{14}^{+}a_{23}^{+}a_{24}^{-}a_{34}^{+}, \quad K_{\{34\}} = a_{12}^{-}a_{13}^{+}a_{14}^{+}a_{23}^{+}a_{24}^{+}a_{34}^{-}, \tag{30}$$

$$\xi_i = k_i\left(x - \frac{x_0}{2}\right) + \delta_i\sqrt{-2\alpha\beta k_i^2 - \gamma k_i^4}\left(t - \frac{t_0}{2}\right),$$
$$a_{ij}^{\pm} = \sqrt{k_i k_j\left(2\gamma k_i^2 + 2\gamma k_2^2 + 2\alpha\beta \pm 3k_i k_j\right) + \omega_i\omega_j}, \quad (i, j = 1, 2, 3, 4). \tag{31}$$

The four-soliton solution is obtained by substituting equations (29) and (30) into equation (10):

$$A = \frac{6\gamma}{\beta}(\ln F_4)_{xx} + b_1(\ln F_4)_{xxx} + b_2(\ln F_4)_{xxt} + \alpha,$$
$$B = \hat{f}A = A(-x + x_0, -t + t_0). \tag{32}$$

After setting $\alpha = 4, \beta = \gamma = k_4 = -1, \delta_1 = \delta_2 = \delta_3 = \delta_4 = k_1 = 1, k_2 = -(4/5), k_3 = (6/5), x_0 = t_0 = 0$, the four solitons can be constructed through equation (32), and these related structures are plotted in Figure 6.

Similar to the two-soliton solution, we also give the second-order breather solution. In this case, we set the parameters in equation (29) as follows:

$$\alpha = -4,$$
$$\beta = \gamma = k_3 = -1,$$
$$\delta_1 = \delta_2 = \delta_3 = \delta_4 = k_1 = 1,$$
$$k_2 = \frac{1}{2},$$
$$k_4 = -\frac{1}{2},$$
$$x_0 = t_0 = 0. \tag{33}$$

Then, $F$ can be written as

$$F = K_{\{\}}\cos\left[\frac{4\sqrt{7} + \sqrt{31}}{4}t\right] + 2K_{\{1\}}\cos\frac{\sqrt{31}t}{4}\cosh(x)$$
$$+ 2K_{\{4\}}\cos(\sqrt{7}t)\cos\left(\frac{x}{2}\right)$$
$$+ K_{\{24\}}\cos\left[\frac{-4\sqrt{7} + \sqrt{31}}{4}t\right] + K_{\{23\}}\cosh\left(\frac{x}{2}\right)$$
$$+ K_{\{34\}}\cosh\left(\frac{3x}{2}\right). \tag{34}$$

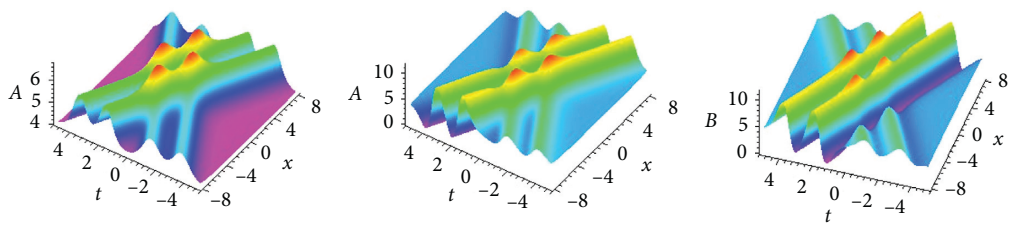
The corresponding second-order $t$-breather structures are depicted in Figure 7.

We set the parameters in equation (29) as follows:

$$\alpha = -4,$$
$$\beta = -1,$$
$$\gamma = -1,$$
$$\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1,$$
$$k_1 = I,$$
$$k_2 = \frac{I}{2},$$
$$k_3 = -I,$$
$$k_4 = -\frac{I}{2},$$
$$x_0 = t_0 = 0. \tag{35}$$

Then, $F$ can be written as

$$F = K_{\{\}}\cosh\left[\frac{12 + \sqrt{33}}{4}t\right] + 2K_{\{1\}}\cosh\frac{\sqrt{33}t}{4}\cos(x)$$
$$+ 2K_{\{2\}}\cos\left(\frac{x}{2}\right)\cosh(t)\left[4\cosh^2(t) - 3\right]$$
$$+ K_{\{24\}}\cosh\left[\frac{-12 + \sqrt{33}}{4}t\right] + K_{\{23\}}\cos\left(\frac{x}{2}\right) + K_{\{34\}}\cos\left(\frac{3x}{2}\right), \tag{36}$$




(a) (b) (c)

FIGURE 6: The four-soliton solution (32) with (a) $b_1 = b_2 = 0$; (b) and (c) $b_1 = b_2 = 10$.

and the corresponding second-order $x$-breather solution is obtained which is shown in Figure 8.

