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Research Article

Finite-Time H_2/H_∞ Control Design for Stochastic Poisson Systems with Applications to Clothing Hanging Device

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In this paper, the design of finite-time H_2/H_∞ controller for linear Itô stochastic Poisson systems is considered. First, the definition of finite-time H_2/H_∞ control is proposed, which considers the transient performance, H_2 index, and H_∞ index simultaneously in a predetermined finite-time interval. Then, the state feedback and observer-based finite-time H_2/H_∞ controllers are presented and some new sufficient conditions are obtained. Moreover, an algorithm is given to optimize H_2 and H_∞ index, simultaneously. Finally, a simulation example indicates the effectiveness of the results.

1. Introduction

It is known to all that stochastic systems have been studied extensively and applied to biological network [1], power systems [2], financial systems [3, 4], and other fields. There are also many other applications of stochastic systems (see, e.g., [5–7]). In the past few decades, stochastic systems driven by Wiener noise have been widely investigated. For example, Shaikin [8] solved the optimization problem for multiplicative stochastic systems with several external disturbances and vector Wiener processes. Xiang et al. [9] introduced the finite-time properties and state feedback H_∞ control problem for switched stochastic systems with Wiener noise. Yan et al. [10] were concerned with finite-time H_2 control of the Markovian stochastic systems with Wiener noise. However, in the real world, an actual physical system is inevitably affected by Wiener noise and Poisson jump noise. At present, some achievements have been made in the research of stochastic Poisson systems (see, e.g., [11–14]).

On the other hand, H_2/H_∞ optimization control is one of the most important problems in the controlled system. The H_2 optimal control system has good system performance, while the H_∞ control theory can deal with the system robustness problem well. In view of this, Bernstein and Haddad [15] proposed the H_2/H_∞ mixed control problem, which can solve both the problems of system performance and robustness. Since then, the H_2/H_∞ control has been developed and used extensively (see, e.g., [16–19]). Besides, in some engineering research, such as communication system [20–23], robotic operating system [24], and industrial production system [25], more attention should be paid to the system transient performance. In order to describe system transient performance clearly, the concepts of finite-time stability (FTS) and finite-time boundedness (FTB) are proposed, which reflect the specific system behavior in a relatively short time interval. Nowadays, the problems of FTS and FTB have been deeply investigated (see, e.g., [26–36]). In consideration of the merits of FTB and H_2/H_∞ control, the finite-time H_2/H_∞ control for stochastic systems with Wiener noise is first presented in [37], which satisfies both FTB and H_2/H_∞ performance index. However, in many practical systems, it is not only disturbed by Wiener noise, but also by Poisson noise. So far, there are few literature studies to investigate this problem of stochastic Poisson systems affected by both Wiener and Poisson noises.

Motivated by aforementioned discussions, the problems of finite-time H_2/H_∞ control for stochastic Poisson systems with both Wiener noise and Poisson noise are considered in this paper. The main work of this paper consists of the following three aspects:

(i) Unlike the model considered in [37], this paper studies the model of stochastic Poisson systems with Wiener and Poisson noises. The former considers only Wiener noise, and the latter considers both Wiener and Poisson noises. Moreover, in the former model, the measurement output $y(t)$ is composed of only the state, but the measurement output considered in the latter model is composed of both the state and external interference. The latter model is more general than the former model in [37], which is used to model many real systems.

(ii) The two theorems (Theorems 2 and 4) are obtained to guarantee the existence of state feedback finite-time (SFFT) and observer-based finite-time (OBFT) H_2/H_∞ controllers, respectively. The two theorems (Theorems 2 and 4) contain the parameters both α and Poisson jump intensity λ, which are complex than the corresponding conditions in [37]. By adjusting the two parameters, the most satisfying finite-time H_2/H_∞ controllers will be designed.

(iii) A new optimization algorithm constrained by matrix inequality is proposed to demonstrate the relationships among α, λ, and optimal H_2/H_∞ index, which is more complex than that in [37].

Notations: the notations presented in this work are standard. For specific contents, one can refer to [37].

2. Preliminaries

Consider a continuous-time stochastic Poisson system

$$\begin{cases} dx(t) = [A_{11}x(t) + B_{11}v(t) + F_1 r(t)]\mathrm{d}t + [A_{12}x(t) + B_{12}v(t) + F_2 r(t)]\mathrm{d}\mathscr{W}(t) \\ +[A_{13}x(t) + B_{13}v(t) + F_3 r(t)]\mathrm{d}\mathscr{N}(t), \\ y(t) = C_{11}x(t) + D_{11}r(t), \\ z(t) = C_{12}x(t) + D_{12}v(t), \\ x(0) = x_0 \in R^n, \end{cases} \tag{1}$$

where A_{11}, A_{12}, A_{13}, B_{11}, B_{12}, B_{13}, C_{11}, C_{12}, D_{11}, D_{12}, F_1, F_2, and F_3 are known constant matrices. $x(t) \in \mathscr{R}^l$, $y(t) \in \mathscr{R}^q$, $z(t) \in \mathscr{R}^s$, and $v(t) \in \mathscr{R}^n$ are the state vector, measurement output, control output, and control input, respectively. x_0 is the initial condition of the system. $\mathscr{W}(t)$ presents one-dimensional standard Wiener process and $\mathscr{N}(t)$ is the marked Poisson process with Poisson jump intensity λ. $r(t) \in \mathscr{R}^p$ is the disturbance input which satisfies the following equation:

$$\mathscr{E}\int_0^t r'(s)r(s)\mathrm{d}s < f, \quad (f > 0). \tag{2}$$

Next, the definition of mean-square FTB of system (1) is introduced.

Definition 1. Given some scalars $b_2 > b_1 > 0$ and $T > 0$ and a matrix $R > 0$, the above stochastic system (1) with $v(t) \equiv 0$ is mean-square FTB w.r.t. (b_1, b_2, T, R, f), if

$$\mathscr{E}[x'(0)Rx(0)] \le b_1 \Longrightarrow \mathscr{E}[x'(t)Rx(t)] < b_2, \quad \forall t \in [0, T]. \tag{3}$$

Remark 1. From Definition 1, we can know that the concept of FTB describes the specific behavior of the stochastic system (1) in a prescribed time interval.

Lemma 1 (see [38]). *Let $\overline{\mathscr{V}}(t,x) \in C^2(R^1, R^n)$ and $\overline{\mathscr{V}}(t,x) > 0$. Consider the following system*

$$\mathrm{d}x(t) = A_1(x)\mathrm{d}t + A_2(x)\mathrm{d}\mathscr{W}(t) + A_3(x)\mathrm{d}\mathscr{N}(t), \tag{4}$$

its stochastic differential of $\overline{\mathscr{V}}(t,x)$ is given by

$$\mathrm{d}\overline{\mathscr{V}}(t,x) = \mathscr{L}\overline{\mathscr{V}}(t,x)\mathrm{d}t + \frac{\partial\overline{\mathscr{V}}'(t,x)}{\partial x}A_2(x)\mathrm{d}\mathscr{W}(t) + \left[\overline{\mathscr{V}}(t, x + A_3(x)) - \overline{\mathscr{V}}(t,x)\right]\mathrm{d}\mathscr{N}(t), \tag{5}$$

where

$$\mathscr{L}\overline{\mathscr{V}}(t,x) = \frac{\partial\overline{\mathscr{V}}(t,x)}{\partial t} + \frac{\partial\overline{\mathscr{V}}(t,x)}{\partial x}A_1(x) + \frac{1}{2}A_2'(x)\frac{\partial^2\overline{\mathscr{V}}(t,x)}{\partial x^2}A_2(x) + \lambda\left[\overline{\mathscr{V}}(t, x + A_3(x)) - \overline{\mathscr{V}}(t,x)\right]. \tag{6}$$

3. Design of SFFT H_2/H_∞ Controller

In this section, a SFFT H_2/H_∞ controller for system (1) is designed. Consider a linear SF controller

$$v(t) = Kx(t), \tag{7}$$

where K is the required SF gain matrix.

Substituting (7) into (1), the following closed-loop system is obtained:

$$\begin{cases} \mathrm{d}x(t) = \left[\widetilde{A}_{11}x(t) + F_1 r(t)\right]\mathrm{d}t + \left[\widetilde{A}_{12}x(t) + F_2 r(t)\right]\mathrm{d}\mathscr{W}(t) + \left[\widetilde{A}_{13}x(t) + F_3 r(t)\right]\mathrm{d}\mathscr{N}(t), \\ y(t) = C_{11}x(t) + D_{11}r(t), \\ z(t) = \widetilde{C}x(t), \\ x(0) = x_0 \in R^n, \end{cases} \tag{8}$$

where $\widetilde{A}_{11} = A_{11} + B_{11}K$, $\widetilde{A}_{12} = A_{12} + B_{12}K$, $\widetilde{A}_{13} = A_{13} + B_{13}K$, and $\widetilde{C} = C_{12} + D_{12}K$.

