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Research Article

A Possibilistic Portfolio Model with Fuzzy Liquidity Constraint

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Investors are concerned about the reliability and safety of their capital, especially its liquidity, when investing. This paper sets up a possibilistic portfolio selection model with liquidity constraint. In this model, the asset return and liquidity are fuzzy variables which follow the normal possibility distributions. Liquidity is measured as the turnover rate of the asset. On the basis of possibility theory, we transform the model into a quadratic programming problem to obtain its solution. We illustrate that, in the process of investment, investors can make better use of capital by choosing their investment portfolios according to their expected return and asset liquidity.

1. Introduction

Recently, portfolio selection has received extensive attentions [1–4]. The portfolio selection model studies how to allocate investment funds among different assets to guarantee profits and disperse investment risk. Markowitz [5] proposed a mean-variance model (MVM) for portfolio selection, which played an important role in the development of modern portfolio selection theory. The MVM uses mean and variance to describe, respectively, the expected return and risk of a portfolio. The basic rule is the investors' trade-off between expected return and risk.

In the MVM, using the variance of return of a portfolio as a risk measure has some limitations and computational difficulties to construct large-scale portfolio. In order to overcome the difficulty, some researchers extended the MVM in various ways. Examples are the semivariance model [6], mean absolute deviations model [7], semiabsolute deviation model [8], Value-at-Risk (VaR) model [9], Conditional Value-at-Risk (CVaR) model [10], and so on. But whether there is a risk measure that is best for all portfolios is still an open problem [11]. The main reason is that each measure performs the best in its domain, but not when considered in another measure domain [12].

The MVM combines probability theory with optimization techniques to model investment behavior under uncertainty. However, Black and Litterman [13] showed that the solutions of the MVM are very sensitive to perturbations of input parameters. This means that a small uncertainty in the parameters can make the usual optimal solution practically meaningless. Therefore, it is necessary to develop models that are immune, as far as possible, to data uncertainty. Robust optimization is a good tool to solve this problem [14]. In order to systematically counter the sensitivity of the optimal portfolio to statistical and modeling errors in the estimation of relevant market parameters, Goldfarb and Iyengar [15] established a robust portfolio selection problem. They introduced "uncertainty structures" for the market parameters and transformed the robust portfolio selection problem corresponding to these uncertainty structures into a second-order cone program. Piri et al. [16] studied the robust models of the mean-Conditional Value-at-Risk (M-CVaR) portfolio selection problem, in which the mean return risk is estimated in both the interval and ellipsoidal uncertainty sets. The corresponding robust models are a linear programming problem and a quadratic programming problem, respectively. Kara et al. [17] established a robust optimization problem based on the data. In order to obtain the robust optimal solution of the

portfolio, they used the method of Robust Conditional Value-at-Risk under Parallelepiped Uncertainty, an evaluation, and a numerical finding of the robust optimal portfolio allocation. Kang and Li [18] proposed a robust mean-risk optimization problem with multiple risk measures under ambiguous distribution in which variance, value of risk (VaR), and conditional value of risk (CVaR) were simultaneously used as a triple-risk measure. And a closed-form expression of the portfolio was obtained. Khanjarpanah and Pishvaee [19] established a robust flexible portfolio optimization model and extended the model by introducing an improved robust flexible method. Khodamoradi et al. [20] established a robust cardinality constraints mean-variance optimization model and proved that the model was equivalent to a mixed integer second-order cone program. Through numerical examples, they found that, in both robust and nonrobust models, short selling and risk-neutral interest rates would increase sharpe ratios. Robust optimization solves the effect of parameter uncertainty on the optimal solution well. In addition, in real financial markets, investors' behaviors are influenced by such uncertainty factors as politics, psychology, and social cognition. Fuzzy set theory [21] is one of the effective methods to deal with such uncertainty. Many scholars have tried to employ fuzzy variables to manage portfolio selection problem and build many fuzzy portfolio models [22–24]. Wang and Zhu [25] introduced the fuzzy portfolio selection models. In 2012, Tsaur [26] constructed a fuzzy portfolio model with the parameters of fuzzy-input return rates and fuzzy-output proportions. Zhou et al. [27] proposed the concept of fuzzy semientropy. They used the semientropy to quantify the downside risk and set up two mean-semientropy portfolio selection models. To obtain the optimal solution, they used the genetic algorithm. Possibility theory is an important theory of fuzzy sets which was first proposed by Zadeh [28] and developed by Dubois and Prade [29]. In possibility theory, the relationship between fuzzy variables and possibility distributions is the same as that between random variables and probability distributions in probability theory. Tanaka et al. [30] firstly proposed the possibilistic portfolio selection model. In this model, the fuzzy variables were considered to follow the exponential possibility distributions. In 1999, Tanaka and Guo [31] believed that the upper and lower possibility distributions can be used to reflect experts' knowledge in the portfolio selection model. Carlsson and Fullér [32] introduced the notions of lower and upper possibilistic mean values as well as the notations for crisp possibilistic mean value and variance of continuous possibility distributions. Based on that, Carlsson et al. [33] assumed that the return of securities is trapezoidal fuzzy variables and found the optimal portfolio model with the highest utility score by scoring the utility. Zhang and Nie [34] extended the possibilistic mean and variance concepts proposed by Carlsson and Fullér and presented the notions of upper and lower possibilistic variances and covariances of fuzzy numbers. Then, in 2007, Zhang [35] proposed a possibilistic mean-standard deviation model in which the investment proportion has lower bound constrained. Zhang et al. [36] proposed two kinds of portfolio selection models based on upper and lower possibilistic means and variances, and the notions of upper and lower possibilistic efficient portfolios. In order to better integrate an uncertain decision environment with vagueness and ambiguity, in 2009, Zhang et al. [37] proposed a possibilistic mean-variance portfolio selection model based on the definitions of the possibilistic return and possibilistic risk. In this model, the returns of assets were fuzzy variables with LR-type possibility distributions. Based on the definitions of the upper and lower possibilistic mean, Zhang et al. [38] defined the interval-valued possibilistic mean and possibilistic variance. Based on these, a new portfolio selection model with the maximum utility was established. Li et al. [39] developed a possibilistic portfolio model under VaR and risk-free investment constraints. Liu and Zhang [40] defined the product of multiple fuzzy numbers' possibilistic mean and variance, and, then, based on these definitions, they developed three multiperiod fuzzy portfolio optimization models. To solve these models, they applied a novel fuzzy programming approach-based self-adaptive differential evolution algorithm.