*3.2. Symmetry-Breaking Lump Solutions to the AB-BO System.* Based on the idea of generating the lump solution for one nonlinear equation, we derive this kind of solutions of AB-BO systems (7a) and (7b) with the function $F$ written as

$$F = F_N = \sum_{m=0}^{n(n+1)/2} \sum_{j=0}^{m} a_{j,m} \left(x - \frac{x_0}{2}\right)^{2j} \left(t - \frac{t_0}{2}\right)^{2m-2j} \tag{37}$$

with suitable constants $a_{j,m}$ [39].

For $N = 1$, equation (37) possesses the form

$$F_1 = a_{0,0} + a_{0,1}\left(t - \frac{t_0}{2}\right)^2 + a_{1,1}\left(x - \frac{x_0}{2}\right)^2, \tag{38}$$

where

$$\begin{aligned} a_{0,1} &= -\frac{4\alpha^2\beta^2 a_{0,0}}{3\gamma}, \\ a_{1,1} &= -\frac{2\alpha\beta a_{0,0}}{3\gamma}. \end{aligned} \tag{39}$$

After setting

$$\begin{aligned} t_0 &= x_0 = 0, \\ \alpha &= \beta = \gamma = -1, \\ a_{0,0} &= 1, \end{aligned} \tag{40}$$

the solution (10) becomes

$$\begin{aligned} A = &-\frac{16x\left(9 + 12t^2 - 2x^2\right)b_1}{\left(3 + 4t^2 + 2x^2\right)^3} - \frac{32t\left(3 - 6x^2 + 4t^2\right)b_2}{\left(3 + 4t^2 + 2x^2\right)^3} \\ &- \frac{16t^4 - 72t^2 + 16t^2x^2 - 63 + 60x^2 + 4x^4}{\left(3 + 4t^2 + 2x^2\right)^2}, \end{aligned} \tag{41a}$$

$$\begin{aligned} B = &\frac{16x\left(9 + 12t^2 - 2x^2\right)b_1}{\left(3 + 4t^2 + 2x^2\right)^3} + \frac{32t\left(3 - 6x^2 + 4t^2\right)b_2}{\left(3 + 4t^2 + 2x^2\right)^3} \\ &- \frac{16t^4 - 72t^2 + 16t^2x^2 - 63 + 60x^2 + 4x^4}{\left(3 + 4t^2 + 2x^2\right)^2}. \end{aligned} \tag{41b}$$

The invariant condition $F(x,t) = F(-x + x_0, -t + t_0)$ of this function (38) is satisfied. Figure 9(a) is a normal first-order lump structure ($b_1 = b_2 = 0$) for the solution $A = B = -(4x^4 + 60x^2 - 63/(3 + 2x^2)^2)$ at time $t = 0$. Figures 9(b) and 9(c) are two symmetry-breaking lump structures for the solution $A/B = -(4x^4 + 60x^2 - 63/(3 + 2x^2)^2) \mp (160x(9 - 2x^2)/(3 + 2x^2)^3)$, with the parameters $b_1 = b_2 = 10$ at time $t = 0$. As these solutions are all rational functions, these functions describe the symmetry-breaking lump structures.

