Research Article

Statistical Characteristics of the First Passage Time Analysis for the Gene Regulatory Circuit in *Bacillus subtilis* by Cell Mapping Method

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Received 15 October 2019; Revised 23 December 2019; Accepted 8 January 2020; Published 5 February 2020

Academic Editor: Alejandro F. Villaverde

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In this paper, we will explore the stochastic exit problem for the gene regulatory circuit in *B. subtilis* affected by colored noise. The stochastic exit problem studies the state transition in *B. subtilis* (from competent state to vegetative state in this case) through three different quantities: the probability density function of the first passage time, the mean of first passage time, and the reliability function. To satisfy the Markov nature, we convert the colored noise system into the equivalent white noise system. Then, the stochastic generalized cell mapping method can be used to explore the stochastic exit problem. The results indicate that the intensity of noise and system parameters have the effect on the transition from competent to vegetative state in *B. subtilis*. In addition, the effectiveness of the stochastic generalized cell mapping method is verified by Monte Carlo simulation.

1. Introduction

*B. subtilis*, which is a single-celled creature, is simpler and easier than multicellular creature [1]. It includes three physiological states: vegetative state, competent state, and sporulation. When there is sufficient nutrition, *B. subtilis* is in vegetative state and performs normal life activities. In the case of nutrient limitation, a minority of *B. subtilis* cells become competent state, while most of them form sporulation. Sporulation is a dormant body rather than a propagule. Once sporulation forms, *B. subtilis* maintains this state until the environment improves [2]. The competent cell can switch to vegetative state with a certain probability. The state transition can be attributed to the excitability in the gene regulatory circuit.

Cells can use the gene regulatory circuit to implement the state transition [3]. This process is affected by the fluctuations [4], which arise from processes intrinsic or extrinsic to gene expression [5]. Kohar and Lu studied the effect of both the intrinsic and extrinsic noises and parameters on the dynamics of gene regulatory circuits through the randomization-based approach [6]. Intrinsic noise results from stochasticity in the biochemical reactions at an individual gene. In 2010, Dandach and Khammash had presented a novel numerical method for the analysis of stochastic biochemical events and then studied the genetic circuit regulating competence under the excitation of intrinsic noise in *B. subtilis* [7]. Extrinsic noise is the additional variation originating from the fluctuation in the cellular components and has a global effect on the dynamics of the system. In addition, the Ornstein–Uhlenbeck noise (the colored noise), which is a time-correlated noise with zero mean [8], is closer to the actual cell behavior [2]. Thus, we mainly consider the extrinsic colored noise in *B. subtilis* in this paper.

To confirm the state of *B. subtilis*, it is necessary to understand ComK gene. The ComK gene, a regulatory transcription unit, is essential for the development of genetic competence in *B. subtilis* [9]. When the concentration of ComK protein is high, *B. subtilis* is in the competent state.
Conversely, when the concentration of ComK protein is low, *B. subtilis* is in the vegetative state. Therefore, we can judge which state *B. subtilis* is in by analyzing the concentration of ComK protein.

The stochastic exit problem is usually used to analyze such state transition through three quantities: the probability density function of first passage time (PDF of FPT), the mean of first passage time (MFPT), and the reliability function [10, 11]. FPT refers to the time that it takes for a trajectory in the system to start from a stable equilibrium point and then cross the boundary of security region and reach the target region for the first time. Broadly stated, the FPT refers to the time that *B. subtilis* takes to finish the transition from the stable competent state to stable vegetative state in this paper. In this context, the PDF of FPT refers to the probability that the system finishes the transition as time goes on. Mathematically, the MFPT is the expectation of the FPT. Concretely, the MFPT refers to the mean time that *B. subtilis* needs to achieve the state transition. The reliability function indicates the change relation between the reliability of system and time, which represents the probability that *B. subtilis* remains in the competent state as time goes on. In the paper, we study the stochastic exit problem for the gene regulatory circuit in *B. subtilis* with colored noise through the above three different quantities.