Next, we choose the following H_2 cost function:

$$J_1(x(t), v(t)) = \mathscr{E}\int_0^T \left[x'(t)G_1x(t) + v'(t)G_2v(t)\right]\mathrm{d}t, \tag{9}$$

where $G_1 > 0$ and $G_2 > 0$ are known weighting scalars or positive matrices.

Similarly, substituting the SF controller (7) into (9), the following formula is obtained:

$$J_1(x(t)) = \mathscr{E}\int_0^T \left[x'(t)G_1x(t) + x'(t)K'G_2Kx(t)\right]\mathrm{d}t. \tag{10}$$

Given $\gamma > 0$ and assuming zero initial condition, the control output $z(t)$ and the disturbance input $r(t)$ satisfy the following equation:

$$\mathscr{E}\int_0^T z'(t)z(t)\mathrm{d}t < \gamma^2 \mathscr{E}\int_0^T r'(t)r(t)\mathrm{d}t. \tag{11}$$

Based on the above preparations, the definition of the SFFT H_2/H_∞ controller is introduced.

Definition 2. Given positive scalars b_1, b_2, T, and f and a matrix $R > 0$. If a positive scalar J_1^* exists, a SF controller (7) can be designed to make the following conditions hold:

(i) The closed-loop system (8) is mean-square FTB w.r.t. (b_1, b_2, T, R, f)

(ii) The H_2 cost function (10) meets $J_1(x(t)) \le J_1^*$ under $r(t) = 0$ condition

(iii) Assuming that the initial state is zero and the nonzero disturbance input and the control output satisfy inequality (11); then (7) is the SFFT H_2/H_∞ controller for system (1)

Remark 2. Definition 3 implies that a SFFT H_2/H_∞ controller not only makes the closed-loop system FTB, but also gets minimum performance cost and better interference suppression capability. In actual systems, these three aspects really need to be considered. For example, in industrial steel rolling heating furnace, excessive instantaneous furnace temperature cannot be permitted. Moreover, it is hoped that the fuel consumption is less and the anti-interference ability is stronger in the rolling furnace.

Next, the following theorem is given for obtaining the SFFT H_2/H_∞ controller.

Theorem 1. *Given positive scalars b_1, b_2, T, and f and a matrix $R > 0$, if there exist a nonnegative scalar α and two matrices $N > 0$ and K such that*

$$\begin{bmatrix} \mathscr{T}_1 & F_1 & \widetilde{N}\widetilde{A}_{12}' & \sqrt{\lambda}\,\widetilde{N}\left(\widetilde{A}_{13} + I\right)' \\ * & -\gamma^2 I & F_2' & \sqrt{\lambda}\,F_3' \\ * & * & -\widetilde{N} & 0 \\ * & * & * & -\widetilde{N} \end{bmatrix} < 0, \tag{12}$$

$$\begin{bmatrix} \mathscr{T}_2 & \widetilde{N}\widetilde{A}_{12}' & \sqrt{\lambda}\,\widetilde{N}\left(\widetilde{A}_{13} + I\right)' \\ * & -\widetilde{N} & 0 \\ * & * & -\widetilde{N} \end{bmatrix} < 0, \tag{13}$$

$$\frac{b_1}{\lambda_{\min}(N)} + f\gamma^2 < \frac{b_2}{\lambda_{\max}(N)}e^{-\alpha T}, \tag{14}$$

hold, where $\widetilde{N} = R^{-1/2}NR^{-1/2}$, $\mathscr{T}_1 = \widetilde{A}_{11}\widetilde{N} + \widetilde{N}\widetilde{A}_{11}' - \lambda\widetilde{N} - \alpha\widetilde{N} + \widetilde{N}'\widetilde{C}'\widetilde{C}\widetilde{N}$, and $\mathscr{T}_2 = \widetilde{A}_{11}\widetilde{N} + \widetilde{N}\widetilde{A}_{11}' - \lambda\widetilde{N} - \alpha\widetilde{N} + \widetilde{N}G_1\widetilde{N} + \widetilde{N}K'G_2K\widetilde{N}$, then $v(t) = Kx(t)$ is said to be a SFFT H_2/H_∞ controller and we can get the upper bound of H_2 index, that is, $J_{state}^* = \lambda_{max}(N^{-1})b_1e^{\alpha T}$.

Proof. Here are three steps to prove Theorem 1.

Step 1: prove that system (3) is mean-square FTB.

Obviously,

$$\begin{bmatrix} \widetilde{N}\widetilde{C}'\widetilde{C}\widetilde{N} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \widetilde{N}\widetilde{C}' \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \widetilde{N}\widetilde{C}' \\ 0 \\ 0 \\ 0 \end{bmatrix}' \ge 0. \tag{15}$$

Therefore, condition (12) means

$$\begin{bmatrix} \widetilde{\mathcal{T}}_1 & F_1 & \widetilde{N}\widetilde{A}_{12}' & \sqrt{\lambda}\,\widetilde{N}\left(\widetilde{A}_{13}+I\right)' \\ * & -\gamma^2 I & F_2' & \sqrt{\lambda}\,F_3' \\ * & * & -\widetilde{N} & 0 \\ * & * & * & -\widetilde{N} \end{bmatrix} < 0, \tag{16}$$

where $\widetilde{\mathcal{T}}_1 = \widetilde{A}_{11}\widetilde{N} + \widetilde{N}\widetilde{A}_{11}' - \lambda\widetilde{N} - \alpha\widetilde{N}$.

Let $\overline{\mathcal{V}}(x(t)) = x'(t)\widetilde{N}^{-1}x(t)$, and applying Lemma 1 for $\overline{\mathcal{V}}(x(t))$, the $\mathcal{L}_1\overline{\mathcal{V}}(x(t))$ of system (8) is given by

$$\mathcal{L}_1\overline{\mathcal{V}}(x(t)) = \begin{bmatrix} x(t) \\ r(t) \end{bmatrix}' \begin{bmatrix} \overline{Z}_1 & \overline{Z}_2 \\ * & \overline{Z}_3 \end{bmatrix} \begin{bmatrix} x(t) \\ r(t) \end{bmatrix}, \tag{17}$$

where $\overline{Z}_1 = \widetilde{A}_{11}'\widetilde{N}^{-1} + \widetilde{N}^{-1}\widetilde{A}_{11} + \widetilde{A}_{12}'\widetilde{N}^{-1}\widetilde{A}_{12} + \lambda(\widetilde{A}_{13}+I)'\widetilde{N}^{-1}(\widetilde{A}_{13}+I) - \lambda\widetilde{N}^{-1}$, $\overline{Z}_2 = \lambda(\widetilde{A}_{13}+I)'\widetilde{N}^{-1}F_3 + \widetilde{A}_{12}'\widetilde{N}^{-1}F_2 + \widetilde{N}^{-1}F_1$, and $\overline{Z}_3 = \lambda F_3'\widetilde{N}^{-1}F_3 + F_2'\widetilde{N}^{-1}F_2$.

Pre- and postmultiplying (16) by $\operatorname{diag}\{\widetilde{N}^{-1}, I, \widetilde{N}^{-1}, \widetilde{N}^{-1}\}$, we can get the following inequality:

$$\begin{bmatrix} \mathcal{I}_1 & \widetilde{N}^{-1}F_1 & \widetilde{A}_{12}'\widetilde{N}^{-1} & \mathcal{I}_2 \\ * & -\gamma^2 I & F_2'\widetilde{N}^{-1} & \sqrt{\lambda}\,F_3'\widetilde{N}^{-1} \\ * & * & -\widetilde{N}^{-1} & 0 \\ * & * & * & -\widetilde{N}^{-1} \end{bmatrix} < 0, \tag{18}$$

where $\mathcal{I}_1 = \widetilde{A}_{11}'\widetilde{N}^{-1} + \widetilde{N}^{-1}\widetilde{A}_{11} - \lambda\widetilde{N}^{-1} - \alpha\widetilde{N}^{-1}$ and $\mathcal{I}_2 = \sqrt{\lambda}(\widetilde{A}_{13}+I)'\widetilde{N}^{-1}$.