Several modifications to the basic MVM have been suggested in the literature to consider more realistic factors such as liquidity, budget, and lower/upper bound constraints, while deciding the allocation of money among assets. Liquidity refers to the ability to transact a large number of shares at prices that do not vary substantially from past prices unless new information enters the market.

There are various ways of measuring liquidity, among which trading volume, number of trades, transaction amount, turnover rate, and velocity of circulation are commonly used. The turnover rate was introduced by Datar et al. [41]. The turnover rate is the total amount of traded shares divided by the total net asset value of the fund over a particular period. Marshalla and Young [42] argued that liquidity is the most important factor in a portfolio. In our model, turnover rate is controlled through fuzzy chance constraint. Fuzzy portfolio selection with chance constraint, different assumptions, and estimation methods has been discussed in the literature (see [43–46]). Barak et al. [47] developed a mean-variance-skewness fuzzy portfolio model with cardinality constraint and considered the fuzzy chance constraint to measure portfolio liquidity. Furthermore, they designed a genetic algorithm to solve the model.

In real financial markets, short-time and institutional investors hope to not only reach the expected rate of return, but also ensure that the liquidity of the portfolio should not be lower than the expected value. Moreover, in addition to return, risk, and liquidity, the threshold constraint is also the major concern for researchers and practitioners because, in order to manage the portfolio more effectively, it is necessary to limit the upper and lower bounds (threshold constraints) of the capital invested in each asset. The core of this study was inspired by Li et al. [39] and Barak et al. [47] and has been subsequently further promoted and developed by other researchers. Numerous studies have been done about possibilistic portfolio selection but a few papers consider liquidity constraint and regard asset liquidity as fuzzy variables obey the possibility distribution.

Our goal is to analyze the return-risk trade-off with liquidity constraint and threshold constraints under an uncertain market environment. To increase the applicability of the model, the return rate of assets is expressed as a fuzzy variable which is associated with a normal possibility distribution. Liquidity is measured by turnover rate and is also represented by a fuzzy variable associated with a normal possibility distribution. Thus, we established a possibilistic portfolio selection model with fuzzy chance constraints. By using the possibility theory, we transformed the chance-constrained model into a deterministic mathematical model and obtained the solution for the model.

The rest of this paper is organized as follows. In Section 2, we present some basic concepts regarding the possibility theory and the notions of the possibilistic mean and variance of a fuzzy number. At the same time, in this section, we recall the notion of normal possibility distribution and introduce a theorem about it. In Section 3, we propose a possibilistic portfolio model with liquidity constraint and threshold constraints. We suppose the expected rate of return and the turnover rate of the assets are both normally distributed fuzzy variables and, then, provide a solution for the model by a theorem. Section 4 provides a numerical example to illustrate the proposed approach. Section 5 concludes and provides directions for future research.

2. Preliminaries

In this section, we review some definitions and properties.

Let ξ be a fuzzy variable with membership function μ, and let r be a real number. Then, the possibility of ξ is defined as follows:

$$\mathrm{Pos}\{\xi \geq r\} = \sup_{x \geq r} \mu(x). \tag{1}$$

Definition 1. The upper possibilistic mean value of $\widetilde{A}$ with α-level set $A_\alpha = [a(\alpha), b(\alpha)]$ is defined as

$$M^*(\widetilde{A}) = \frac{\int_0^1 \mathrm{Pos}[\widetilde{A} \geq b(\alpha)] b(\alpha) \mathrm{d}\alpha}{\int_0^1 \mathrm{Pos}[\widetilde{A} \geq b(\alpha)] \mathrm{d}\alpha} = 2\int_0^1 \alpha b(\alpha) \mathrm{d}\alpha, \tag{2}$$

where Pos denotes the possibility measure.

