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Research Article

Some Topological Notations via Maki's Λ-Sets

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Our purpose is to present the notions of a β-Λ-set and a β-V-sets in topological space. We discuss the basic properties of β-Λ-sets and β-V-sets. Also, the achievement of the topology defined by these families of sets is obtained. Finally, these results are applied to the case of (X, τ) which is the digital n-space (Z^n, K^n) (cf. Section 4).

1. Introduction and Preliminaries

In the first section, we survey some of the standard facts on the concept of near open sets in the topological spaces. A generalized open set which is called a β-open set is introduced by Abd El-Monsef et al. [1]. Maki [2] developed the definition of Levin [3] and Dunham [4] to generalize closed sets, by introducing the concept of Λ-set in topological spaces as the set that coincides with their kernel. Also, he compares the topology generated by Λ-set and the given topology. The development continued at the hands of Calads and Dontchev [5]. They presented concepts of Λ_s-sets, V_s-sets, and $g.V_s$-sets. The new topology deduced by the concept of pre-Λ-sets, and pre-V-sets were introduced by Ganster et al. [6]. Also, Caldas et al. [7, 8] set up notation of Λ_p-set and Λ_p-continuous function. Λ_p-irresolute functions are defined as an identification of irresolute functions and V_p-closed functions by using Λ_p-sets and V_p-sets. Notions enable us to obtain conditions under which functions and inverse functions preserve Λ_p-sets and V_p-sets. Abd El-Monsef et al. [9] introduced the notion of b-Λ-sets and b-V-sets and obtained new topologies defined by these families of sets. Also, $g \cdot \Lambda_b$-sets, $g \cdot V_b$-sets, and some of its properties were introduced and studied. The digital line is a typical example of a $T_{(1/2)}$ space proved by Khalimsky et al. [10]. The Λ_β-sets and V_β-sets concepts can be applied in a rough set theory of decreasing the boundary region of any subset of an approximation space. The notation of a Λ_β, V_β, $g.\Lambda_\beta$-sets, and $g.V_\beta$-sets shall be introduced. Properties of these sets and some related new separation axioms are investigated. Also, we introduce a new class of topological spaces called T^{V_β}-spaces and as an application with the clarification that the image of T^{V_β}-spaces under a homeomorphism is a T^{V_β}-space and that a T^{V_β}-space is equivalent to pre-R_0 space.

Via this paper, (Y, σ)(or purely Y) represent the topological spaces. For a subset B of Y, $\mathrm{cl}(B)$, $\mathrm{int}(B)$, and B^c denote the closure of B, the interior of B, and the complement of B, respectively. Let us recall the following definitions, which are useful in the sequel.

Definition 1. A subset B of the space Y, which carries topology σ is called

(a) Semiopen [11] if $B \subseteq \mathrm{cl}(\mathrm{int}(B))$
(b) Preopen [12] if $B \subseteq \mathrm{int}(\mathrm{cl}(B))$
(c) α-Open [13] if $B \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B)))$
(d) β-Open [1] (or semi-preopen [14]) if $B \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(B)))$

(e) b-Open [15] if $B \subseteq \operatorname{int}(\operatorname{cl}(B)) \cup \operatorname{cl}(\operatorname{int}(B))$

The class of all semiopen sets (resp., preopen, α-open, β-open, and b-open) is denoted by $So(Y,\sigma)$ (resp., $Po(Y,\sigma)$, $\alpha o(Y,\sigma)$, $\beta o(Y,\sigma)s$, and $Bo(Y,\sigma)$).

The complement of these sets called semiclosed [16] (resp., preclosed [12], α-closed [13], β-closed [1], and b-closed [15]), and the classes of all these sets will be denoted by $Sc(Y,\sigma)$(resp., $Pc(Y,\sigma)$, $\alpha c(Y,\sigma)$, $\beta c(Y,\sigma)$, and $Bc(Y,\sigma)$).