For $N = 2$, the function $F$ of equation (37) can be rewritten regularly as

$$\begin{aligned} F_2 = &a_{0,0} + a_{0,1}t^2 + a_{1,1}x^2 + a_{0,2}t^4 + a_{1,2}x^2t^2 + a_{2,2}x^4 \\ &+ a_{0,3}t^6 + a_{1,3}x^2t^4 + a_{2,3}x^4t^2 + a_{3,3}x^6, \end{aligned} \tag{42}$$

where

$$\begin{aligned} a_{0,1} &= -\frac{76\alpha^2\beta^2 a_{0,0}}{75\gamma}, \\ a_{0,2} &= \frac{272\alpha^4\beta^4 a_{0,0}}{1875\gamma^2}, \\ a_{0,3} &= -\frac{\alpha^6\beta^6 a_{0,0}}{1875\gamma^3}, \\ a_{1,1} &= \frac{2\alpha\beta a_{0,0}}{15\gamma}, \\ a_{1,2} &= \frac{48\alpha^3\beta^3 a_{0,0}}{125\gamma^2}, \\ a_{1,3} &= -\frac{32\alpha^5\beta^5 a_{0,0}}{625\gamma^3}, \\ a_{2,2} &= \frac{4\alpha^2\beta^2 a_{0,0}}{75\gamma^2}, \\ a_{2,3} &= -\frac{16\alpha^4\beta^4 a_{0,0}}{625\gamma^3}, \\ a_{3,3} &= -\frac{8\alpha^3\beta^3 a_{0,0}}{1875\gamma^3}. \end{aligned} \tag{43}$$

After setting

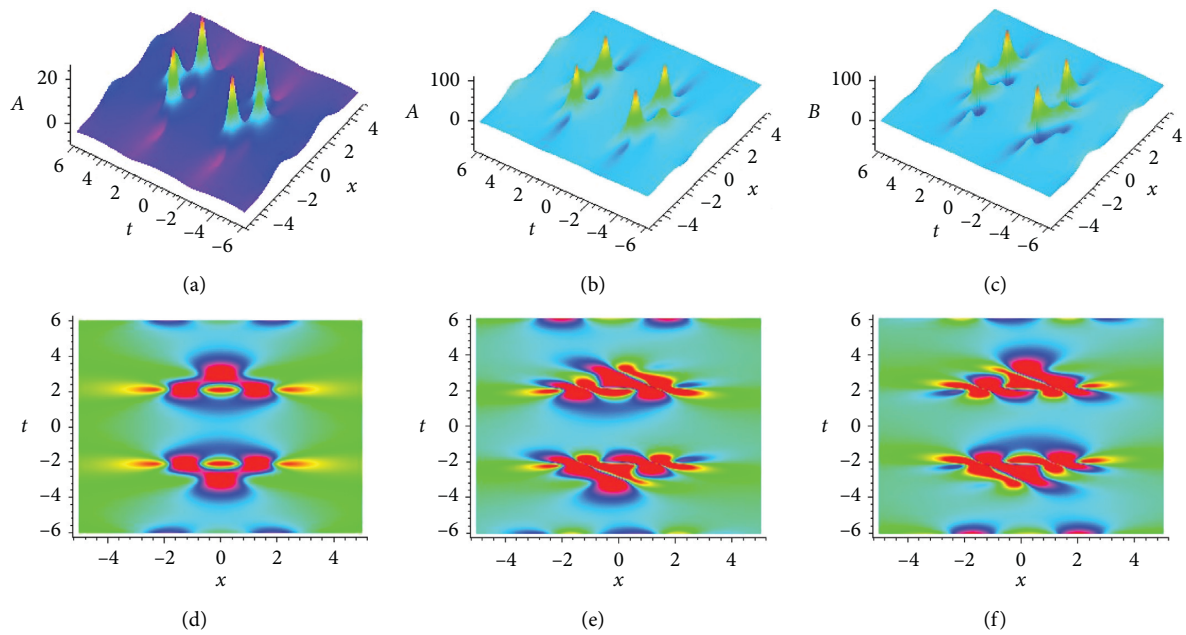



Figure 7: The second-order $t$-breather solution of equation (34) with (a) $b_1 = b_2 = 0$; (b) and (c) $b_1 = b_2 = 5$. (d)–(f) The corresponding density plots of (a)–(c), respectively.