Definition 2. The lower possibilistic mean value of $\widetilde{A}$ is defined as

$$M_*(\widetilde{A}) = \frac{\int_0^1 \mathrm{Pos}[\widetilde{A} \leq a(\alpha)] a(\alpha) \mathrm{d}\alpha}{\int_0^1 \mathrm{Pos}[\widetilde{A} \leq a(\alpha)] \mathrm{d}\alpha} = 2\int_0^1 \alpha a(\alpha) \mathrm{d}\alpha. \tag{3}$$

Definition 3. The possibilistic mean value of $\widetilde{A}$ is defined as

$$M(\widetilde{A}) = \frac{M_*(\widetilde{A}) + M^*(\widetilde{A})}{2} = \int_0^1 \alpha(a(\alpha) + b(\alpha)) \mathrm{d}\alpha. \tag{4}$$

Definition 4. The possibilistic variance of $\widetilde{A}$ is defined as

$$\begin{aligned} Var(\widetilde{A}) &= \int_0^1 Pos[\widetilde{A} \leq a(\alpha)] \left(\left[\frac{a(\alpha) + b(\alpha)}{2} - a(\alpha) \right]^2 \right) d\alpha \\ &\quad + \int_0^1 Pos[\widetilde{A} \geq b(\alpha)] \left(\left[\frac{a(\alpha) + b(\alpha)}{2} - b(\alpha) \right]^2 \right) \mathrm{d}\alpha \\ &= \int_0^1 \alpha \left(\left[\frac{a(\alpha) + b(\alpha)}{2} - a(\alpha) \right]^2 \right) d\alpha \\ &\quad + \int_0^1 \alpha \left(\left[\frac{a(\alpha) + b(\alpha)}{2} - b(\alpha) \right]^2 \right) \mathrm{d}\alpha \\ &= \frac{1}{2} \int_0^1 \alpha [b(\alpha) - a(\alpha)]^2 \mathrm{d}\alpha. \end{aligned} \tag{5}$$

Definition 5. The possibilistic covariance between fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ is defined as

$$\mathrm{Cov}(\widetilde{A}, \widetilde{B}) = \frac{1}{2} \int_0^1 \alpha (b_1(\alpha) - a_1(\alpha)) (b_2(\alpha) - a_2(\alpha)) \mathrm{d}\alpha. \tag{6}$$

Lemma 1 (see[25]). *Let $\alpha, \beta \in R$ and let $\widetilde{A}$ and $\widetilde{B}$ be fuzzy numbers. Then,*

(1) $M(\alpha\widetilde{A} + \beta\widetilde{B}) = \alpha M(\widetilde{A}) + \beta M(\widetilde{B})$,
(2) $Var(\alpha\widetilde{A} + \beta\widetilde{B}) = \alpha^2 Var(\widetilde{A}) + \beta^2 Var(\widetilde{B}) + 2|\alpha\beta| Cov(\widetilde{A}, \widetilde{B})$.

Definition 6. The fuzzy variable ξ is obeying the normal possibility distribution, if its membership function is

$$\mu_\xi(x) = \exp\left\{ -\left(\frac{x - \mu}{\sigma} \right)^2 \right\}. \tag{7}$$

Then, it can be written as $\xi \sim FN(\mu, \sigma^2)$.

Example 1. If the fuzzy variable $\xi \sim FN(\mu, \sigma^2)$, then its λ-level set is

$$(\xi)_k = \left[\mu - \sigma\sqrt{\ln \lambda^{-1}}, \mu + \sigma\sqrt{\ln \lambda^{-1}} \right], \quad \lambda \in (0, 1). \tag{8}$$

In fact, from (4), we obtain

$$\begin{aligned} M(\xi) &= \frac{M_*(\xi) + M^*(\xi)}{2} = \int_0^1 \lambda (a(\lambda) + b(\lambda)) \mathrm{d}\lambda \\ &= \int_0^1 \lambda \left(\mu + \sigma\sqrt{\ln\lambda^{-1}} + \mu - \sigma\sqrt{\ln\lambda^{-1}} \right) \mathrm{d}\lambda \\ &= \int_0^1 2\mu\lambda \mathrm{d}\lambda = \mu. \end{aligned} \tag{9}$$

It follows from (5) that

$$\begin{aligned}\operatorname{Var}(\xi) &= \frac{1}{2}\int_0^1 \lambda\left(\mu+\sigma\sqrt{\ln\lambda^{-1}}-\mu+\sigma\sqrt{\ln\lambda^{-1}}\right)^2 \mathrm{d}\lambda \\ &= \frac{1}{2}\int_0^1 4\lambda\sigma^2\ln\lambda^{-1}\mathrm{d}\lambda \\ &= 2\sigma^2\int_0^1 \lambda\ln\lambda^{-1}\mathrm{d}\lambda \\ &= 2\sigma^2\frac{1}{4}=\frac{1}{2}\sigma^2.\end{aligned} \tag{10}$$

Next, we show the distribution of $\sum_{k=1}^n \lambda_k\xi_k$.