Definition 2. A subset B of a topological space (Y,σ) is called

(a) Λ-set (resp., V-set) [2] if it is an intersection (resp., union) of open supersets of B(resp., closed sets contained in B)

(b) Λ_α-set (resp., V_α-set) [5] if it is an intersection (resp., union) of α-open supersets of B (resp.,α-closed sets contained in B)

(c) Λ_s-set (resp.V_s-set) [5] if it is an intersection (resp., union) of semiopen supersets of B(resp., semiclosed sets contained in B)

(d) Pre-Λ-set (resp., pre-V-set) [6] if it is an intersection (resp., union) of preopen supersets of B(resp., preclosed sets contained in B)

(e) b-Λ-set (resp.b-V-set) [9] if it is an intersection (resp., union) of b-open supersets of B(resp., b-closed sets contained in B)

Remark 1. For some kernels in an ordinary topological space (Y,σ), the following different notations may be used (e.g., [4, 5, 7, 9, 10, 12, 17–20]).

For example, $\sigma - \ker(B) = \ker(B) = B^{\Lambda}$, where $B^{\Lambda} = \cap\{J\colon J \supseteq B, J \in \sigma\}$ and B is called the kernel of B.

$So(Y,\sigma) - \ker(B) = S\ker(B) = B^{\Lambda_s} = \Lambda_s(B)$, where $B^{\Lambda_s} = \cap\{J\colon J \supseteq B, J \in So(Y,\sigma)\}$ and B^{Λ_s} is named the semikernel of B.

$Po(Y,\sigma) - \ker(B) = P\ker(B) = B^{\Lambda_p} = \Lambda_p(B)$, where $B^{\Lambda_p} = \cap\{J\colon J \supseteq B, J \in Po(Y,\sigma)\}$ and B^{Λ_p} is named the prekernel of B.

σ^{α}-$\ker(A) = \alpha\ker(B) = B^{\Lambda_\alpha} = \Lambda_\alpha(B)$, where $B^{\Lambda_\alpha} = \cap\{J\colon J \supseteq B, J \in \sigma^{\alpha}\}$ and B^{Λ_α} is named the α-kernel of B.

$Spo(Y,\sigma)$-$\ker(B) = Sp\ker(B) = B^{\Lambda_{sp}} = \Lambda_{sp}(B)$, where $B^{\Lambda_{sp}} = \cap\{J\colon J \supseteq B, J \in Spo(Y,\sigma)\}$ and $B^{\Lambda_{sp}}$ is named the semi-prekernel of B.

$Bo(Y,\sigma)$-$\ker(B) = B\ker(B) = B^{\Lambda_B} = \Lambda_B(B)$, where $B^{\Lambda_B} = \cap\{J\colon J \supseteq B, J \in Bo(Y,\sigma)\}$ and B^{Λ_B} is named the B-kernel of B.

$Ro(Y,\sigma)$-$\ker(B) = R\ker(B) = B^{\Lambda_R} = \Lambda_R(B)$, where $B^{\Lambda_R} = \cap\{J\colon J \supseteq B, J \in Ro(Y,\sigma)\}$ and B^{Λ_R} is named the R-kernel of B.

Definition 3. A subset B of a topological space (Y,σ) is named

(a) Generalized Λ-set [2] if $\Lambda(B) \subseteq F$, whenever $B \subseteq F$, and $F \in C(Y,\sigma)$

(b) Generalized semi-Λ-set [5] if $\Lambda_s(B) \subseteq F$, whenever $B \subseteq F$, and $F \in Sc(Y,\sigma)$

(c) Generalized pre-Λ-set [6] if $\Lambda_p(B) \subseteq F$, whenever $B \subseteq F$, and $F \in pc(Y,\sigma)$

(d) Generalized b-Λ-set [9] if $\Lambda_b(B) \subseteq F$, whenever $B \subseteq F$, and $F \in Bc(X,\tau)$

Definition 4. A space Y which carries topology σ is named a β-T_1-space [21] if to each pair of distinct points y, z of (Y,σ), there corresponds a β-open set B containing y but not z and a β-open set C containing z but not y. Clearly, a space (Y,σ) is β-T_1 if and only if each singleton is β-closed.