The cell mapping method [12], which is an efficient numerical method to analyze global dynamics, was firstly suggested by Hsu in 1980. Later, many scholars have developed new versions. Among them, the composite cell coordinate system (CCCS) method [13, 14] is usually used to obtain the global properties in the deterministic system, containing attractor, basin of attraction, boundary, and so on. The stochastic generalized cell mapping (SGCM) method [15, 16] is used to obtain the stochastic response [17–19] in the stochastic system.

The rest of this paper is as follows. In Section 2, we give the deterministic model of gene regulatory circuit in *B. subtilis* and then determine the security domain and boundary. The effect of noise on the PDF of FPT, MFPT, and reliability function in the stochastic system are shown in Section 3. We research the effect of system parameters on the three quantities in the stochastic system in Section 4. In Section 5, the conclusions are presented.

### 2. Model Introduction and the Determination of Security Domain

An excitable core module contains positive and negative feedback loops [2], which were found by Suel et al. in 2006. The module can explain the mechanism of entering or exiting the competent state very well [20, 21]. Both the positive and negative feedback loops consist of four important parts: First, the ComK activates itself via a transcriptional autoregulatory positive feedback loop, which results in the basal expression of ComK and the beginning of this core module. Thus, the ComK transcription factor is the heart of this circuit. Second, the ComK proteins can be degraded by MecA via enzymatic degradation. Third, the degradation of ComK is inhibited by the ComS, because ComS can also bind with MecA competitively. Fourth, the overexpression of ComK inhibits the synthesis of ComS. The enzymatic degradation reactions are assumed to be of the standard Michaelis–Menten form [2]:

\[
\begin{align*}
\frac{\gamma_s}{k_s} & \quad \text{MecA} + \text{ComK} \xrightarrow{\gamma_K} \text{MecA} + \text{ComK}, \\
\frac{\gamma_b}{k_b} & \quad \text{MecA} + \text{ComS} \xrightarrow{\gamma_m} \text{MecA} + \text{ComS}.
\end{align*}
\]

In the following, the symbols *K* and *S* represent the concentrations of ComK protein and ComS protein, respectively. The concentrations of free MecA and the complexes MecA–ComK and MecA–ComS can be denoted by *M_j*, *M_k*, and *M_s*, respectively. Meanwhile, the total amount of *M_j*, *M_k*, and *M_s* is determined in an enclosed living environment of *B. subtilis*. Hence, the relationship among free MecA, complexes MecA–ComK, and complexes MecA–ComS can be expressed [2, 22] by

\[
M_j + M_k + M_s = M_{\text{total}} = \text{constant}.
\]

The following equations describe the relationship among *K*, *S*, and MecA [7]. Other symbols are the system parameters, whose description are given in Table 1 [2]:

\[
\begin{align*}
\frac{dK}{dt} &= \alpha_k + \frac{\beta_s K^n}{k_s^n + K^n} - y_s M_j K + y_a M_k, \\
\frac{dS}{dt} &= \frac{\beta_s}{1 + (K/k_s)} - y_b M_j K + y_b M_s, \\
\frac{dM_j}{dt} &= -(y_a + y_j) M_k K + y_a M_j K, \\
\frac{dM_s}{dt} &= -(y_b + y_j) M_k K + y_b M_j S.
\end{align*}
\]

Through dimensionless, the final simplified equations [2] are as follows:

\[
\begin{align*}
\frac{dK}{dt} &= \alpha_k + \frac{b_s K^n}{k_0^n + K^n} - \frac{K}{1 + K + S}, \\
\frac{dS}{dt} &= \frac{b_s}{1 + (K/k_s)} - \frac{S}{1 + K + S}.
\end{align*}
\]

Here, \(\alpha_k = \alpha_k/\left(\delta I_K \Gamma_k\right)\), \(b_s = \beta_s/\left(\delta I_K\right)\), \(b_s = \beta_s/\left(\delta I_K\right)\), \(k_0 = K_0/\Gamma_k\), \(k_1 = K_1/\Gamma_k\), \(\Gamma_k = (y_a + y_j)/y_a\), \(\Gamma_s = (y_b + y_j)/y_b\), \(\delta_k = (\gamma_1 M_{\text{total}})/I_k\), and \(\delta_s = (\gamma_2 M_{\text{total}})/I_s\).