By utilizing Schur complement, (18) is equivalent to

$$\begin{bmatrix} \overline{Z}_1 - \alpha\widetilde{N}^{-1} & \overline{Z}_2 \\ * & \overline{Z}_3 - \gamma^2 I \end{bmatrix} < 0. \tag{19}$$

Taking conditions (17) and (19) into consideration, it follows

$$\mathcal{L}_1\overline{\mathcal{V}}(x(t)) < \alpha\overline{\mathcal{V}}(x(t)) + \gamma^2 r'(t)r(t). \tag{20}$$

Integrating from 0 to t on both sides of (20), then taking mathematical expectation, one has

$$\mathcal{E}\overline{\mathcal{V}}(x(t)) < \mathcal{E}\overline{\mathcal{V}}(x(0)) + \alpha\int_0^t \mathcal{E}\overline{\mathcal{V}}(x(s))\mathrm{d}s + \gamma^2\int_0^t \mathcal{E}r'(s)r(s)\mathrm{d}s. \tag{21}$$

Utilizing Gronwall inequality in [26], it follows

$$\mathcal{E}\overline{\mathcal{V}}(x(t)) < \mathcal{E}\overline{\mathcal{V}}(x(0))e^{\alpha t} + \gamma^2 e^{\alpha t}\int_0^t \mathcal{E}r'(s)r(s)\mathrm{d}s. \tag{22}$$

On the basis of above conditions, we have

$$\begin{aligned}\mathcal{E}\overline{\mathcal{V}}(x(t)) &= \mathcal{E}\left[x'(t)R^{1/2}N^{-1}R^{1/2}x(t)\right] \ge \lambda_{\min} \\ &\quad\cdot\left(N^{-1}\right)\mathrm{E}\left[x'(t)Rx(t)\right],\end{aligned} \tag{23}$$

$$\begin{aligned}\mathcal{E}\overline{\mathcal{V}}(x(0))e^{\alpha t} &= \mathcal{E}\left[x'(0)R^{1/2}N^{-1}R^{1/2}x(0)\right]e^{\alpha t} \\ &\le \lambda_{\max}\left(N^{-1}\right)\mathrm{E}\left[x'(0)Rx(0)e^{\alpha t}\right] \\ &\le \lambda_{\max}\left(N^{-1}\right)b_1 e^{\alpha T},\end{aligned} \tag{24}$$

$$\gamma^2 e^{\alpha t}\int_0^t \mathcal{E}r'(s)r(s)\mathrm{d}s < e^{\alpha T}f\gamma^2. \tag{25}$$

From (22) to (25), the following inequality is obtained:

$$\mathcal{E}\left[x'(t)Rx(t)\right] < \lambda_{\max}(N)e^{\alpha T}\left[\frac{b_1}{\lambda_{\min}(N)} + f\gamma^2\right]. \tag{26}$$

According to condition (14), we get that (26) leads to $\mathcal{E}[x'(t)Rx(t)] < b_2$ for all $t \in [0, T]$. So, system (8) is mean-square FTB w.r.t. (b_1, b_2, T, R, f).

Step 2: prove that the H_2 cost function (10) satisfies $J_1(x(t)) \le J_1^*$ under $r(t) = 0$ condition.

When $r(t) = 0$, we get that the $\mathcal{L}_2\overline{\mathcal{V}}(x(t))$ of system (8) is given by

$$\begin{aligned}\mathcal{L}_2\overline{\mathcal{V}}(x(t)) &= x'(t)\Big[\widetilde{A}_{11}'\widetilde{N}^{-1} + \widetilde{N}^{-1}\widetilde{A}_{11} + \widetilde{A}_{12}'\widetilde{N}^{-1}\widetilde{A}_{12} \\ &\quad + \lambda\left(\widetilde{A}_{13}+I\right)'\widetilde{N}^{-1}\left(\widetilde{A}_{13}+I\right) - \lambda\widetilde{N}^{-1}\Big]x(t).\end{aligned} \tag{27}$$

By Schur complement, the equivalent condition of (13) is given by

$$\begin{aligned}&\widetilde{N}\widetilde{A}_{11}' + \widetilde{A}_{11}\widetilde{N} + \widetilde{N}\widetilde{A}_{12}'\widetilde{N}^{-1}\widetilde{A}_{12}\widetilde{N} \\ &\quad + \lambda\widetilde{N}\left(\widetilde{A}_{13}+I\right)'\widetilde{N}^{-1}\left(\widetilde{A}_{13}+I\right)\widetilde{N} + \widetilde{N}G_1\widetilde{N} \\ &\quad + \widetilde{N}K'G_2K\widetilde{N} - \lambda\widetilde{N} - \alpha\widetilde{N} < 0.\end{aligned} \tag{28}$$

Pre- and postmultiplying (28) by $\widetilde{N}^{-1}$, it yields

$$\begin{aligned}&\widetilde{A}_{11}'\widetilde{N}^{-1} + \widetilde{N}^{-1}\widetilde{A}_{11} + \widetilde{A}_{12}'\widetilde{N}^{-1}\widetilde{A}_{12} + G_1 + K'G_2K \\ &\quad + \lambda\left(\widetilde{A}_{13}+I\right)'\widetilde{N}^{-1}\left(\widetilde{A}_{13}+I\right) - \lambda\widetilde{N}^{-1} - \alpha\widetilde{N}^{-1} < 0.\end{aligned} \tag{29}$$

According to (27) and (29), we get

$$\mathcal{L}_2\overline{\mathcal{V}}(x(t)) + x'(t)\left(G_1 + K'G_2K\right)x(t) - \alpha\overline{\mathcal{V}}(x(t)) < 0. \tag{30}$$

Integrating from 0 to t on both sides of (30), then taking mathematical expectation, the following inequality is obtained:

$$\begin{aligned}&\mathscr{E}\int_0^t x'(t)\left(G_1+K'G_2K\right)x(t)\mathrm{d}t+\mathscr{E}\overline{\mathscr{V}}(x(t))\\&\quad<\mathscr{E}\overline{\mathscr{V}}(x(0))+\alpha\mathscr{E}\int_0^t\overline{\mathscr{V}}(x(t))\mathrm{d}t.\end{aligned}\tag{31}$$

From (31), we get

$$\mathscr{E}\overline{\mathscr{V}}(x(t))<\mathscr{E}[\overline{\mathscr{V}}(x(0))]+\alpha\mathscr{E}\int_0^t\overline{\mathscr{V}}(x(t))\mathrm{d}t,\tag{32}$$

$$J_1(x(t))<\alpha\mathscr{E}\int_0^t\overline{\mathscr{V}}(x(t))\mathrm{d}t+\mathscr{E}[\overline{\mathscr{V}}(x(0))].\tag{33}$$

From (32), by Gronwall inequality, one has

$$\mathscr{E}\overline{\mathscr{V}}(x(t))<\mathscr{E}\overline{\mathscr{V}}(x(0))e^{\alpha t}.\tag{34}$$

Combining (33) and (34), it is obtained that

$$\begin{aligned}J_1(x(t))&<\alpha\mathscr{E}\int_0^t x'(0)R^{1/2}N^{-1}R^{1/2}x(0)e^{\alpha t}\mathrm{d}t\\&\quad+\mathscr{E}\left[x'(0)\widetilde{N}^{-1}x(0)\right]\\&=\mathscr{E}\left[x'(0)R^{1/2}N^{-1}R^{1/2}x(0)e^{\alpha t}\right]\\&<\lambda_{\max}\left(N^{-1}\right)b_1e^{\alpha T}=J_1^*.\end{aligned}\tag{35}$$

Step 3: prove that the nonzero disturbance and the control output satisfy inequality (11).

Pre- and postmultiplying (12) respectively by $\operatorname{diag}\left\{\widetilde{N}^{-1},I,\widetilde{N}^{-1},\widetilde{N}^{-1}\right\}$, and then using Schur complement, we have

$$\begin{bmatrix}\mathscr{T}_3 & \overline{Z}_2\\ * & -\gamma^2I+\overline{Z}_2\end{bmatrix}<0,\tag{36}$$

where $\mathscr{T}_3=\widetilde{A}_{11}'\widetilde{N}^{-1}+\widetilde{N}^{-1}\widetilde{A}_{11}+\widetilde{A}_{12}'\widetilde{N}^{-1}\widetilde{A}_{12}+\lambda(\widetilde{A}_{13}+I)'\widetilde{N}^{-1}(\widetilde{A}_{13}+I)+\widetilde{C}'\widetilde{C}-\lambda\widetilde{N}^{-1}-\alpha\widetilde{N}^{-1}$.