Definition 5. A space Y which carries topology σ is said to be

(i) β-Connected [22, 23] if Y cannot be expressed as the union of two nonempty disjoint β-open sets of Y

(ii) β-R_0 [24] if for any β-open set U and any point $y \in U$, $\beta\mathrm{Cl}(\{y\}) \subseteq U$

2. Λ_β-Sets and V_β-Sets

In the second part, we introduce the concept of a β-Λ-set and a β-V-set in topological space which is denoted by Λ_β, V_β respectively, and we proceed with the study of its characterization.

Definition 6. Let B be a subset of a space Y which carries topology σ. We define the subsets $\Lambda_\beta(B)$ and $V_\beta(B)$ as follows:

(a) $\Lambda_\beta(B) = B^{\Lambda_\beta} = \cap\{G\colon G \supseteq B, G \in \beta O(Y,\sigma)\}$

(b) $V_\beta(B) = B^{V_\beta} = \cup\{F\colon F \subseteq B, F^c \in \beta o(Y,\sigma)\}$

As an illustration, consider the following example.

Example 1. Let $Y = \{1,2,3,4\}$ and $\sigma = \{Y, \phi, \{1\}, \{2,3\}, \{1,2,3\}\}$; then, $\beta O(Y,\sigma) = \{Y, \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \{1,3,4\}\}$, and $\Lambda_\beta(\{1\}) = \{1\}, \Lambda_\beta(\{2,4\}) = \{2,4\}, \Lambda_\beta(\{2\}) = \{2\}, \Lambda_\beta(\{1,3,4\}) = \{1,3,4\}$, $V_\beta(\{1\}) = \{1\}, V_\beta(\{2,4\}) = \{2,4\}, V_\beta(\{1,3\}) = \{1,3\}$, and $V_\beta(\{1,2,3\}) = \{1,2,3\}$.

In the first result, we summarize the fundamental properties of the set β-Λ-sets.

Proposition 1. *Let B, C, and $\{C_\lambda\colon \lambda \in \Omega\}$ be subsets of a space Y which carries topology σ. Then, the following properties are valid:*

(a) $C \subseteq \Lambda_\beta(C)$

(b) If $B \subseteq C$, *then* $\Lambda_\beta(B) \subseteq \Lambda_\beta(C)$

(c) $\Lambda_\beta(\Lambda_\beta(C)) = \Lambda_\beta(C)$

(d) If $B \in \beta O(Y,\sigma)$, *then* $B = \Lambda_\beta(B)$ *(i.e., B is an* Λ_β*-set)*

(e) $\Lambda_\beta(\cup_{\lambda\in\Omega} C_\lambda) = \cup_{\lambda\in\Omega}(\Lambda_\beta(C_\lambda))$

(f) $\Lambda_\beta(C^c) = (V_\beta(C))^c$

(g) $\Lambda_\beta(\cap_{\lambda\in\Omega} C_\lambda) \subseteq \cap_{\lambda\in\Omega}(\Lambda_\beta(C_\lambda))$

Proof

(a) It is obvious.

(b) Suppose that $y \notin \Lambda_\beta(C)$. Then, there exists a subset $J \in \beta O(Y,\sigma)$ such that $J \supseteq C$ with $y \notin G$ since $C \supseteq B$; then $y \notin \Lambda_\beta(C)$, and thus, $\Lambda_\beta(B) \subseteq \Lambda_\beta(C)$.

(c) It follows from (a) and (b) that $\Lambda_\beta(C) \subseteq \Lambda_\beta(\Lambda_\beta(C))$. If $y \notin \Lambda_\beta(C)$, then there exists $J \in \beta o(Y,\sigma)$ such that $C \supseteq J$, and $y \notin J$. Hence, $\Lambda_\beta(C) \subseteq J$, and so, we have $y \notin \Lambda_\beta(\Lambda_\beta(C))$. Then, $\Lambda_\beta(\Lambda_\beta(C)) = \Lambda_\beta(C)$.

(d) By Definition 6, and since $B \in \beta O(Y,\sigma)$, we have $\Lambda_\beta(B) \subseteq B$. By (a), we have that $A = \Lambda_\beta(B)$.