The primary task is to obtain the global properties through the CCCS method, which includes attractor, basin of attraction, and boundary. Thus, the security domain, target domain, and boundary can be determined. The attractor, which represents the steady response of the dynamical system, corresponds to the stable state of *B. subtilis*. The basin of attraction indicates the region which is either security or target region. The security region and the target region are divided by the boundary. In the following, \(\Omega = \{(K, S) | 0 \leq K \leq 0.6, 0 \leq S \leq 10\}\) is selected the interesting domain, and the parameters are taken [2] as \(\alpha_k = 0.004, k_0 = 0.2, k_1 = 0.222, n = 2, \text{ and } p = 5\).
Table 1: Description of parameters in the dynamical model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_k$</td>
<td>Basal expression rate of ComK</td>
</tr>
<tr>
<td>$\beta_k$</td>
<td>Saturating expression rate of ComK positive feedback</td>
</tr>
<tr>
<td>$\beta_s$</td>
<td>Unrepressed expression rate of ComS</td>
</tr>
<tr>
<td>$k_n$</td>
<td>ComK concentration for half-maximal ComK activation</td>
</tr>
<tr>
<td>$k_s$</td>
<td>ComK concentration for half-maximal ComS repression</td>
</tr>
<tr>
<td>$k_k$</td>
<td>Unrepressed degradation rate of ComK</td>
</tr>
<tr>
<td>$\delta_k$</td>
<td>Unrepressed degradation rate of ComS</td>
</tr>
<tr>
<td>$\Gamma_k$</td>
<td>ComK concentration for half-maximal degradation</td>
</tr>
<tr>
<td>$\Gamma_s$</td>
<td>ComS concentration for half-maximal degradation</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Hill coefficient of ComK positive feedback</td>
</tr>
<tr>
<td>$p$</td>
<td>Hill coefficient of ComS repression by ComS</td>
</tr>
</tbody>
</table>

Without loss of generality, this work is focused on the transition from competent to vegetative state in *B. subtilis*. As shown in Figure 1, the blue triangle A1 represents attractor, which means the stable vegetative state of *B. subtilis*. Similarly, the red diamond A2 is another attractor that denotes the stable competent state. The cyan region B1 and yellow region B2 represent the basin of attraction of attractors A1 and A2, respectively. The two basins of attraction are divided by the boundary shown by the dark green curve. In this paper, we choose attractor A2 as the start point. The yellow region B2 is the security domain, while the cyan region B1 represents the target region. Hence, the FPT refers to the time it takes for the trajectory to start from A2, then cross the boundary (the dark green curve), and reach B1 for the first time. The PDF of FPT represents the possibility that the trajectory crosses the boundary for the first time as time goes on. The MFPT refers to the average time that the trajectory takes to cross the boundary for the first time. The reliability function refers to the possibility that the trajectory remains in the security region B2 over time.

3. Effect of Noise on the FPT Statistics and Reliability of System

In fact, an excitable system is sensitive to noise. According to the experimental results [4], the colored noise [8] may be more compatible with actual fluctuations than the white noise (4). The gene regulatory circuit in *B. subtilis* affected by colored noise is as follows [2]:

\[
\begin{align*}
\frac{dK}{dt} &= \alpha_k + \frac{\beta_k K^n}{k_n + K^n} - \frac{K}{1 + K + S}, \\
\frac{dS}{dt} &= \frac{b_s}{1 + (k_1/k_s)^p} - \frac{S}{1 + K + S} + \xi(t). \\
\end{align*}
\]

(5)

Here, $\xi(t)$ is the Ornstein–Uhlenbeck noise and the properties of $\xi(t)$ can be written as $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = D\lambda \exp(-\lambda|t-t'|)$, where $D$ is the noise strength and $\lambda$ is the inverse of the correlation time $\tau_c$.