Combining (17), (36), we get

$$\mathscr{L}_1\overline{\mathscr{V}}(x(t))<\alpha\overline{\mathscr{V}}(x(t))+\gamma^2r'(t)r(t)-z'(t)z(t).\tag{37}$$

Pre- and postmultiplying (37) by $e^{-\alpha t}$, one has

$$e^{-\alpha t}\mathscr{L}_1\overline{\mathscr{V}}(x(t))<\alpha e^{-\alpha t}\overline{\mathscr{V}}(x(t))+e^{-\alpha t}\left[\gamma^2r'(t)r(t)-z'(t)z(t)\right].\tag{38}$$

By applying Lemma 1, we obtain

$$\mathscr{L}_1\left[e^{-\alpha t}\overline{\mathscr{V}}(x(t)\right]=-\alpha e^{-\alpha t}\overline{\mathscr{V}}(x(t))+e^{-\alpha t}\mathscr{L}_1\overline{\mathscr{V}}(x(t)).\tag{39}$$

□

According to (38) and (39), it yields

$$\mathscr{L}_1\left[e^{-\alpha t}\overline{\mathscr{V}}x(t)\right]<e^{-\alpha t}\left[\gamma^2r'(t)r(t)-z'(t)z(t)\right].\tag{40}$$

Because $e^{-\alpha t}$ is between 0 and 1, for (40), we have

$$\mathscr{L}_1\left[e^{-\alpha t}\overline{\mathscr{V}}(x(t)\right]<\gamma^2r'(t)r(t)-z'(t)z(t).\tag{41}$$

Integrating from 0 to t on both sides of (41), then taking mathematical expectation, the following inequality can be obtained under zero initial condition:

$$e^{-\alpha t}\mathscr{E}\overline{\mathscr{V}}(x(t))<\gamma^2\mathscr{E}\int_0^t r'(s)r(s)\mathrm{d}s-\mathscr{E}\int_0^t z'(s)z(s)\mathrm{d}s.\tag{42}$$

We know that $e^{-\alpha t}\mathscr{E}\overline{\mathscr{V}}(x(t))>0$, so it yields

$$\mathscr{E}\int_0^t z'(s)z(s)\mathrm{d}s<\gamma^2\mathscr{E}\int_0^t r'(s)r(s)\mathrm{d}s.\tag{43}$$

This completes the proof.

It is obvious that conditions (12)–(14) are not linear matrix inequalities. In order to simplify the solving process, the following theorem is given.

Theorem 2. *Given positive scalars b_1, b_2, T, and f and a matrix $R>0$, if there exist two scalars $m>0$ and $\alpha\ge 0$ and two matrices $N>0$ and Y such that*

$$\begin{bmatrix}\mathscr{T}_4 & F_1 & \mathscr{T}_5 & \mathscr{T}_6 & \mathscr{T}_7\\ * & -\gamma^2I & F_2' & F_3' & 0\\ * & * & -\widetilde{N} & 0 & 0\\ * & * & * & -\widetilde{N} & 0\\ * & * & * & * & -I\end{bmatrix}<0,\tag{44}$$

$$\begin{bmatrix}\mathscr{T}_4 & \mathscr{T}_5 & \mathscr{T}_6 & \widetilde{N} & Y'\\ * & -\widetilde{N} & 0 & 0 & 0\\ * & * & -\widetilde{N} & 0 & 0\\ * & * & * & -G_1^{-1} & 0\\ * & * & * & * & -G_2^{-1}\end{bmatrix}<0,\tag{45}$$

$$\begin{bmatrix}f\gamma^2-b_2e^{-\alpha T} & \sqrt{b_1}\\ * & -m\end{bmatrix}<0,\tag{46}$$

$$mI<N<I,\tag{47}$$

hold, where $\mathscr{T}_4=A_{11}\widetilde{N}+\widetilde{N}A_{11}'-\lambda\widetilde{N}-\alpha\widetilde{N}+B_{11}Y+Y'B_{11}'$, $\mathscr{T}_5=\widetilde{N}A_{12}'+Y'B_{12}'$, $\mathscr{T}_6=\sqrt{\lambda}(\widetilde{N}A_{13}'+Y'B_{13}'+\widetilde{N})$, and $\mathscr{T}_7=\widetilde{N}C_{12}'+Y'D_{12}'$, then $v(t)=Kx(t)=Y\widetilde{N}^{-1}x(t)$ is said to be a SFFT H_2/H_∞ controller and we can get the upper bound of H_2 index, that is, $J_1^*=m^{-1}b_1e^{\alpha T}$.

Proof. Let $Y=K\widetilde{N}$, inequalities (12) and (13) can be obtained from (44) and (45), respectively, and (14) in Theorem 1 can be obtained from (46) and (47) easily. This ends the proof. □

Remark 3. In Theorem 2, when α is fixed, (44)–(47) can be treated as LMIs which are easy to solve.

4. Design of OBFT H_2/H_∞ Controller

In some practical cases, not all states can be measured directly. Therefore, the design of OBFT H_2/H_∞ controller is necessary. Typically, an OB dynamic controller is given by

$$\begin{cases} \mathrm{d}\hat{x}(t) = \left[A_{11}\hat{x}(t) + B_{11}v(t) + L\left(y(t) - C_{11}\hat{x}(t)\right)\right]\mathrm{d}t, \\ v(t) = K\hat{x}(t), \\ \hat{x}(0) = 0, \end{cases} \tag{48}$$

where $\hat{x}(t) \in R^n$ is the estimation of $x(t)$ and L is the desired estimator gain.

Substituting the OB controller (48) into system (1), we will obtain the following closed-loop system:

$$\begin{cases} \mathrm{d}\tilde{x}(t) = \left[\overline{A}_{11}\tilde{x}(t) + W_1 r(t)\right]\mathrm{d}t + \left[\overline{A}_{12}\tilde{x}(t) + W_2 r(t)\right]\mathrm{d}\mathscr{W}(t) + \left[\overline{A}_{13}\tilde{x}(t) + W_3 r(t)\right]\mathrm{d}\mathscr{N}(t), \\ \tilde{z}(t) = H\tilde{x}(t), \end{cases} \tag{49}$$

and then we get the closed-loop cost function

$$J_2(\tilde{x}(t)) = \mathscr{E}\int_0^T \tilde{x}'(t)\Xi\tilde{x}(t)\mathrm{d}t, \tag{50}$$

where

$$\begin{aligned} \tilde{x}(t) &= \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}, \\ \overline{A}_{11} &= \begin{bmatrix} A_{11} & B_{11}K \\ LC_{11} & A_{11} + B_{11}K - LC_{11} \end{bmatrix}, \\ W_1 &= \begin{bmatrix} F_1 \\ LD_{11} \end{bmatrix}, \\ \overline{A}_2 &= \begin{bmatrix} A_{12} & B_{12}K \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{51}$$

$$\begin{aligned} W_2 &= \begin{bmatrix} F_2 \\ 0 \end{bmatrix}, \\ \overline{A}_{13} &= \begin{bmatrix} A_{13} & B_{13}K \\ 0 & 0 \end{bmatrix}, \\ W_3 &= \begin{bmatrix} F_3 \\ 0 \end{bmatrix}, \\ H &= \left[C_{12}' K' D_{12}'\right]', \\ \Xi &= \begin{bmatrix} G_1 & 0 \\ 0 & K'G_2K \end{bmatrix}. \end{aligned} \tag{52}$$

Assuming that the initial state is zero, the control output $\tilde{z}(t)$ and the arbitrary nonzero disturbance input $r(t)$ satisfy the following equation:

$$\mathscr{E}\int_0^T \tilde{z}'(t)\tilde{z}(t)\mathrm{d}t < \gamma^2\mathscr{E}\int_0^T r'(t)r(t)\mathrm{d}t. \tag{53}$$

Then, we give the definition of OBFT H_2/H_∞ control.

Definition 3. Given positive scalars b_1, b_2, T, and f and a matrix $\tilde{R} > 0$. If a positive scalar J_2^* exists, an OBFT controller (48) can be designed to make the following conditions hold:

(i) System (49) is mean-square FTB w.r.t. $(b_1, b_2, T, \tilde{R}, f)$, that is, $\mathscr{E}[x'(0)\tilde{R}x(0)] \le b_1 \Longrightarrow \mathscr{E}[x'(t)\tilde{R}x(t)] < b_2$, where $0 < b_1 < b_2$, $T > 0$ and $\tilde{R} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$

(ii) The H_2 cost function (50) meets $J_2(\tilde{x}(t)) \le J_2^*$ under $r(t) = 0$ condition

(iii) Assuming that the initial state is zero, the nonzero disturbance input and the control output satisfy inequality (53); then (48) is an OBFT H_2/H_∞ controller for system (1)

Next, the following theorem is given for obtaining the OBFT H_2/H_∞ controller for system (1).