(e) Suppose that there exists a point y such that $y \notin \Lambda_\beta(\cup_{\lambda\in\Omega} C_\lambda)$. Then, there exists a β-open set J such that $(\cup_{\lambda\in\Omega} C_\lambda) \subseteq J$ and $y \in J$. Thus, for each $\lambda \in \Omega$, we have $y \notin \Lambda_\beta(C_\lambda)$. This implies that $y \notin \cup_{\lambda\in\Omega}\Lambda_\beta(C_\lambda)$. Conversely, suppose that there exists a point $y \in Y$ such that $y \notin \cup_{\lambda\in\Omega}\Lambda_\beta(C_\lambda)$. Then, by Definition 6, there exists a subset $J_\lambda \in \beta O(Y,\sigma)$ (for all $\lambda \in \Omega$) such that $y \notin J_\lambda, C_\lambda \subseteq J_\lambda$. Let $J = \cup_{\lambda\in\Omega} J_\lambda$. Then, we have that $y \notin \cup_{\lambda\in\Omega} J_\lambda$, $\cup_{\lambda\in\Omega} C_\lambda \subseteq J$, and $J \in \beta O(Y,\sigma)$. This implies that $y \notin \Lambda_\beta(\cup_{\lambda\in\Omega} C_\lambda)$. Then, the proof (e) is completed.

(f) $(V_\beta(C))^c = \cap\{F^c : C^c \subseteq F^c, F^c \in \beta O(Y,\sigma)\} = \Lambda_\beta(C^c)$.

(g) Suppose that there exist a point $y \notin \cap_{\lambda\in\Omega}\Lambda_\beta(C_\lambda)$; then, there exists $\lambda \in \Omega$ such that $y \notin \Lambda_\beta(C_\lambda)$; hence, there exist $\lambda \in \Omega$ and $J \in \beta O(Y,\sigma)$ such that $J \supseteq C_\lambda$ and $y \notin J$. Thus, $y \notin \Lambda_\beta(\cup_{\lambda\in\Omega} C_\lambda)$. By using Proposition 1(f), one can easily verify our result. □

Proposition 2. *For subsets B, C, and $\{C_\lambda : \lambda \in \Omega\}$ of a topological space (Y,σ), the following properties hold:*

(a) $V_\beta(C) \subseteq C$

(b) If $B \subseteq C$, *then* $V_\beta(B) \subseteq V_\beta(C)$

(c) $V_\beta(V_\beta(C)) = V_\beta(C)$

(d) If C is β-closed in (Y,σ), *then* $C = V_\beta(C)$

(e) $V_\beta(\cap_{\lambda\in\Omega} C_\lambda) = \cap_{\lambda\in\Omega}(V_\beta(C_\lambda))$

(f) $V_\beta(\cup_{\lambda\in\Omega} C_\lambda) \supseteq \cup_{\lambda\in\Omega}(C_\beta(C_\lambda))$

Proof. (a), (b), and (c) are produced directly from Definition 6 and Proposition 1. To prove (d), let C be β-closed in (Y,σ); then, $C^c \in \beta O(Y,\sigma)$. By (d) and (f) of Proposition 1, $C^c = \Lambda_\beta(C^c) = (V_\beta(C))^c$. Hence, $C = V_\beta(C)$.

(f) can be demonstrated by using (d) and Proposition 1 (f). We have $V_\beta(\cup_{\lambda\in}\Omega C_\lambda) = [\Lambda_\beta(\cup_{\lambda\in\Omega} C_\lambda)^c]^c = [\Lambda_\beta(\cap_{\lambda\in\Omega} C_\lambda^c)]^c \subseteq [\cap_{\lambda\in\Omega}\Lambda_\beta(C_\lambda^c)]^c = [\cap_{\lambda\in\Omega}(V_\beta(C_\lambda))^c]^c = \cup_{\lambda\in\Omega}(V_\beta(C_\lambda))$. □

Remark 2. $\Lambda_\beta(B_1 \cap B_2) \neq \Lambda_\beta(B_1) \cap \Lambda_\beta(B_2)$ is generally not true, as the following example shows.

Example 2. Let $Y = \{1,2,3\}$ and $\sigma = \{Y, \phi, \{1\}\}$. Let $B_1 = \{2\}$ and $B_2 = \{3\}$. Then, $\Lambda_\beta(B_1 \cap B_2) = \phi$, but $\Lambda_\beta(B_1) \cap \Lambda_\beta(B_2) = Y$.