Then, the SGCM method will be used to compute the PDF of FPT, MFPT, and reliability function of system (5). The Markov nature should be satisfied before the SGCM method is used. However, the colored noise obviously does not satisfy this nature. In order to resolve this contradiction, we convert the stochastic dynamical system with the colored noise into the equivalent one with white noise [8], which satisfies the Markov nature well. The equivalent stochastic dynamical system is as follows:

\[
\begin{align*}
\frac{dK}{dt} &= \alpha_k + \frac{b_k K^n}{k_n + K^n} - \frac{K}{1 + K + S}, \\
\frac{dS}{dt} &= \frac{b_s}{1 + (K/k_s)^p} - \frac{S}{1 + K + S} + \eta, \\
\frac{d\eta}{dt} &= h(\eta) + \lambda g_w(t). \\
\end{align*}
\]

Here, $h(\eta) = -\lambda \eta$ and $g_w$ is the Gaussian white noise with properties $\langle g_w(t) \rangle = 0$ and $\langle g_w(t) g_w(t') \rangle = 2 D \delta(t-t')$. Moreover, $\eta(t)$ satisfies $\langle \eta(t) \rangle = 0$ and $\langle \eta(t) \eta(t') \rangle = D \lambda \exp(-\lambda|t-t'|)$, and the initial distribution is $P(\eta_0) = (2\pi D\lambda)^{-1/2} \exp(-\eta_0^2/2 D)$.

In this part, we will study the effect of noise on the PDF of FPT, MFPT, and reliability function of system (6). More concretely, we analyze how the noise strength $D$ and the inverse of the correlation time $\lambda$ affect the three
quantities through the SGCM method. The stable competent state A2, the security domain B2, the target region B1, and boundary are shown in Figure 1. Note that the symbols $f(t)$, $M(t)$, and $P(t)$, respectively denote the PDF of FPT, MFPT, and reliability function in the following figures.

The PDF of FPT is shown in Figure 2(a) with $b_k = 0.13$, $b_g = 0.75$, and $\lambda = 0.30$. For $f(t)$ with different noise strength $D$, there is the same trend, i.e., as the time $t$ increases, all $f(t)$ increases from initial value $f(0) = 0$ to the peak value, then gradually decreases, and finally returns to 0. The difference is the peak of $f(t)$. As the noise strength $D$ increases from 0.10 to 0.70, the peak of $f(t)$ gradually increases, the time corresponding to the peak reduces and the time that $f(t)$ takes to drop from the peak to zero also becomes short. These results show that probability that $B. subtilis$ achieves the state transition becomes large as $D$ increases. As shown in Figure 2(b), we can find that $M(t)$ reduces from 31 to 10 as $D$ varies from 0.10 to 0.80. Moreover, the corresponding reliability function is shown in Figure 2(c). As the noise strength $D$ increases, the

Figure 2: (a) PDF of FPT of system (6): $\lambda = 0.30$. (b) MFPT of system (6): $\lambda = 0.30$. (c) Reliability function of system (6): $\lambda = 0.30$. Complexity
The effect of $\lambda$ on $f(t)$, $M(t)$, and $P(t)$ is shown in Figure 3 with $b_h = 0.13$, $b_s = 0.75$, and $D = 0.20$. In Figure 3(a), it can be found that the value of $f(t)$ increases firstly and then gradually decreases. Obviously, as $\lambda$ increases, the time that $f(t)$ takes to drop from the peak to zero becomes short, while the time corresponding to the peak is about the same, which shows probability that $B. subtilis$ achieves the transition from the competent to vegetative state becomes large as $\lambda$ increases. As shown in Figure 3(b), we can find that $M(t)$ reduces from 76 to 12 as
λ varies from 0.05 to 1.00. The corresponding reliability function \( P(t) \) is shown in Figure 3(c). The curvature of curve increases with the increase of \( \lambda \), which indicates the reliability of the system is weakened. These results indicate that it becomes easy for a trajectory to pass through the boundary and reach the target region B1. And the reliability of the system is weakened with the increase of \( \lambda \), which means the transition (from competent to vegetative state) in \( B. subtilis \) becomes easy. In other words, the larger the inverse of the correlation time, the easier the state transition in \( B. subtilis \). Similarly, the results obtained by the SGCM method are in accordance with the MC simulation.