Theorem 1. *If $\xi_k, k=1,2,\ldots,n$, are n normally distributed fuzzy variables expressed as $\xi_k \sim FN(\mu_k,\sigma_k^2)$ and $\lambda_k, k=1,2,\ldots,n$, are n real numbers, then*

$$\sum_{k=1}^n \lambda_k\xi_k \sim FN\left(\sum_{k=1}^n \lambda_k\mu_k, \frac{1}{2}\left(\sum_{k=1}^n |\lambda_k|\sigma_k\right)^2\right). \tag{11}$$

Proof. According to Lemma 1 and (9), one has

$$M\left(\sum_{k=1}^n \lambda_k\xi_k\right) = \sum_{k=1}^n \lambda_k M(\xi_k) = \sum_{k=1}^n \lambda_k\mu_k. \tag{12}$$

From (6), we deduce

$$\begin{aligned}\operatorname{Cov}(\xi_i,\xi_j) &= \frac{1}{2}\int_0^1 \alpha 2\sigma_i\sqrt{\ln\alpha^{-1}}2\sigma_j\sqrt{\ln\alpha^{-1}}\mathrm{d}\alpha \\ &= 2\sigma_i\sigma_j\int_0^1 \alpha\ln\alpha^{-1}\mathrm{d}\alpha \\ &= 2\sigma_i\sigma_j\frac{1}{4}=\frac{1}{2}\sigma_i\sigma_j, \quad i,j=1,2,\ldots,n.\end{aligned} \tag{13}$$

It follows from Lemma 1 and (10) that

$$\begin{aligned}\operatorname{Var}\left(\sum_{k=1}^n \lambda_k\xi_k\right) &= \sum_{k=1}^n \lambda_k^2\operatorname{Var}(\xi_k) + 2\sum_{i>j=1}^n |\lambda_i\lambda_j|\operatorname{Cov}(\xi_i,\xi_j) \\ &= \sum_{k=1}^n \lambda_k^2\frac{1}{2}\sigma_k^2 + 2\sum_{i>j=1}^n \frac{1}{2}\sigma_i\sigma_j|\lambda_i\lambda_j| \\ &= \frac{1}{2}\left(\sum_{k=1}^n \lambda_k^2\sigma_k^2 + 2\sum_{i>j=1}^n \sigma_i\sigma_j|\lambda_i\lambda_j|\right) \\ &= \frac{1}{2}\left(\sum_{k=1}^n |\lambda_k|\sigma_k\right)^2.\end{aligned} \tag{14}$$

Thus,

$$\sum_{k=1}^n \lambda_k\xi_k \sim FN\left(\sum_{k=1}^n \lambda_k\mu_k, \frac{1}{2}\left(\sum_{k=1}^n |\lambda_k|\sigma_k\right)^2\right). \tag{15}$$

This completes the proof. □

3. Model Foundation

3.1. Possibilistic Portfolio Model with Fuzzy Liquidity Constraint and Risk-Free Investment. Suppose that there are n risky assets and one risk-free asset available for investment. Let $\tilde{r}_k$ be the return rate of asset $k, k=1,2,\ldots,n$, which is a fuzzy number. Let x_k represent the proportion invested in asset $k, k=1,2,\ldots,n$, and let r_f be the return of the risk-free asset. Thus, the return $\tilde{R}$ on the portfolio can be written as

$$\tilde{R} = \sum_{k=1}^n x_k\tilde{r}_k + r_f\left(1-\sum_{k=1}^n x_k\right). \tag{16}$$

Obviously $\tilde{R}$ is a fuzzy number.

To establish a new model, we need the following values. The possibilistic mean of the portfolio return $\tilde{R}$ is given by

$$M(\tilde{R}) = \sum_{k=1}^n x_k M(\tilde{r}_k) + r_f\left(1-\sum_{k=1}^n x_k\right). \tag{17}$$

The possibilistic variance of $\tilde{R}$ is written as

$$\begin{aligned}\operatorname{Var}(\tilde{R}) &= \sum_{k=1}^n x_k^2\operatorname{Var}(\tilde{r}_k) + 2\sum_{i>j=1}^n |x_ix_j|\operatorname{Cov}(\tilde{r}_i,\tilde{r}_j) \\ &= \sum_{k=1}^n x_k^2\operatorname{Var}(\tilde{r}_k) + 2\sum_{i>j=1}^n x_ix_j\operatorname{Cov}(\tilde{r}_i,\tilde{r}_j).\end{aligned} \tag{18}$$

Now, we can establish the following possibilistic portfolio selection model with fuzzy liquidity constraint:

$$(\mathrm{PL1})\begin{cases}\min \operatorname{Var}(\tilde{R}) = \sum_{k=1}^n x_k^2\operatorname{Var}(\tilde{r}_k) + 2\sum_{i>j=1}^n x_ix_j\operatorname{Cov}(\tilde{r}_i,\tilde{r}_j), \\ \text{s.t.} \sum_{k=1}^n x_k\mathrm{M}(\tilde{r}_k) + r_f\left(1-\sum_{k=1}^n x_k\right) \ge \mu, \\ \operatorname{Pos}\left[\sum_{k=1}^n \tilde{l}_k x_k \ge l_0\right] \ge 1-\alpha, \\ \sum_{k=1}^n x_k \le 1, \\ 0 \le d_k \le x_k \le g_k, k=1,2,\ldots,n,\end{cases} \tag{19}$$

where μ is the underestimated expected return rate and l_0 is the predetermined value. d_k and g_k represent, respectively, the lower and the upper bounds on investment in asset k , $k=1,2,\ldots,n$. $\tilde{l}_k$ is the turnover rate of asset $k, k=1,2,\ldots,n$, which reflects the liquidity of the asset. α reflects the sensitivity of the investor. If the value of α is close to 0, then the investor is sensitive to liquidity of the portfolio. Otherwise, the investor is not sensitive to the portfolio's liquidity.