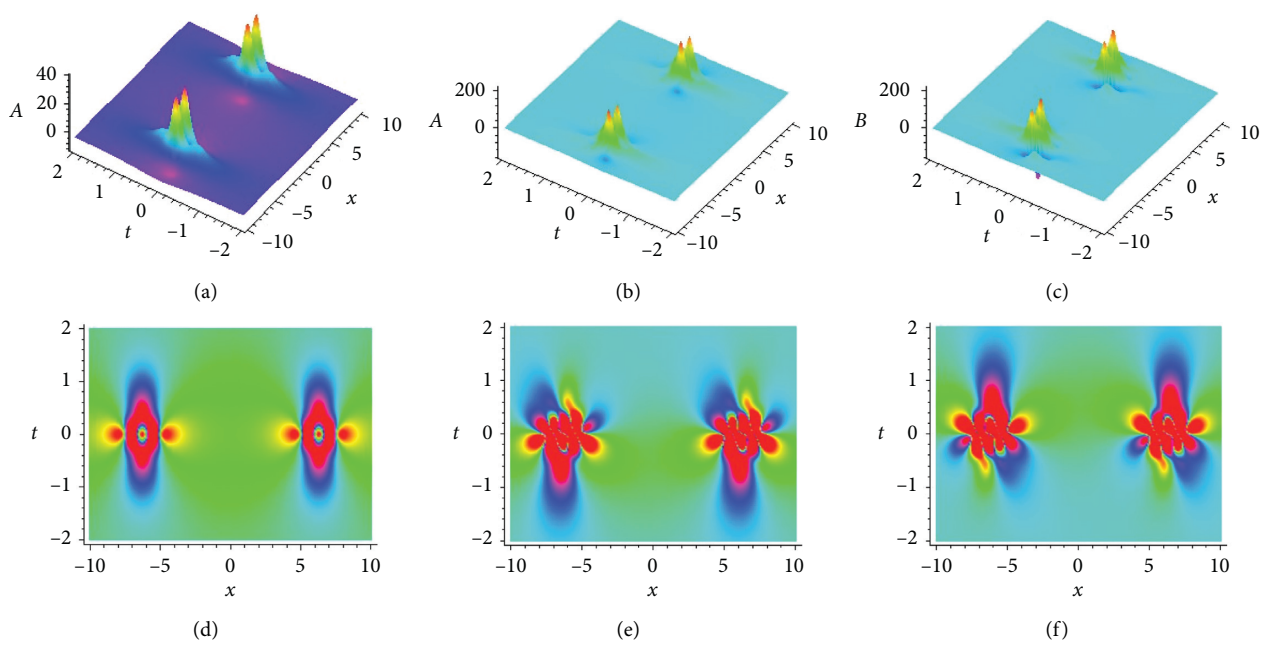



Figure 8: The second-order $x$-breather solution of equation (36) with (a) $b_1 = b_2 = 0$; (b) and (c) $b_1 = b_2 = 5$. (d)–(f) The corresponding density plots of (a)–(c), respectively.

$$\begin{aligned} t_0 &= x_0 = 0, \\ \alpha &= \beta = \gamma = -1, \\ a_{0,0} &= 1, \end{aligned} \tag{44}$$

we have

$$\begin{aligned} a_{0,1} &= \frac{76}{75}, \\ a_{0,2} &= \frac{272}{1875}, \\ a_{0,3} &= \frac{64}{1875}, \\ a_{1,1} &= -\frac{2}{15}, \\ a_{1,2} &= \frac{48}{125}, \\ a_{1,3} &= \frac{32}{625}, \\ a_{2,2} &= \frac{4}{75}, \\ a_{2,3} &= \frac{16}{625}, \\ a_{3,3} &= \frac{8}{1875}. \end{aligned} \tag{45}$$

Figure 10 shows the second-order lump structure under the invariant condition $F(x,t) = F(-x + x_0, -t + t_0)$ for the solution of AB-BO systems (7a) and (7b).

For $N = 3$, the function $F$ of equation (37) is

$$\begin{aligned} F_3 = {} & a_{0,1}t^2 + a_{1,1}x^2 + a_{0,2}t^4 + a_{1,2}x^2t^2 + a_{2,2}x^4 + a_{0,3}t^6 \\ & + a_{1,3}x^2t^4 + a_{2,3}x^4t^2 + a_{3,3}x^6 + a_{0,4}t^8 \\ & + a_{1,4}x^2t^6 + a_{2,4}x^4t^4 + a_{3,4}x^6t^2 + a_{4,4}x^8 + a_{0,5}t^{10} \\ & + a_{1,5}x^2t^8 + a_{2,5}x^4t^6 + a_{3,5}x^6t^4 + a_{4,5}x^8t^2 \\ & + a_{5,5}x^{10} + a_{0,6}t^{12} + a_{1,6}x^2t^{10} + a_{2,6}x^4t^8 + a_{3,6}x^6t^6 \\ & + a_{4,6}x^8t^4 + a_{5,6}x^{10}t^2 + a_{6,6}x^{12} + a_{0,0}. \end{aligned} \tag{46}$$