Definition 7. In a space Y which carries topology σ, a subset C is called a Λ_β-set (resp., V_β-set) of (Y,σ) if $C = \Lambda_\beta(C)$ (resp., $C = V_\beta$-set).

Example 3. Let (Y,σ) be a topological space as in Example 1. Then, $A = \{1\}$ is Λ_β-set and $C = \{2,3\}$ is V_β-set.

Remark 3. By Proposition 1 (d) and Proposition 2 (d), we have that

(a) If C is a Λ-set or if $C \in \beta O(Y,\sigma)$, then C is a Λ_β-set

(b) If C is a V-set or if C is β-closed set, then C is a V_β-set

The converse of this remark is not true in general as shown in Example 4.

Example 4. Let (R,τ) be a usual topology, and a singleton $\{y\}$ is not β-open but Λ_β-set.

Proposition 3. *For a space Y which carries topology σ, the following statements hold:*

(a) ϕ and Y are Λ_β-sets and V_β-sets, where ϕ and Y are subsets of Y

(b) Union of any number of Λ_β-sets (resp., V_β-sets) is Λ_β-set (resp., V_β-set)

(c) Intersection of any number of Λ_β-sets (resp., V_β-sets) is Λ_β-set (resp., V_β-set)

(d) A subset C is a Λ_β-set if and only if C^c is a V_β-set

Proof. We will only take a case of Λ_β-sets.

(a) Obvious. To prove (b), let $\{C_\lambda : \lambda \in \Omega\}$ be a family of Λ_β-sets in (Y,σ). If $C = \cup\{C_\lambda : \lambda \in \Omega\}$, then by Proposition 1 C is Λ_β-sets. To prove (c), let $\{C_\lambda : \lambda \in \Omega\}$ be a family of Λ_β-set in (Y,σ). Then, by Proposition 1 (g) and Definition 6, we have $\Lambda_\beta(\cup_{\lambda\in\Omega} B_\lambda) \subseteq \cup_{\lambda\in\Omega} V_\beta(C_\lambda) = \cap_{\lambda\in\Omega} C_\lambda$; hence, by Proposition 1 (a), $\cap_{\lambda\in\Omega} C_\lambda = \cap_{\lambda\in\Omega} C_\lambda = \Lambda_\beta(\cap_{\lambda\in\Omega} C_\lambda)$. □

Remark 4. Let σ^{Λ_β} and σ^{V_β} be the family of all Λ_β-sets and V_β-sets from Y. Then, σ^{Λ_β} and σ^{V_β} are a topology on Y containing all β-open and β-closed sets, respectively. Clearly, $(Y,\sigma^{\Lambda_\beta})$ and (Y,σ^{V_β}) are Alexandroff Space [25], that is, arbitrary intersections of open sets are open.

Proposition 4. *For a topological space (Y,σ), the following statements hold:*

(a) $\sigma^{\Lambda} \subseteq \sigma^{\Lambda_\beta}$

(b) $\sigma^{\Lambda_\beta} \subseteq \sigma^{\Lambda_s}, \sigma^{\Lambda_\beta} \subseteq \sigma^{\Lambda_p}$ *and* $\sigma^{\Lambda_\beta} \subseteq \sigma^{\Lambda_b}$

(c) $\sigma^{\Lambda_s} = (\sigma^{\beta})^{\Lambda_s}$ *and* $\sigma^{\Lambda_s} \subseteq (\sigma^{\Lambda_\beta})^{\Lambda_s}$

Proof

(a) Let B be a subset of Y and B is Λ-set or $(B \in \sigma^{\Lambda})$. Then, $B = \cap\{U : B \subseteq U, U \in \sigma\}$. Since every open set is β-open, then $B = \cap\{U : B \subseteq U, U\ is\ \beta-\text{open}\}$, so $\sigma^{\Lambda} \subseteq \sigma^{\Lambda_\beta}$.

(b) Since $\beta O(Y,\sigma) \subseteq SO(Y,\sigma)$ [25], by similar way of (a), then $\sigma^{\Lambda_\beta} \subseteq \sigma^{\Lambda_s}$. Also, since $\beta O(Y,\sigma) \subseteq PO(Y,\sigma) \subseteq BO(Y,\sigma)$, then $\sigma^{\Lambda_\beta} \subseteq \sigma^{\Lambda_p} \subseteq \sigma^{\Lambda_b}$.