4. Effect of System Parameters on the FPT Statistics and Reliability of System

In this section, we analyze how the system parameters \( b_k \) and \( b_s \) affect the PDF of FPT, MFPT, and reliability function of system (5). The noise parameters are taken as \( D = 0.20 \) and \( \lambda = 0.20 \) in the following.

The effect of parameter \( b_k \) is discussed firstly. Figure 4 shows clearly the global properties of system (5) for different \( b_k \). It is obvious that the area of security domain B2 becomes large with the increase of \( b_k \), while the area of the target region B1 becomes small. In theory, the phenomenon indicates that the state transition (from competent to vegetative state) becomes difficult.

Figure 5(a) presents the change of the PDF of FPT as \( b_k \) varies. We can see that \( f(t) \) still has only one peak. As \( b_k \) increases from 0.10 to 0.13, the peak of \( f(t) \) reduces and the time that \( f(t) \) takes to drop from the peak to zero becomes long. In Figure 5(b), \( M(t) \) increases from 4 to 34 as \( b_k \) varies from 0.10 to 0.135. The corresponding reliability function is shown in Figure 5(c). The curve becomes flat as the parameter \( b_k \) increases, which means that the reliability of the system is enhanced. These results indicate that as \( b_k \) increases, it becomes difficult for a trajectory to pass through the boundary and reach the target region B1.
That is, the state transition in *B. subtilis* becomes difficult, which is consistent with the above phenomenon in Figure 4.

Next, the effect of parameter $b_s$ is explored. Figure 6 shows the global dynamical character of system (5) for different $b_s$. As the $b_s$ increases, the area of security domain B2 becomes large and the area of the target region B1 becomes small. This phenomenon theoretically indicates that the state transition becomes difficult.

Figure 7(a) shows the change of the PDF of FPT. As $b_s$ increases, the peak of $f(t)$ gradually reduces, and the time that $f(t)$ takes to drop from the peak to zero becomes long. In Figure 7(b), $M(t)$ increases from 11 to 26 as $b_s$ varies from 0.68 to 0.725. The corresponding reliability function is shown in Figure 7(c). The curvature of curve decreases as $b_s$ increases, which indicates that the reliability of the system is enhanced. Thus, with the increase of $b_s$, it becomes difficult for a trajectory to pass through the boundary and reach the target region B1, which means the state transition in *B. subtilis* becomes difficult. The conclusion is in accordance with the above phenomenon in Figure 6.

**Figure 5:** (a) Probability density function of FPT of system (6): $D = 0.20, \lambda = 0.20$. (b) MFPT of system (6): $D = 0.20, \lambda = 0.20$. (c) Reliability function of system (6): $D = 0.20, \lambda = 0.20$. 
Figure 6: The global dynamic character of system (4): $b_x = 0.14$, (a) $b_x = 0.68$, (b) $b_x = 0.69$, (c) $b_x = 0.70$, and (d) $b_x = 0.71$.

Figure 7: Continued.
Conclusions

In this paper, we focus on the time of transition from competent state to vegetative state in *B. subtilis*. The stochastic exit problem of the gene regulatory circuit in *B. subtilis* affected by the colored noise is investigated through the three quantities related to the time: the PDF of FPT, MFPT, and the reliability function. The results show that noise parameters and system parameters have significant effect on the state transition (from competent to vegetative state). As noise parameter \( D \) or \( \lambda \) increases, the state transition in *B. subtilis* becomes easy and the reliability of the system is weakened. Conversely, with the increase of system parameters \( b_k \) or \( b_s \), the state transition in *B. subtilis* becomes difficult and the reliability of the system is enhanced. In addition, the effectiveness of the SGCM method is verified by MC simulation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

The study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Acknowledgments

This research was supported by National Natural Science Foundation of China (Grant nos.11972289, 11672230, 11672231, and 11672232), Natural Science Basic Research Plan in Shaanxi Province (Grant no. 2018J1043), and the Central University Fundamental Research Fund (Grant no. 3102018ZY043).

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