In this section, we suppose that the return rate of asset $k, k=1,2,\ldots,n$, is a normally distributed fuzzy variable denoted as $\tilde{r}_k \sim FN(\mu_k,\sigma_k^2)$, and its membership function is

$$A_{\tilde{r}_k}(x) = \exp\left\{-\left(\frac{x-\mu_k}{\sigma_k}\right)^2\right\}. \tag{20}$$

The turnover rate of asset $k, k = 1, 2, \ldots, n$, is also a normally distributed fuzzy variable denoted as $\tilde{l}_k \sim FN(a_k, b_k^2)$. So, its membership function is

$$A_{\tilde{l}_k}(x) = \exp\left\{-\left(\frac{x - a_k}{b_k}\right)^2\right\}. \tag{21}$$

From Theorem 1, we obtain

$$\tilde{R} \sim FN\left(\sum_{k=1}^{n} x_k\mu_k + r_f\left(1 - \sum_{k=1}^{n} x_k\right), \frac{1}{2}\left(\sum_{k=1}^{n} x_k\sigma_k\right)^2\right),$$
$$\sum_{k=1}^{n} \tilde{l}_k x_k \sim FN\left(\sum_{k=1}^{n} x_k a_k, \frac{1}{2}\left(\sum_{k=1}^{n} x_k b_k\right)^2\right). \tag{22}$$

Then, the membership function of $\sum_{k=1}^{n} \tilde{l}_k x_k$ is defined by

$$A(u) = \exp\left\{-\frac{\left(u - \sum_{k=1}^{n} x_k a_k\right)^2}{(1/2)\left(\sum_{k=1}^{n} x_k b_k\right)^2}\right\}. \tag{23}$$

Furthermore,

$$\operatorname{Pos}\left[\sum_{k=1}^{n} \tilde{l}_k x_k \geq l_0\right] = \sup_{u \geq l_0} \exp\left\{-\frac{\left(u - \sum_{k=1}^{n} x_k a_k\right)^2}{(1/2)\left(\sum_{k=1}^{n} x_k b_k\right)^2}\right\}$$
$$= \begin{cases} \exp\left\{-\dfrac{\left(l_0 - \sum_{k=1}^{n} x_k a_k\right)^2}{(1/2)\left(\sum_{k=1}^{n} x_k b_k\right)^2}\right\}, & \text{for } \sum_{k=1}^{n} x_k a_k < l_0, \\ 1, & \text{for } \sum_{k=1}^{n} x_k a_k \geq l_0. \end{cases} \tag{24}$$

From (19) and (24), we have

$$-\left(l_0 - \sum_{k=1}^{n} a_k x_k\right)^2 \geq \frac{1}{2}\ln(1-\alpha)\left(\sum_{k=1}^{n} x_k b_k\right)^2. \tag{25}$$

From (22) and (25), (PL1) can be transformed into

$$(\text{NPL1})\begin{cases} \min \operatorname{Var}(\tilde{R}) = \dfrac{1}{2}\left(\sum_{k=1}^{n} x_k\sigma_k\right)^2, \\ \text{s.t.} \sum_{k=1}^{n} x_k\left(\mu_k - r_f\right) + r_f \geq \mu, \\ -\left(l_0 - \sum_{k=1}^{n} a_k x_k\right)^2 \geq \dfrac{1}{2}\ln(1-\alpha)\left(\sum_{k=1}^{n} x_k b_k\right)^2, \\ \sum_{k=1}^{n} x_k \leq 1, \\ 0 \leq d_k \leq x_k \leq g_k, \quad k = 1, 2, \ldots, n. \end{cases} \tag{26}$$

3.2. Possibilistic Portfolio Model without Risk-Free Investment. Similar to the previous section, we propose a portfolio selection model without risk-free investment as follows:

$$(\text{PL2})\begin{cases} \min \operatorname{Var}(\tilde{R}) = \sum_{k=1}^{n} x_k^2 \operatorname{Var}(\tilde{r}_k) + 2\sum_{i>j=1}^{n} x_i x_j \operatorname{Cov}(\tilde{r}_i, \tilde{r}_j), \\ \text{s.t.} \sum_{k=1}^{n} x_k M(\tilde{r}_k) \geq \mu, \\ \operatorname{Pos}\left[\sum_{k=1}^{n} \tilde{l}_k x_k \geq l_0\right] \geq 1 - \alpha, \\ \sum_{k=1}^{n} x_k = 1, \\ 0 \leq d_k \leq x_k \leq g_k, \quad k = 1, 2, \ldots, n. \end{cases} \tag{27}$$