Theorem 3. *Given positive scalars b_1, b_2, T, and f and a matrix $\tilde{R} > 0$, if there exist a nonnegative scalar β and a positive matrix P such that*

$$\begin{bmatrix} \mathscr{H}_1 & \tilde{P}W_1 & \overline{A}_{12}'\tilde{P} & \sqrt{\lambda}\left(\overline{A}_{13} + I\right)'\tilde{P} \\ * & -\gamma^2 I & W_2'\tilde{P} & \sqrt{\lambda}W_3'\tilde{P} \\ * & * & -\tilde{P} & 0 \\ * & * & * & -\tilde{P} \end{bmatrix} < 0, \tag{54}$$

$$\begin{aligned} &\overline{A}_{11}'\tilde{P} + \tilde{P}\overline{A}_{11} + \overline{A}_{12}'\tilde{P}\overline{A}_{12} + \lambda\left(\overline{A}_{13} + I\right)'\tilde{P}\left(\overline{A}_{13} + I\right) \\ &\quad - \lambda\tilde{P} - \beta\tilde{P} + \Xi < 0, \end{aligned} \tag{55}$$

$$\lambda_{\max}(P)b_1 + f\gamma^2 < \lambda_{\min}(P)b_2 e^{-\beta T}, \tag{56}$$

hold, where $\tilde{P} = \tilde{R}^{1/2}P\tilde{R}^{1/2}$ and $\mathscr{H}_1 = \overline{A}_{11}'\tilde{P} + \tilde{P}\overline{A}_{11} + H'H - \beta\tilde{P} - \lambda\tilde{P}$, then (48) is said to be an OBFT H_2/H_∞ controller and we can get the upper bound of H_2 index, that is, $J_2^* = \lambda_{\max}(P)b_1 e^{\beta T}$.

Proof. Here are three steps to prove the theorem.

Step 1: prove that system (49) is mean-square FTB.

Let $\overline{\mathscr{V}}(\tilde{x}(t)) = \tilde{x}'(t)\tilde{P}\tilde{x}(t)$ where $\tilde{P} > 0$. Applying generalized Itô formula for $\overline{\mathscr{V}}(\tilde{x}(t))$, the $\mathscr{L}_3\overline{\mathscr{V}}(\tilde{x}(t))$ of system (49) is given by

$$\mathscr{L}_3\overline{\mathscr{V}}(\tilde{x}(t)) = \begin{bmatrix} \tilde{x}(t) \\ r(t) \end{bmatrix}' \begin{bmatrix} \tilde{Z}_1 & \tilde{Z}_2 \\ * & \tilde{Z}_3 \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ r(t) \end{bmatrix}, \tag{57}$$

where $\tilde{Z}_1 = \overline{A}_{11}'\tilde{P} + \tilde{P}\overline{A}_{11} + \overline{A'}_{12}\tilde{P}\overline{A}_{12} + \lambda(\overline{A}_{13} + I)'\tilde{P}(\overline{A}_{13} + I) - \lambda\tilde{P}$, $\tilde{Z}_2 = \tilde{P}W_1 + A'_{12}\tilde{P}W_2 + \lambda(\overline{A}_{13} + I)'\tilde{P}W_3$, and $\tilde{Z}_3 = W_2'\tilde{P}W_2 + \lambda W_3'\tilde{P}W_3$.

Note that

$$\begin{bmatrix} H'H & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} H' \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} H' \\ 0 \\ 0 \\ 0 \end{bmatrix}' \geq 0. \tag{58}$$

Therefore, inequality (54) means

$$\begin{bmatrix} \mathscr{H}_2 & \tilde{P}W_1 & \overline{A}_{12}'\tilde{P} & \sqrt{\lambda}(\overline{A}_{13} + I)'\tilde{P} \\ * & -\gamma^2 I & W_2'\tilde{P} & \sqrt{\lambda}W_3'\tilde{P} \\ * & * & -\tilde{P} & 0 \\ * & * & * & -\tilde{P} \end{bmatrix} < 0, \tag{59}$$

where $\mathscr{H}_2 = \overline{A}_{11}'\tilde{P} + \tilde{P}\overline{A}_{11} - \beta\tilde{P} - \lambda\tilde{P}$.

By utilizing Schur complement, (59) can be converted into

$$\begin{bmatrix} \tilde{Z}_1 - \beta\tilde{P} & \tilde{Z}_2 \\ * & -\gamma^2 I + \tilde{Z}_3 \end{bmatrix} < 0. \tag{60}$$

Combining (57) and (60), we get

$$\mathscr{L}_3\overline{\mathscr{V}}(\tilde{x}(t)) < \beta\overline{\mathscr{V}}(\tilde{x}(t)) + \gamma^2 r'(t)r(t). \tag{61}$$

Integrating from 0 to t on both sides of (61), then taking mathematical expectation, the following inequality is obtained:

$$\begin{aligned} \mathscr{E}\overline{\mathscr{V}}(\tilde{x}(t)) &< \mathscr{E}\overline{\mathscr{V}}(\tilde{x}(0)) + \beta\int_0^t \mathscr{E}\overline{\mathscr{V}}(\tilde{x}(s))ds \\ &\quad + \gamma^2\int_0^t \mathscr{E}r'(s)r(s)ds. \end{aligned} \tag{62}$$

According to Gronwall inequality, it yields

$$\mathscr{E}\overline{\mathscr{V}}(\tilde{x}(t)) < \mathscr{E}\overline{\mathscr{V}}(\tilde{x}(0))e^{\beta t} + \gamma^2 e^{\beta t}\int_0^t \mathscr{E}r'(s)r(s)ds. \tag{63}$$

According to known conditions, it yields

$$\begin{aligned} \mathscr{E}\overline{\mathscr{V}}(\tilde{x}(t)) &= \mathscr{E}\left[\tilde{x}'(t)\tilde{R}^{1/2}P\tilde{R}^{1/2}\tilde{x}(t)\right] \\ &\geq \lambda_{\min}(P)\mathscr{E}\left[\tilde{x}'(t)\tilde{R}\tilde{x}(t)\right], \end{aligned} \tag{64}$$

$$\begin{aligned} \mathscr{E}\overline{\mathscr{V}}(\tilde{x}(0))e^{\beta t} &= \mathscr{E}\left[\tilde{x}'(0)\tilde{R}^{1/2}P\tilde{R}^{1/2}\tilde{x}(0)\right]e^{\beta t} \\ &\leq \lambda_{\max}(P)\mathscr{E}\left[\tilde{x}'(0)\tilde{R}\tilde{x}(0)e^{\beta t}\right] \\ &\leq \lambda_{\max}(P)b_1 e^{\beta T}, \end{aligned} \tag{65}$$

$$\gamma^2 e^{\beta t}\int_0^t \mathscr{E}r'(s)r(s)ds < e^{\beta T}f\gamma^2. \tag{66}$$

From (63) to –(66), we obtain

$$\mathscr{E}\left[\tilde{x}'(t)\tilde{R}\tilde{x}(t)\right] < \frac{\lambda_{\max}(P)b_1 e^{\beta T} + f\gamma^2 e^{\beta T}}{\lambda_{\min}(P)}. \tag{67}$$

According to (56) and (67), we get $\mathscr{E}[\tilde{x}'(t)\tilde{R}\tilde{x}(t)] < b_2$ for all $t \in [0, T]$. So, system (49) is FTB w.r.t. $(b_1, b_2, T, \tilde{R}, f)$.

Step 2: prove that the H_2 cost function (50) satisfies $J_2(\tilde{x}(t)) \leq J_2^*$ under $r(t) = 0$ condition.