(c) Since $SO(Y,\sigma) = SO(Y,\sigma^\beta)$, then $\sigma^{\Lambda_s} = (\sigma^\beta)^{\Lambda_s}$, and since $\sigma^\beta \subseteq \sigma^{\Lambda_\beta}$, so $\sigma^{\Lambda_s} \subseteq (\sigma^{\Lambda_\beta})^{\Lambda_s}$. □

Proposition 5. *If (Y,ζ) and (Y,σ) are two topological spaces such that $\beta o(Y,\zeta) = \beta o(Y,\sigma)$, then $\sigma^\Lambda \subseteq \zeta^{\Lambda_\beta}$.*

Proof. Since for a two spaces (Y,ζ) and (Y,σ), if $\beta o(Y,\zeta) = \beta o(Y,\sigma)$, then $\sigma \subseteq \zeta^\beta$, and $So(\sigma^\Lambda) \subseteq \zeta^{\Lambda_\beta}$. □

Remark 5. We note that in the above proposition if $\zeta^{\Lambda_\beta} \neq \sigma^{\Lambda_\beta}$, it not necessarily leads to $\sigma^\Lambda \subseteq \zeta^{\Lambda_\beta}$, as the following example.

Example 5. Let $Y = \{1,2,3\}$, $\zeta = \{Y, \phi, \{1\}, \{1,2\}\}$, and $\sigma = \{Y, \phi, \{1\}, \{1,3\}\}$; then, $\zeta^{\Lambda_\beta} \neq \sigma^{\Lambda_\beta} \neq \beta o(Y)$ and $\zeta^{\Lambda_\beta} = \{Y, \phi, \{1\}, \{1,2\}, \{1,3\}\}$, but $\sigma^\Lambda = \{Y, \phi, \{1\}, \{1,3\}\}$.

Proposition 6. *If $B \in PO(Y,\sigma)$(resp., $B \in \sigma^{\Lambda_p}$) and $C \in \beta O(Y,\sigma)$ (resp., $C \in \sigma^{\Lambda_\beta}$, then $B \cap C \in \sigma^{\Lambda_\beta}(B,\sigma_B)$).*

Proof. If B is a preopen set and C is a β-open set; then, $B \cap C \in \beta O(B,\sigma_B)$. So $B \cap C \in \sigma^{\Lambda_\beta}(B,\sigma_B)$. If $B \in \sigma^{\Lambda_p}$, then $B \in \cap\{J\colon B \subseteq J, J \in PO(Y,\sigma)\}$, and if $C \in \sigma^\beta(Y,\sigma)$, then $C = \cap\{U\colon C \subseteq U, U \in \beta o(Y,\sigma)\}$. So $B \cap C = \cap\{J \cap U\colon B \cap C \subseteq J \cap U, J \in Po(Y,\sigma), U \in \beta o(Y,\sigma)\} = \cap\{S\colon S \in \beta O(B,\tau), B \cap C \subset S\}$. Then, $B \cap C \in \sigma^{\Lambda_\beta}(B,\sigma_B)$. □

Proposition 7. *For a space Y which carries topology σ, the following properties are equivalents:*

(a) (Y,σ) is β-T_1

(b) Every subset B of Y is Λ_β–set

(c) Every subset B of Y is V_β-set

Proof

($a \Longrightarrow c$): let $B \subseteq Y$. Since $B = \cup\{\{y\}\colon y \in B\}$, B is a union of β-closed sets; hence, B is V_β-set

($b \Longrightarrow c$): clearly given by Proposition 1

($c \Longrightarrow a$): Since by (c), we have that every singleton is the union of β-closed sets, that is, it is β-closed; then, (Y,σ) is a β-T_1-space

Recall that a subset B of a topological space (Y,σ) is said to be generalized closed (briefly g-closed) [3] if $\mathrm{Cl}(B) \subset U$ whenever $B \subset U$ and $U \in \sigma$.