If the return rate of asset $k, k = 1, 2, \ldots, n$, is a normally distributed fuzzy variable denoted as $\tilde{r}_k \sim FN(\mu_k, \sigma_k^2)$ and the turnover rate of asset $k, k = 1, 2, \ldots, n$, is also a normally distributed fuzzy variable denoted as $\tilde{l}_k \sim FN(a_k, b_k^2)$, then the model (PL2) can be transformed into

$$(\text{NPL2})\begin{cases} \min \operatorname{Var}(\tilde{R}) = \dfrac{1}{2}\left(\sum_{k=1}^{n} x_k\sigma_k\right)^2, \\ \text{s.t.} \sum_{k=1}^{n} x_k\mu_k \geq \mu, \\ -\left(l_0 - \sum_{k=1}^{n} a_k x_k\right)^2 \geq \dfrac{1}{2}\ln(1-\alpha)\left(\sum_{k=1}^{n} x_k b_k\right)^2, \\ \sum_{k=1}^{n} x_k = 1, \\ 0 \leq d_k \leq x_k \leq g_k, \quad k = 1, 2, \ldots, n. \end{cases} \tag{28}$$

(NPL1) and (NPL2) are quadratic programming problems and can be solved by MATLAB, Lingo, etc.

4. Numerical Example

In this section, we give a real portfolio example to illustrate our approach. In this example, we selected eight stocks from the Shanghai Stock Exchange. We collected data on monthly returns and turnover rate for each of the eight stocks over the period of January 2006 to December 2006 from the RESSET Financial Research Database. We use SPSS to generate the frequency distributions of the monthly returns and turnover rates.

Table 1 shows the transaction codes and the frequency distributions of the monthly returns, while Table 2 shows the transaction codes and the frequency distributions of the monthly turnover rates.

Table 1: The possibility distribution of returns.

Stock code	μ_k	σ_k
600058	0.06	0.167
600028	0.09	0.102
600089	0.10	0.207
600115	0.04	0.111
600170	0.03	0.049
600495	0.10	0.080
600526	0.01	0.086
600662	0.03	0.076

Table 2: The possibility distribution of turnover rates (%).

Stock code	a_k	b_k
600058	40.70	31.762
600028	1.77	0.802
600089	32.51	10.681
600115	9.75	4.384
600170	14.86	8.796
600495	73.85	51.658
600526	65.47	16.565
600662	40.14	19.587

Let the risk-free asset be a bank deposit. We use the three-month deposit interest rates as the return on risk-free assets. So, we get the return on risk-free assets $r_f = 2.8\%$. If the possibility of the portfolio's turnover rate must be more than 0.002 (in more than 90% times); that is, $l_0 = 0.2\,(\%)$ and $\alpha = 10\%$, then the lower bound of investment ratio x_k must be $d = \{0.02, 0, 0.1, 0, 0.02, 0.1, 0.1, 0.05\}$ and the upper bound $g = \{0.3, 0.2, 0.3, 0.2, 0.2, 0.4, 0.4, 0.3\}$. By solving the model (NPL1) and (NPL2), the possibilistic efficient portfolios for the different μs are obtained as shown in Tables 3 and 4.

From these two tables we can see that the risk increases as μ increases. We also can see that when the value of μ is low (e.g., $\mu = 3\%$), the proportion that invests is low. For example, when the value of μ is equal to 3%, the investment proportion with risk-free is the lower bound but the investment proportion without risk-free is 0.02, 0.0702, 0.1, 0, 0.2, 0.1, 0.2098, and 0.3. Figure 1 shows these two efficient portfolios. From Figure 1, we can see that when the risk-free investment is included in the portfolio, the risk is lower at the same value ofμ. And from Figure 1, we also can draw the conclusion that the risk of the portfolios can be spread by diversification. The impact of liquidity constraint on the portfolio is shown in Figure 2. Figure 2 shows some possibilistic efficient portfolios with and without liquidity constraints.

It can be seen from Figure 2 that, under the same expected return, those investors who have requirements for financial liquidity need to take on greater risks. In other words, if investors want to keep the risks they take constant, the demand for financial liquidity will reduce the expected return of the portfolio. In addition, the impact of liquidity floor l_0 on the portfolio is shown in Figure 3. Figure 3 shows some possibilistic efficient portfolios with different l_0. The impact of l_0 on the portfolio depends on the slope and the intercept of the l_0 line and the position of the possibilistic efficient frontier.

As shown in Figure 3, when the position of the possibilistic efficient frontier and the slope of the l_0 line remain unchanged, the intercept changes of the l_0 line will affect the optimal solution of the model. As l_0 moves from "a" point to "b" point, the optimal solution of the model would not change. But when l_0 moves from "a" point to "c" point, this change would affect the optimal solution of the model.

Analogous to the above, if the turnover rate of the portfolio must be more than 0.002, i.e., $l_0 = 0.2\,(\%)$, and the expected rate of return must reach 8% or more, then the lower and upper bound of investment ratio x_k should be $d = \{0.02, 0, 0.1, 0, 0.02, 0.1, 0.1, 0.05\}$ and $g = \{0.3, 0.2, 0.3, 0.2, 0.2, 0.4, 0.4, 0.3\}$, respectively. By solving the model (NPL1) with different investor sensitivity parameter values, we obtain the possibilistic efficient portfolios in Table 5. The relationship between investor sensitivity and the portfolio risk is shown in Figure 4.

From Figure 4, we can see that an increase in investor sensitivity is associated with a decrease in the portfolio risk, and when investor sensitivity reaches a certain threshold, the portfolio risk no longer reduces. That is to say, in this case, if investors seek at least 8% of the profits, regardless of the investors' preference for liquidity, the minimum risk of the portfolio is 0.5126 (or 51.26%).