When $r(t) = 0$, we get that the $\mathscr{L}_4\overline{\mathscr{V}}(\tilde{x}(t))$ of system (49) is given by

$$\begin{aligned} \mathscr{L}_4\overline{\mathscr{V}}(\tilde{x}(t)) &= \tilde{x}'(t)\left[\left[\overline{A}_{11}'\tilde{P} + \tilde{P}\overline{A}_{11} + \overline{A}_{12}'\tilde{P}\overline{A}_{12}\right.\right. \\ &\quad \left.+\lambda(\overline{A}_{13} + I)'\tilde{P}(\overline{A}_{13} + I) - \lambda\tilde{P}\right]\tilde{x}(t). \end{aligned} \tag{68}$$

According to (55), we have

$$\mathscr{L}_4\overline{\mathscr{V}}(\tilde{x}(t)) - \beta\overline{\mathscr{V}}(\tilde{x}(t)) + \tilde{x}'(t)\Xi\tilde{x}(t) < 0. \tag{69}$$

Integrating from 0 to t on both sides of (69), then taking mathematical expectation, it yields

$$\begin{aligned} &\mathscr{E}\overline{\mathscr{V}}(\tilde{x}(t)) + \mathscr{E}\int_0^t \tilde{x}'(t)\Xi\tilde{x}(t)dt \\ &\quad < \mathscr{E}\overline{\mathscr{V}}(\tilde{x}(0)) + \beta\mathscr{E}\int_0^t \overline{\mathscr{V}}(\tilde{x}(t))dt. \end{aligned} \tag{70}$$

From (70), we have

$$\mathscr{E}\overline{\mathscr{V}}(\tilde{x}(t)) < \mathscr{E}\overline{\mathscr{V}}(x(0)) + \beta\mathscr{E}\int_0^t \overline{\mathscr{V}}(\tilde{x}(t))dt, \tag{71}$$

$$\mathscr{E}\int_0^t \tilde{x}'(t)\Xi\tilde{x}(t)dt < \mathscr{E}\overline{\mathscr{V}}(\tilde{x}(0)) + \beta\mathscr{E}\int_0^t \overline{\mathscr{V}}(\tilde{x}(t))dt. \tag{72}$$

Using Gronwall inequality for (71), one has

$$\mathscr{E}\overline{\mathscr{V}}(\tilde{x}(t)) < \mathscr{E}\overline{\mathscr{V}}(\tilde{x}(0))e^{\beta t}. \tag{73}$$

From (72) and (73), we have

$$\begin{aligned} J_2(\tilde{x}(t)) &< \beta\mathscr{E}\int_0^t \overline{\mathscr{V}}(\tilde{x}(t))\mathrm{d}t + \mathscr{E}\overline{\mathscr{V}}(\tilde{x}(0)) \\ &< \beta\mathscr{E}\int_0^t \overline{\mathscr{V}}(\tilde{x}(0))e^{\beta t}\mathrm{d}t + \mathscr{E}\overline{\mathscr{V}}(\tilde{x}(0)) \\ &= \mathscr{E}\overline{\mathscr{V}}(\tilde{x}(0))e^{\beta t} < \lambda_{\max}(P)b_1e^{\beta T} = J_2^*. \end{aligned} \tag{74}$$

Step 3: prove that the nonzero disturbance and the control output satisfy the inequality (53).

By using Schur complement, we can obtain the following equivalent conditions of (54):

$$\begin{bmatrix} \widetilde{Z}_1 + H'H - \beta\widetilde{P} & \widetilde{Z}_2 \\ * & \widetilde{Z}_3 - \gamma^2 I \end{bmatrix} < 0. \tag{75}$$

According to (57) and (75), we get

$$\mathscr{L}_3\overline{\mathscr{V}}(\tilde{x}(t)) < \beta\overline{\mathscr{V}}(\tilde{x}(t)) + \gamma^2 r'(t)r(t) - z'(t)z(t). \tag{76}$$

Repeating the proof process of Step 3 in Theorem 1, it yields

$$\mathscr{E}\int_0^t z'(s)z(s)\mathrm{d}s < \gamma^2\mathscr{E}\int_0^t r'(s)r(s)\mathrm{d}s. \tag{77}$$

This completes the proof.

Because the nonlinear problem of inequalities (54)–(56) in Theorem 3 is difficult to solve, we transform the inequalities (54)–(56) into LMIs. □

Theorem 4. *Given positive scalars b_1, b_2, T, and f, if there exist two positive scalars β and ζ and three matrices $\widetilde{P}_{11} > 0$, $\widetilde{P}_{22} > 0$, and M such that*

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ * & \Sigma_{22} & \Sigma_{23} \\ * & * & \Sigma_{33} \end{bmatrix} < 0, \tag{78}$$

$$\begin{bmatrix} \Upsilon_{11} - (\beta+\lambda)\widetilde{P}_{11} & \Upsilon_{12} \\ * & \Upsilon_{22} - (\beta+\lambda)\widetilde{P}_{22} \end{bmatrix} < 0, \tag{79}$$

$$e^{\beta T}\left(\zeta b_1 + f\gamma^2\right) - b_2 < 0, \tag{80}$$

$$I < \mathrm{diag}\{\widetilde{P}_{11}, \widetilde{P}_{22}\} < \zeta I, \tag{81}$$

hold, where $\Sigma_{11} = A'_{11}\widetilde{P}_{11} + \widetilde{P}_{11}A_{11} + A'_{12}\widetilde{P}_{11}A_{12} + C'_{12}C_{12} + \lambda(A'_{13}+I)\widetilde{P}_{11}(A_{13}+I) - (\beta+\lambda)\widetilde{P}_{11}$, $\Sigma_{12} = C'_{11}M' + \widetilde{P}_{11}B_{11}K + A'_{12}\widetilde{P}_{11}B_{12}K + C'_{12}D_{12}K + \lambda(A'_{13}+I)\widetilde{P}_{11}B_{13}K$, $\Sigma_{22} = (A_{11}+B_{11}K)'\widetilde{P}_{22} + \widetilde{P}_{22}(A_{11}+B_{11}K) + K'B'_{12}\widetilde{P}_{11}B_{12}K + \lambda K'B'_{13}\widetilde{P}_{11}B_{13}K + \lambda\widetilde{P}_{22} - MC_{11} - C'_{11}M' + K'D'_{12}D_{12}K - (\beta+\lambda)\widetilde{P}_{22}$, $\Sigma_{13} = \widetilde{P}_{11}F_1 + A'_{12}\widetilde{P}_{11}F_2 + \lambda(A'_{13}+I)\widetilde{P}_{11}F_3$, $\Sigma_{23} = MD_{11} + K'B'_{12}P_{11}F_2 + \lambda K'B'_{13}P_{11}F_3$, $\Sigma_{33} = F'_2\widetilde{P}_{11}F_2 + \lambda F'_3\widetilde{P}_{11}F_3 - \gamma^2 I$, $\Upsilon_{11} = A'_{11}\widetilde{P}_{11} + \widetilde{P}_{11}A_{11} + A'_{12}\widetilde{P}_{11}A_{12} + \lambda(A'_{13}+I)\widetilde{P}_{11}(A_{13}+I) + G_1$, $\Upsilon_{12} = C'_{11}M' + \widetilde{P}_{11}B_{11}K + A'_{12}\widetilde{P}_{11}B_{12}K + \lambda(A'_{13}+I)\widetilde{P}_{11}B_{13}K$, $\Upsilon_{22} = (A_{11}+B_{11}K)'\widetilde{P}_{22} + \widetilde{P}_{22}(A_{11}+B_{11}K) + K'B'_{12}\widetilde{P}_{11}B_{12}K - MC_{11} - C'_{11}M' + \lambda K'B'_{13}\widetilde{P}_{11}B_{13}K + \lambda\widetilde{P}_{22} + K'G_2K$, *then (48) is said to be an OBFT H_2/H_∞ controller and we can get the upper bound of H_2 index, that is, $J_2^* = \zeta b_1 e^{\beta T}$. Furthermore, the estimator gain matrix $L = \widetilde{P}_{22}^{-1}M$ is obtained.*

Proof. Let $\widetilde{P} = \mathrm{diag}\{\widetilde{P}_{11}, \widetilde{P}_{22}\}$ and $M = \widetilde{P}_{22}L$, by substituting (51) and (52) into (54) and (55), (78) and (79) can be easily derived, respectively. From (80) and (81), we can deduce that (49) holds. This ends the proof. □

5. Algorithm

In this section, we propose an algorithm to optimize H_2 index and H_∞ index.

Analysis: in Theorem 2, let $J_1^* < \xi$, the following inequality is derived:

$$\frac{Ne^{-\alpha T}}{b_1} - \xi^{-1}I > 0, \tag{82}$$

where $0 < b_1 < b_2$, $T > 0$, $\alpha \geq 0$, and $m > 0$.

The main purpose of the algorithm is to check whether inequalities (44)–(47) in Theorem 2 have feasible solutions by changing the value of α. If there exist feasible solutions, then ξ and γ^2 are optimized to get the minimum values. The detailed algorithm will be given as follows (Algorithm 1).

6. Examples

In this section, system (8) can be used to simulate a clothing hanging device and the parameters are as follows:

$$\begin{aligned} A_{11} &= \begin{bmatrix} -15 & -9 \\ 8 & -12 \end{bmatrix}, \\ A_{12} &= \begin{bmatrix} -0.6 & 1 \\ 1.3 & -1.2 \end{bmatrix}, \\ A_{13} &= \begin{bmatrix} -1.8 & 1 \\ 1.4 & -1.5 \end{bmatrix}, \\ B_{11} &= [-9 \;\; 5]', \\ B_{12} &= [8.7 \;\; 2.3]', \\ B_{13} &= [-2.8 \;\; 1]', \\ F_1 &= [-0.6 \;\; 0.5]', \\ F_2 &= [0.3 \;\; -0.2]', \\ F_3 &= [0.4 \;\; -0.2]', \\ C_{11} &= [-0.9 \;\; -1.5], \\ C_{12} &= [-1.8 \;\; -2.5], \\ x(0) &= [-0.7 \;\; 0.7]', \end{aligned} \tag{83}$$

and $G_1 = 5$, $G_2 = 4$, $D_{11} = 8$, $D_{12} = 10$, $b_1 = 1$, $b_2 = 4$, $T = 1$, $R = I$, $f = 0.4$, and $\lambda = 2.5$.