The constrained constants are

$$\begin{aligned} a_{0,1} &= -\frac{3480\alpha^2\beta^2 a_{0,0}}{847\gamma}, \\ a_{0,2} &= \frac{642288\alpha^4\beta^4 a_{0,0}}{717409\gamma^2}, \\ a_{0,3} &= -\frac{4382976\alpha^6\beta^6 a_{0,0}}{25109315\gamma^3}, \\ a_{0,4} &= \frac{1997568\alpha^8\beta^8 a_{0,0}}{175765205\gamma^4}, \\ a_{0,5} &= -\frac{534528\alpha^{10}\beta^{10} a_{0,0}}{878826025\gamma^5}, \\ a_{0,6} &= \frac{36864\alpha^{12}\beta^{12} a_{0,0}}{878826025\gamma^6}, \\ a_{1,1} &= -\frac{12\alpha\beta a_{0,0}}{11\gamma}, \\ a_{1,2} &= \frac{432\alpha^3\beta^3 a_{0,0}}{9317\gamma^2}, \\ a_{1,3} &= \frac{3456\alpha^5\beta^5 a_{0,0}}{717409\gamma^3}, \\ a_{1,4} &= \frac{105984\alpha^7\beta^7 a_{0,0}}{2282665\gamma^4}, \\ a_{1,5} &= -\frac{525312\alpha^9\beta^9 a_{0,0}}{175765205\gamma^5}, \\ a_{1,6} &= \frac{110592\alpha^{11}\beta^{11} a_{0,0}}{878826025\gamma^6}, \\ a_{2,2} &= -\frac{60\alpha^2\beta^2 a_{0,0}}{847\gamma^2}, \\ a_{2,3} &= -\frac{25920\alpha^4\beta^4 a_{0,0}}{717409\gamma^3}, \\ a_{2,4} &= \frac{123264\alpha^6\beta^6 a_{0,0}}{5021863\gamma^4}, \\ a_{2,5} &= -\frac{672768\alpha^8\beta^8 a_{0,0}}{175765205\gamma^5}, \\ a_{2,6} &= \frac{27648\alpha^{10}\beta^{10} a_{0,0}}{175765205\gamma^6}, \\ a_{3,3} &= -\frac{96\alpha^3\beta^3 a_{0,0}}{46585\gamma^3}, \\ a_{3,4} &= \frac{21888\alpha^5\beta^5 a_{0,0}}{3587045\gamma^4}, \\ a_{3,5} &= -\frac{4608\alpha^7\beta^7 a_{0,0}}{2282665\gamma^5}, \\ a_{3,6} &= \frac{18432\alpha^9\beta^9 a_{0,0}}{175765205\gamma^6}, \\ a_{4,4} &= \frac{432\alpha^4\beta^4 a_{0,0}}{3587045\gamma^4}, \\ a_{4,5} &= -\frac{79488\alpha^6\beta^6 a_{0,0}}{175765205\gamma^5}, \\ a_{4,6} &= \frac{6912\alpha^8\beta^8 a_{0,0}}{175765205\gamma^6}, \\ a_{5,5} &= -\frac{576\alpha^5\beta^5 a_{0,0}}{17935225\gamma^5}, \\ a_{5,6} &= \frac{6912\alpha^7\beta^7 a_{0,0}}{878826025\gamma^6}, \\ a_{6,6} &= \frac{576\alpha^6\beta^6 a_{0,0}}{878826025\gamma^6}. \end{aligned} \tag{47}$$