The impact of liquidity sensitivity α on the portfolio is shown in Figure 5. Figure 5 shows some possibilistic efficient portfolios with different α. In Figure 5, when the position of the possibilistic efficient frontier and the intercept of the α line remain unchanged, the slope changes of the α line will affect the optimal solution of the model. As α moves from "a" point to "b" point, the optimal solution of the model

Table 3: Possibilistic efficient portfolios with risk-free investment for the different μs with $l_0 = 0.2\,(\%)$ and $\alpha = 10\%$.

μ (%)	600058	600028	600089	600115	600170	600495	600526	600662	Risk	$\sum_{k=1}^{8} x_k$
3	0.02	0	0.1	0	0.02	0.1	0.1	0.05	0.0901	0.39
5	0.02	0	0.1	0	0.02	0.2330	0.1	0.05	0.1441	0.523
6.5	0.02	0.0717	0.1	0	0.02	0.3737	0.1	0.05	0.2654	0.7354
7	0.02	0.1521	0.1	0	0.02	0.3717	0.1	0.05	0.3287	0.8138
7.5	0.02	0.2	0.1497	0	0.02	0.3482	0.1	0.05	0.4408	0.8879
8	0.02	0.2	0.2725	0	0.02	0.2929	0.1	0.05	0.6485	0.9554
8.12	0.02	0.2	0.3	0.0197	0.02	0.2779	0.1	0.05	0.7252	0.9876

Table 4: Possibilistic efficient portfolios without risk-free investment for the different μs with $l_0 = 0.2\,(\%)$ and $\alpha = 10\%$.

μ (%)	600058	600028	600089	600115	600170	600495	600526	600662	Risk	$\sum_{k=1}^{8} x_k$
3	0.02	0.0702	0.1	0	0.2	0.1	0.2098	0.3	0.3607	1
5	0.02	0.0760	0.1	0	0.2	0.1547	0.1493	0.3	0.3640	1
6.5	0.02	0.1554	0.1	0	0.2	0.2868	0.1	0.1378	0.4004	1
7	0.02	0.1906	0.1	0	0.2	0.3281	0.1	0.0613	0.4165	1
7.5	0.02	0.2	0.1817	0	0.1386	0.3097	0.1	0.05	0.5309	1
8	0.02	0.2	0.2853	0	0.0671	0.2776	0.1	0.05	0.6910	1
8.12	0.02	0.2	0.3	0.03	0.024	0.276	0.1	0.05	0.7252	1

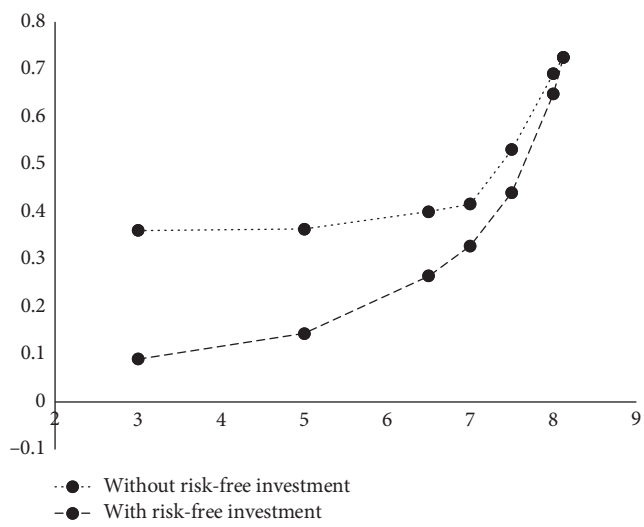

Figure 1: Variation of portfolio risks with μ.

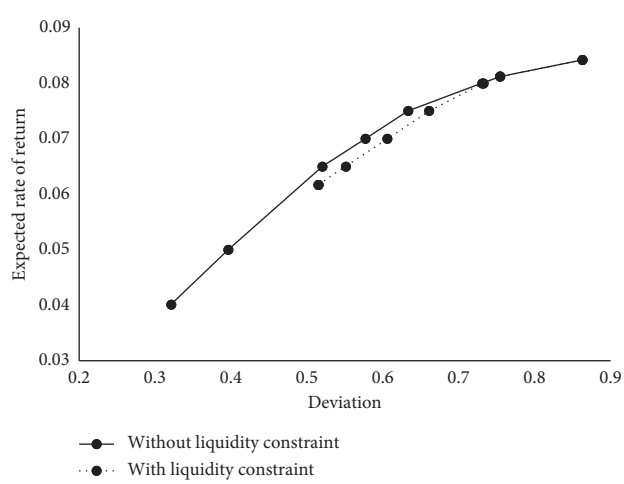

Figure 2: Some possibilistic efficient portfolios with and without liquidity constraint.