6.1. Design of SFFT H_2/H_∞ Controller. By using the above algorithm in Section 5, the relationships of α and ξ (Figure 1), α and γ (Figure 2), and ξ and γ (Figure 3) are derived, respectively. It can be seen from Figure 1 that the value of ξ increases with the increase of α. Besides, it is obvious that

Step 1: given b_1, b_2, R, T, f, and λ.
Step 2: take an appropriate step size d_α for α, and then the values of α are expressed as α_i.
Step 3: let $i = 1$.
Step 4: if α_i makes the following problems $\min_{\text{s.t.}\,(37)-(40),(74),N>0}\xi$ and $\min_{\text{s.t.}\,(37)-(40),(74),N>0}\gamma^2$ feasible, then store α_i into $U(i)$, $\xi_{\min}$ into $V(i)$, and $\gamma_{\min}$ into $W(i)$, and let $\alpha_{i+1} = \alpha_i + d_\alpha$, loop. Otherwise, go to Step 5.
Step 5: exit.

Algorithm 1: Optimization algorithm.

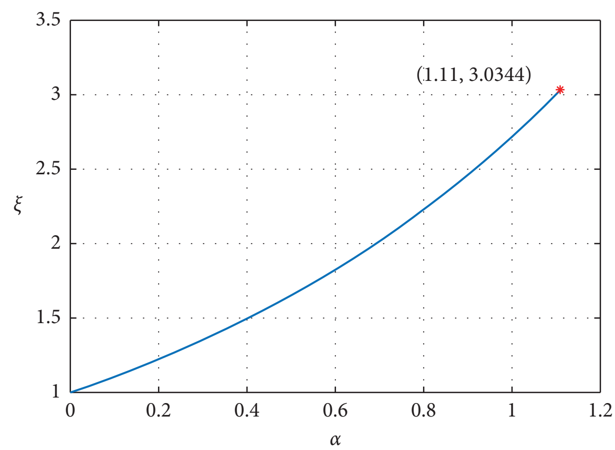

Figure 1: ξ versus α.

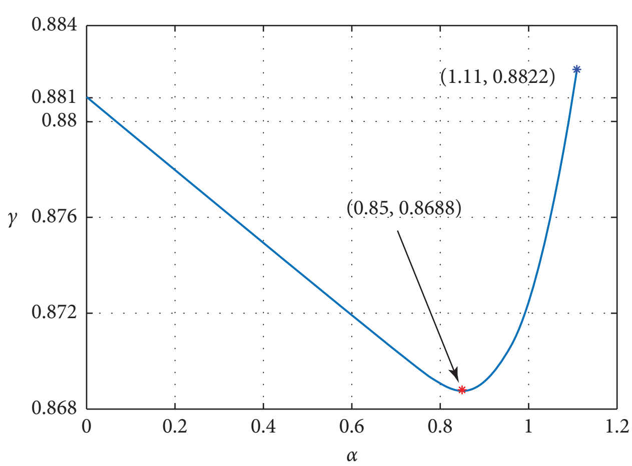

Figure 2: γ versus α.

$\xi = 1$ when $\alpha = 0$ and $\xi = 3.0344$ when $\alpha = 1.11$, that is, the minimum and maximum values of H_2 performance index are 1 and 3.0344, respectively. Also, the range of α is [0, 1.11].

As you can see from Figure 2, the value of γ decreases first and then increases with the increase of α. When $\alpha = 0.85$, γ can take the minimum value of 0.8688; at this point, we can get the optimal value of H_∞ performance index. When $\alpha = 1.11$, γ can take the minimum value of 0.8822. Also, α can be taken within [0, 1.11].

In fact, Figure 3 reflects the relation between ξ and γ. As shown in Figure 3, with the increase of ξ, the value of γ decreases first, and at the point of $\xi = 2.3397$, the value of γ begins to increase. From Figures 1 to 3, we can see how to choose the right state feedback finite-time H_2/H_∞ controller. If the cost problem is mainly considered, a smaller α can be selected. If the ability to suppress interference is mainly considered, we need to refer to Figure 2 to select the appropriate α.

Next, substituting $\alpha = 0$ into Theorem 2, we get

$$N = \begin{bmatrix} 0.8673 & -0.0135 \\ -0.0135 & 0.8946 \end{bmatrix}, Y = \begin{bmatrix} -0.0366 & 0.0770 \end{bmatrix}, \quad m = 0.5237. \tag{84}$$

Then, we get the controller gain matrix as follows:

$$K = \begin{bmatrix} -0.0409 \; 0.0854 \end{bmatrix}. \tag{85}$$

Because the state $x(t)$ in this example is two-dimensional, we make $x(t) = [x_1(t)x_2(t)]'$. Figure 4 describes the trajectories of $x_1(t)$, $x_2(t)$, and $E[x'(t)Rx(t)]$ with stochastic fluctuation driven by both Wiener and Poisson noises in Figures 5 and 6 versus the dimensionless time λt. From Figure 4, we can see that the trajectory of $E[x'(t)Rx(t)]$ does not exceed $b_2 = 4$ in the time interval $\lambda T = 2.5$. Obviously, when the time interval is $T = 1$, the trajectory does not exceed the given range, so we conclude that system (8) is mean-square FTB w.r.t. (1, 4, 1, I, 0.4). Among them, we assume that $r(t) = \sin t\,(\int_0^1 \sin^2 t\mathrm{d}t < f = 0.4)$.

6.2. Design of OBFT H_2/H_∞ Controller. As in the case of state feedback, similar results can be obtained in the case of observer-based finite-time H_2/H_∞ control. The relationships of β and ξ (Figure 7), β and γ (Figure 8), and ξ and γ (Figure 9) are derived, respectively. It can be seen from Figure 7 that the value of ξ increases with the increase of β. Besides, it is obvious that $\xi = 1$ when $\beta = 0$ and $\xi = 2.5857$ when $\beta = 0.95$, that is, the minimum and maximum values of H_2 performance index are 1 and 2.5857, respectively. Also, the range of β is [0, 0.95].

As you can see from Figure 8, the value of γ decreases first and then increases with the increase of β. When $\beta = 0.72$, γ can take the minimum value of 1.1456, and at this point, we can get the optimal value of H_∞ performance index. Besides, the maximum value of H_∞ performance index is 1.1650 when $\beta = 0.95$. Also, the range of β is [0, 0.95].

In fact, Figure 9 reflects the relation between ξ and γ. As shown in Figure 9, with the increase of ξ, the value of γ decreases first, and at the point of $\xi = 2.0544$, the value of γ begins to increase. From Figures 7 to 9, we can see how to

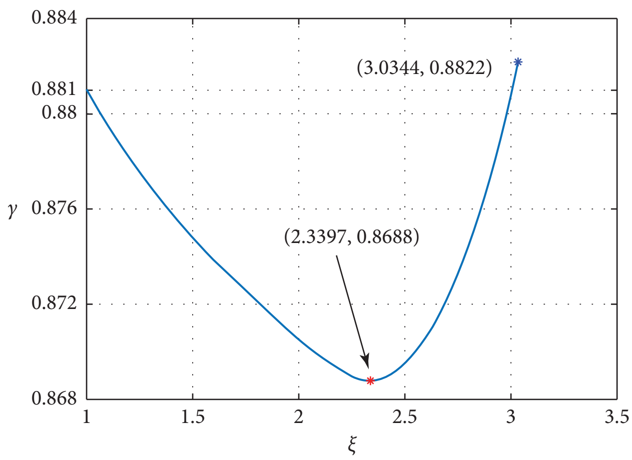

Figure 3: γ versus ξ.