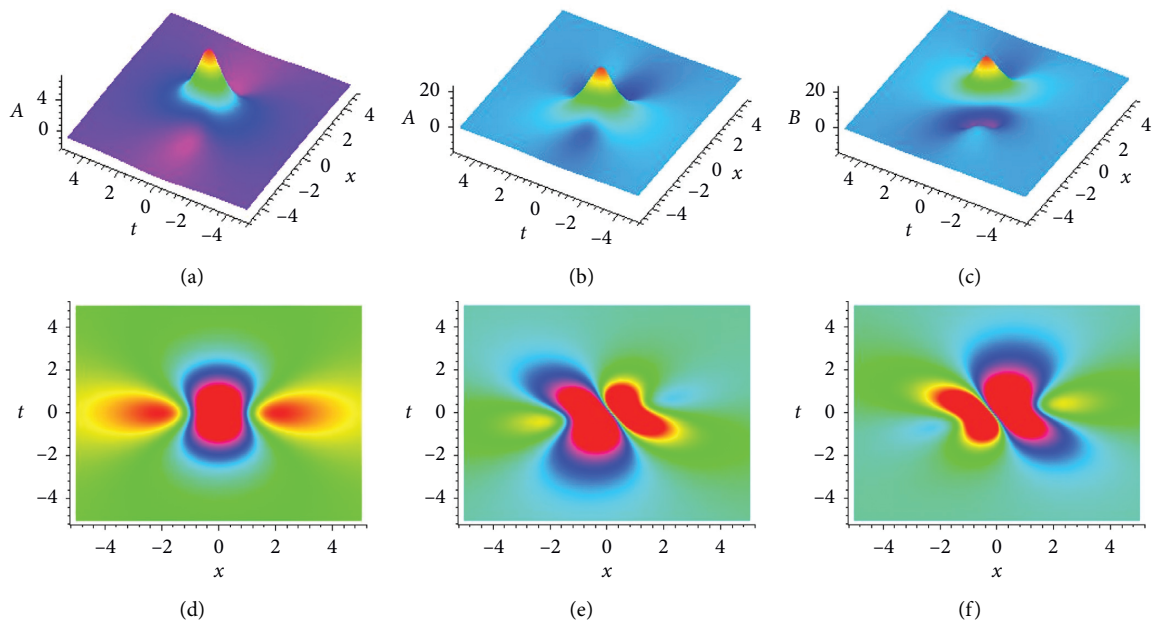



Figure 9: The one-order symmetry-breaking lump solutions (41a) and (41b) with (a) $b_1 = b_2 = 0$; (b) and (c) $b_1 = b_2 = 10$. (d)–(f) The corresponding density plots of (a)–(c), respectively.

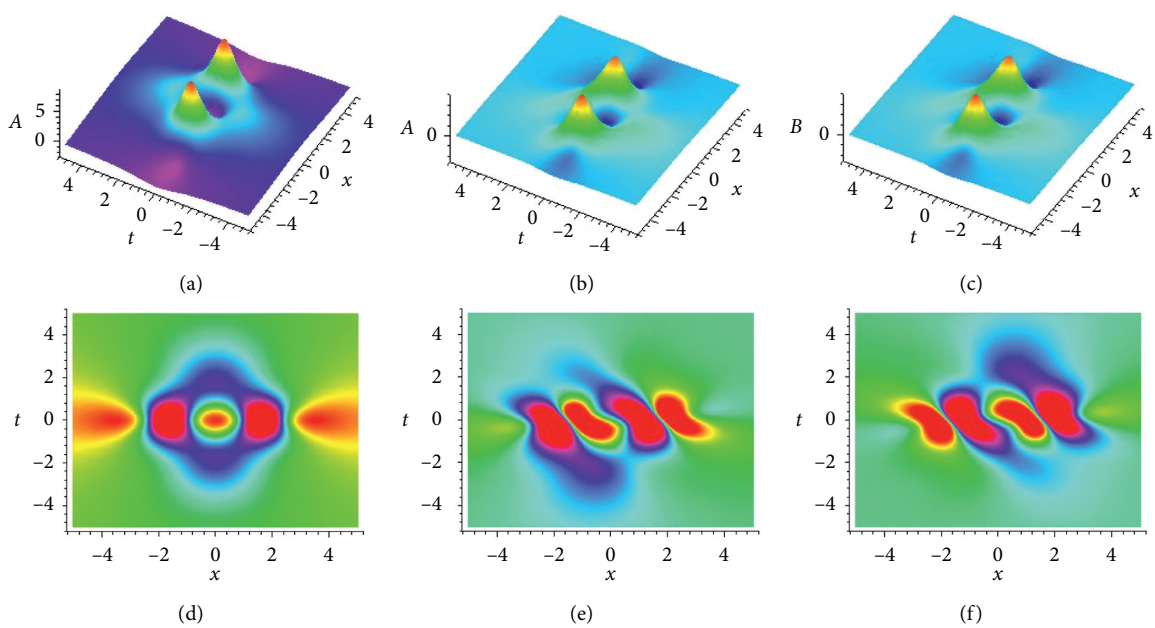



Figure 10: The second-order symmetry-breaking lump solution under the invariant condition $F(x,t) = F(-x + x_0, -t + t_0)$ of AB-BO systems (7a) and (7b) with (a) $b_1 = b_2 = 0$; (b) and (c) $b_1 = b_2 = 10$. (d)–(f) The corresponding density plots of (a)–(c), respectively.