A topological space (Y,σ) is called $T_{1/2}$ if every g-closed subset of Y is closed. Dunham and Maki [26, 27] pointed out that (Y,σ) is $T_{1/2}$ if and only if for each $y \in Y$, the singleton $\{y\}$ is open or closed. □

Theorem 1. *For a space Y which carries topology σ, the following properties hold:*

(a) $(Y,\sigma^{\Lambda_\beta})$ and (Y,σ^{V_β}) are $T_{1/2}$

(b) If (Y,σ) is β-T_1, then both $(Y,\sigma^{\Lambda_\beta})$ and (Y,σ^{V_β}) are discrete spaces

Proof

(a) Let $y \in Y$. Then, $\{y\}$ is either preclosed or open and hence $\{y\}$ is either β-closed or β-open. If $\{y\}$ is β-open, $\{y\} \in \Lambda_\beta$. If $\{y\}$ is β-closed in (Y,σ), then $Y\backslash\{y\}$ is β-open, and hence, $Y\backslash\{y\} \in \Lambda_\beta$. Therefore, $\{y\}$ is closed in $(Y,\sigma^{\Lambda_\beta})$. Hence, $(Y,\sigma^{\Lambda_\beta})$ and (Y,σ^{V_β}) are $T_{1/2}$ spaces.

(b) This follows from Proposition 7.

Relationship between some types of Λ-sets is summarized in Figure 1. □

Proposition 8. *For a topological space (Z,ζ), the following statements are equivalent:*

(a) (Z,ζ) is a β-T_1

(b) The singleton $\{z\}$ is a Λ_β-set, for each $z \in Z$

(c) The singleton $\{z\}$ is a β-closed, for each $z \in Z$

Proof

($a \Longrightarrow b$): let $z \in Z$ be an arbitrary point. For any point y in which $y \neq z$, there exists $J \in \beta o(Z,\zeta)$ such that $z \in J$ and $y \notin J$. Therefore, we have $y \notin \Lambda_\beta(\{z\})$. This shows that $\Lambda_\beta(\{z\}) \subseteq \{z\}$. Since $\{z\} \subseteq \Lambda_\beta(\{z\})$, we have $\Lambda_\beta(\{z\}) = \{z\}$.

($b \Longrightarrow c$): let $z \in Z$. For any $y \in Z - \{z\}$, $\{y\} = \Lambda_\beta(\{y\})$, and hence, there exists $J_y \in \beta O(Z,\zeta)$ such that $z \notin J_y$ and $y \in J_y$. Therefore, we obtain $y \in J_y \subset Z - \{z\}$, and hence, $Z - \{z\} = \cup\{J_y\colon y \in Z - \{z\}\}$ which is β-open. Thus, $\{z\}$ is β-closed.

($c \Longrightarrow a$): For any distinct points z and y of Z, the singletons $\{z\}$ and $\{y\}$ are β-closed which implies to (Z,ζ) which is β-T_1. □

Proposition 9. *For topological spaces (Y,σ) and $(Y,\sigma^{\Lambda_\beta})$, the following properties hold:*

(a) A space (Y,σ) is β-T_1 if and only if $(Y,\sigma^{\Lambda_\beta})$ is the discrete space

(b) The identity function $f\colon (Y,\sigma^{\Lambda_\beta}) \longrightarrow (Y,\sigma)$ is strongly β-irresolute

(c) If $(Y,\sigma^{\Lambda_\beta})$ is connected, then (Y,σ) is β-connected

Proof

(a) Suppose that (Y,σ) is β-T_1 and y be any point of Y. By Proposition 8, $\{y\}$ is a Λ_β-set and $\{y\} \in \sigma^{\Lambda_\beta}$. For any subset B of Y, by Proposition 3, $B \in \sigma^{\Lambda_\beta}$. This shows that $(Y,\sigma^{\Lambda_\beta})$ is discrete. Conversely, for each $y \in Y$, $\{y\} \in \sigma^{\Lambda_\beta}$, and hence, $\{y\}$ is a Λ_β-set. Proposition 8 (Y,σ) is β-T_1.