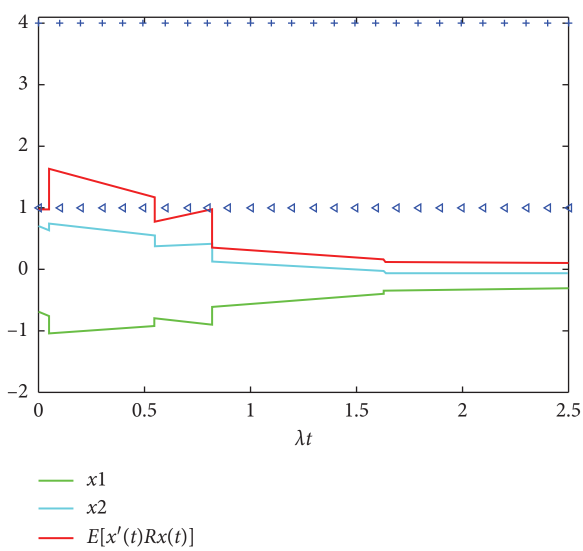

Figure 4: The trajectory for $E[x'(t)Rx(t)]$.

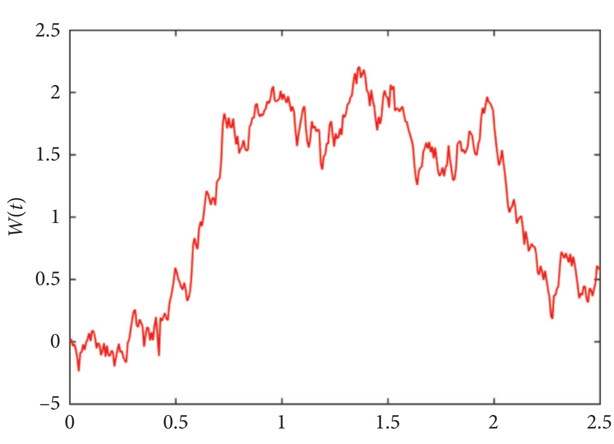

Figure 5: The time evolution of Wiener process.

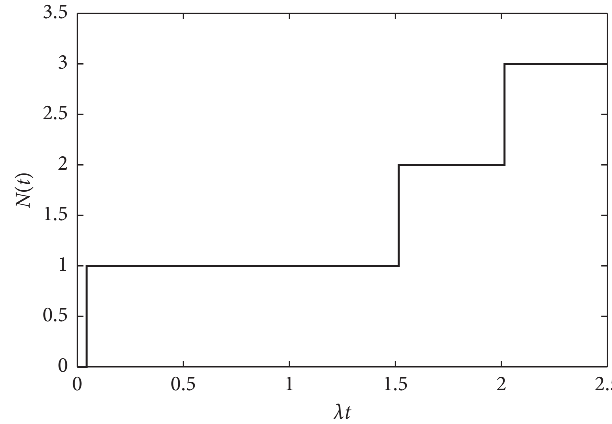

Figure 6: The time evolution of Poisson counting process.

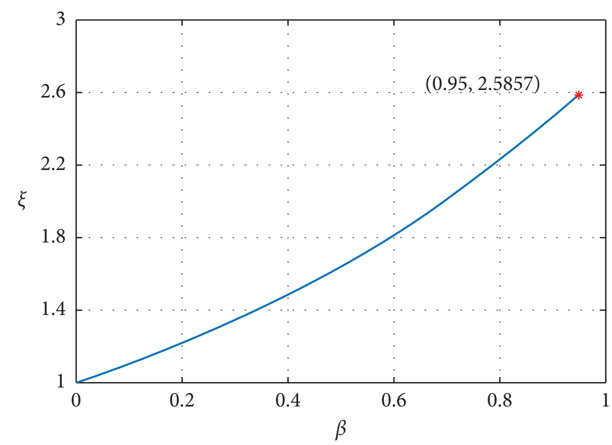

Figure 7: ξ versus β.

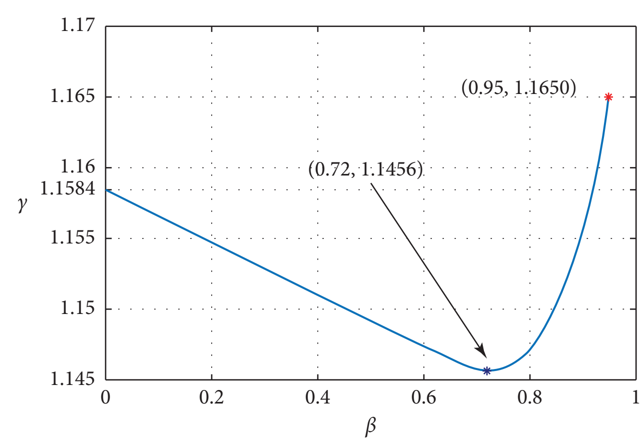

Figure 8: γ versus β.

choose the right OBFT H_2/H_∞ controller. If the cost problem is mainly considered, select the smaller β with reference to Figure 7. If the ability to suppress interference is mainly considered, we need to refer to Figure 8 to select the appropriate β.

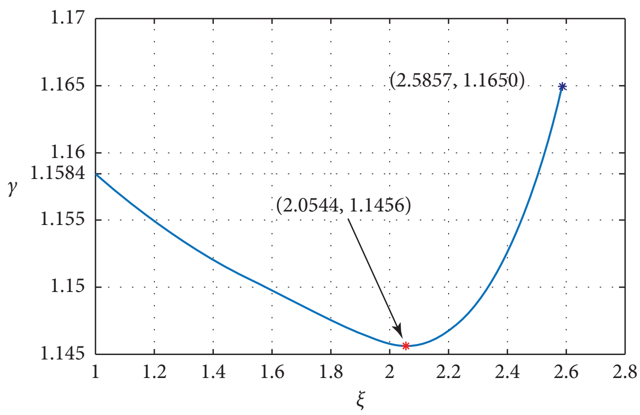

FIGURE 9: γ versus ξ.

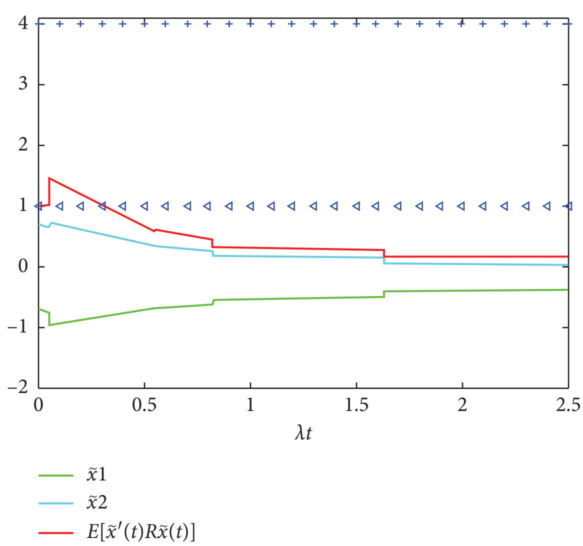

FIGURE 10: The trajectory for $E[x'(t)R\tilde{x}(t)]$.

Substituting $\beta = 0$ into Theorem 4, we have

$$\begin{aligned} P_{11} &= \begin{bmatrix} 1.3326 & 0.0458 \\ 0.0458 & 1.5739 \end{bmatrix}, \\ P_{22} &= \begin{bmatrix} 1.4821 & -0.0298 \\ -0.0298 & 1.7362 \end{bmatrix}, \\ M &= \begin{bmatrix} -0.0326 \\ 0.1348 \end{bmatrix}, \\ \zeta &= 2.7271. \end{aligned} \tag{86}$$

Then, we obtain the following observer gain matrix:

$$L = \begin{bmatrix} -0.0205 \\ 0.0773 \end{bmatrix}. \tag{87}$$

Because the state $\tilde{x}$ in this example is two-dimensional, we make $\tilde{x} = [\tilde{x}_1 \tilde{x}_2]'$. Figure 10 describes the trajectories of $\tilde{x}_1$, $\tilde{x}_2$, and $E[x'(t)R\tilde{x}(t)]$ with stochastic fluctuation driven by both Wiener and Poisson noises in Figures 5 and 6 versus the dimensionless time λt. From Figure 10, it is obvious that the trajectory of $E[x'(t)R\tilde{x}(t)]$ does not exceed $b_2 = 4$ in the time interval $\lambda T = 2.5$. Obviously, when the time interval is $T = 1$, the trajectory does not exceed the given range, so we draw a conclusion that system (8) is mean-square FTB w.r.t. (1, 4, 1, I, 0.4). Among them, we assume that $r(t) = \sin t\,(\int_0^1 \sin^2 t\mathrm{d}t < f = 0.4)$.

7. Conclusions

In this paper, state feedback and observer-based finite-time H_2/H_∞ controllers for stochastic Poisson systems have been designed, respectively. Two sufficient conditions for guaranteeing the existence of controllers have been proposed and converted to matrix inequality constrained optimization problems, and an algorithm for all Theorems has been provided to derive the optimal H_2 index and H_∞ index under the condition of the finite-time boundedness.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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