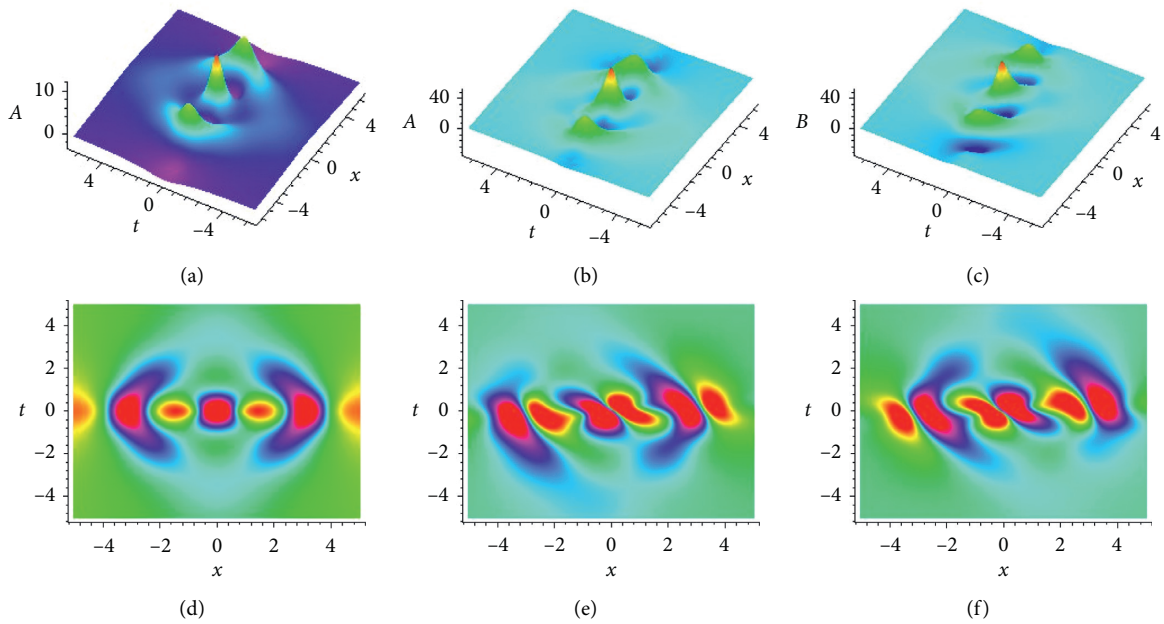



Figure 11: The third-order symmetry-breaking lump solution under the invariant condition $F(x,t) = F(-x + x_0, -t + t_0)$ of AB-BO systems (7a) and (7b) with (a) $b_1 = b_2 = 0$; (b) and (c) $b_1 = b_2 = 10$. (d)–(f) The corresponding density plots of (a)–(c), respectively.

Figure 11 shows the third-order lump structures when $t_0 = x_0 = 0, \alpha = \beta = \gamma = -1, a_{0,0} = 1$, which are symmetry breaking.

## 4. Summary and Conclusion

It is believed that the two-place correlated physical events widely exist in the field of natural science, and discussing AB physics has a profound influence on other scientific fields. In this article, we studied the nonlocal BO equation coupled with an AB system. First of all, one established a special AB-BO system via the parity with a shift of the space variable and time reversal with a delay. At the same time, with the derived extended Bäcklund transformation and the corresponding Hirota bilinear form, the symmetry-breaking soliton, symmetry-breaking breather, and symmetry-breaking lump solutions were presented. Finally, by choosing special parameters, these solutions of the AB-BO system were discussed in detail. The coefficients with shifted parity and delayed time reversal in the nonlocal AB-BO system were discussed, from which the abundant symmetry-breaking solutions were illustrated by changing parameters of events $A$ and $B$.

## Data Availability

The data used to support the findings of this study are included within the article. For more details, the data are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Acknowledgments

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