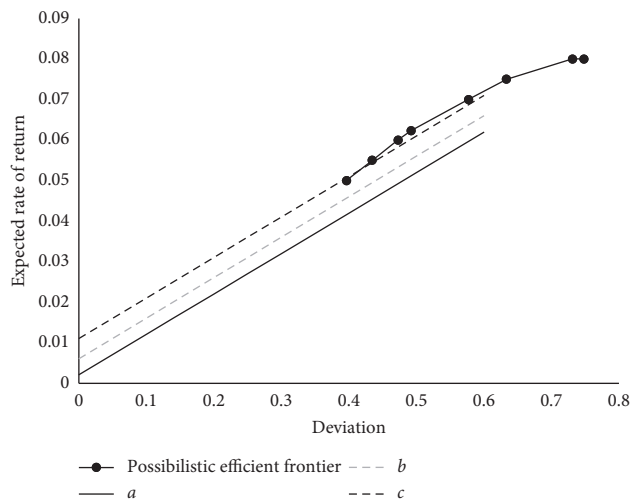

Figure 3: Some possibilistic efficient portfolios with different l_0.

Table 5: Possibilistic efficient portfolios with risk-free investment for the different α values with $l_0 = 0.2\,(\%)$and $\mu = 8\,(\%)$.

α (%)	600058	600028	600089	600115	600170	600495	600526	600662	Risk	$\sum_{k=1}^{8} x_k$
0.1	0.02	0.2	0.2725	0	0.02	0.2929	0.1	0.05	0.6485	0.9554
0.2	0.02	0.2	0.2544	0	0.02	0.3110	0.1	0.05	0.6241	0.9564
0.3	0.02	0.2	0.2315	0	0.02	0.3339	0.1	0.05	0.5942	0.9605
0.4	0.02	0.2	0.2012	0	0.02	0.3642	0.1	0.05	0.5560	0.9554
0.5	0.02	0.2	0.1654	0	0.02	0.4	0.1	0.05	0.5126	0.9554
0.6	0.02	0.2	0.1654	0	0.02	0.4	0.1	0.05	0.5126	0.9554

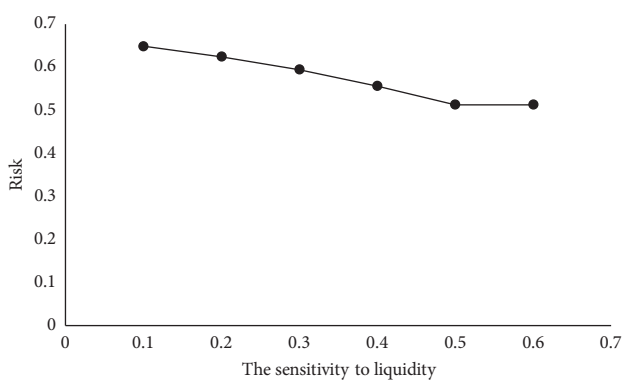

Figure 4: Variation of portfolio risk with investor sensitivity α.

would not change. But when α moves from "a" point to "c" point, this change would affect the optimal solution of the model.

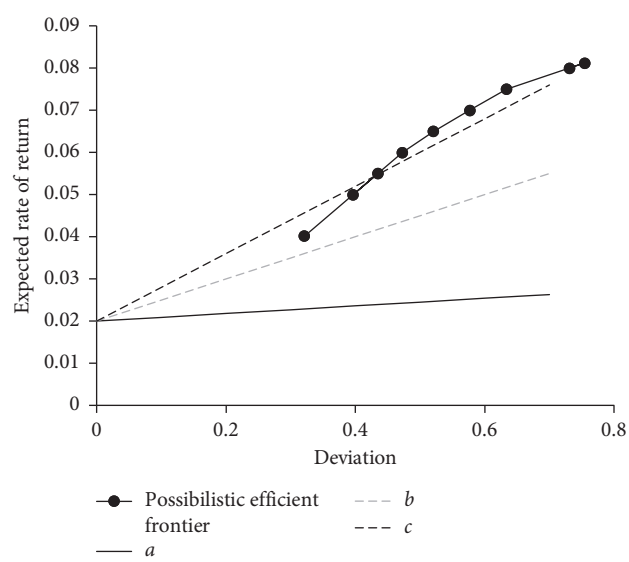

Figure 5: Some possibilistic efficient portfolios with different α.

5. Conclusions and Directions for Future Research

In this paper, we proposed a possibilistic portfolio model, which is different from the MVM proposed by Markowitz. Unlike the MVM, we measured the liquidity of asset as the turnover rate. Besides, we assumed that the expected rate of returns and turnover rates of assets are fuzzy variables, which follow the normal possibility distribution. We, then, applied the possibility theory and fuzzy chance constraint to consider the turnover rate and obtained the solution for the model. Furthermore, we illustrated our proposed effective approaches for the portfolio construction using numerical examples. In these examples, we analyzed the risk between the portfolios with and without risk-free constraints. We also analyzed changes in the portfolio's risk with respect to

changes in investor sensitivity. Finally, we drew a conclusion that our method can play a leading role in financial markets.

Future research can extend our model in the following ways. Firstly, researchers can use other methods to solve the problem and compare the results. Secondly, our fuzzy portfolio model can be extended to a multiperiod case. Thirdly, our model can be considered in a fuzzy random environment. Finally, with the development of behavioral finance, investor behavior has received more attention. We believe that investor behavior can be considered in fuzzy portfolio theory.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

This manuscript was presented in 2018 International Conference on Applied Finance, Macroeconomic Performance, and Economic Growth.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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