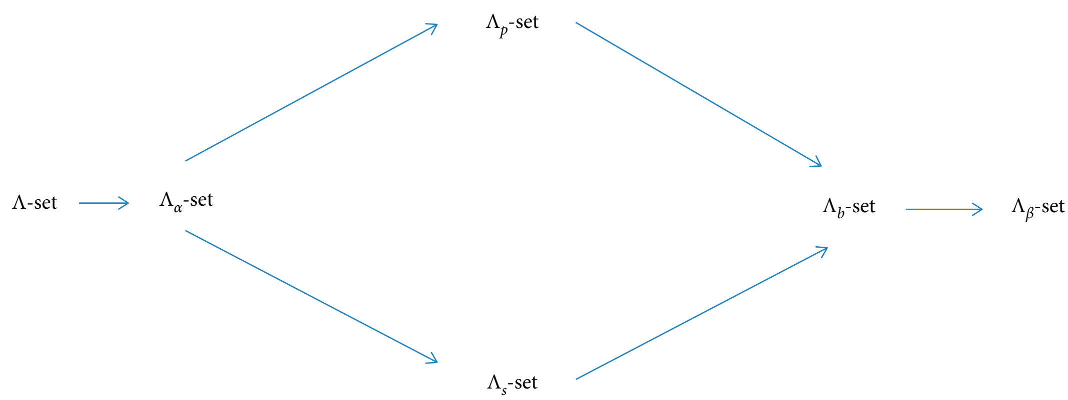

FIGURE 1: Relationship between some forms of near Λ-sets.

(b) Let J be any β-open set of (Y, σ). By Proposition 3, $f^{-1}(J) = J \in \sigma^{\Lambda_\beta}$, and hence, f is strongly β-irresolute.

(c) Suppose that (Y, σ) is not β-connected. There exist nonempty β-open sets J_1, J_2 of (Y, σ) such that $J_1 \cap J_2 = \phi$ and $J_1 \cup J_2 = Y$. Therefore, $f^{-1}(J_1)$ and $f^{-1}(J_2)$ are the disjoint nonempty open sets in $(Y, \sigma^{\Lambda_\beta})$ and $f^{-1}(J_1) \cup f^{-1}(J_2) = Y$. This shows that $(y, \sigma^{\Lambda_\beta})$ is not connected. □

Theorem 2. *For a topological space (Y, σ), the following properties are equivalent:*

(a) (Y, σ) is β-T_1

(b) (Y, σ) is β-R_0

(c) $\beta cl(\{y\}) = \Lambda_\beta(\{y\})$, for each $y \in Y$

Proof

(a) $\Longrightarrow$ (b): for any β-open set G, and any point $y \in J$, by Proposition 8, $\beta\mathrm{Cl}(\{y\}) = \{y\} \subset J$. Hence, (Y, σ) is β-R_0.

(b) $\Longrightarrow$ (a): for each $y \in Y$, $\{y\}$ is preopen or preclosed [21]. Let $z \in Y$ such that $y \neq z$. In case $\{y\}$ is preopen, we have $\{y\} \in \beta O(Y, \sigma)$ and $z \notin \{y\}$. In case $\{y\}$ preclosed, since $z \in Y$-$\{y\} \in \beta O(Y, \sigma)$, $\beta\mathrm{Cl}(\{z\}) \subset Y$-$\{y\}$, and hence, $y \in Y$-$\beta\mathrm{Cl}(\{z\}) \in \beta O(Y, \sigma)$. Since $z \notin \{Y\}$-$\beta\mathrm{Cl}(\{z\})$, then there exists $G \in \beta o(Y, \sigma)$ such that $y \in G$ and $z \notin J$; consequently, there exists $H \in \beta O(Y, \sigma)$ such that $z \in H$ and $y \notin H$. Hence, (Y, σ) is β-T_1.

(b) $\Longrightarrow$ (c): let $y \in Y$ and $J \in \beta O(Y, \sigma)$ containing y. Then, by (a), $\beta cl(\{y\}) \subset J$ and $\beta cl(\{y\}) \subset \Lambda_\beta(\{y\})$. Suppose that $z \in \Lambda_\beta(\{y\})$. Then, $z \in J$ for every β-open set J containing y, and hence, $y \in \beta cl(\{z\})$. Now, we show that $y \in \beta cl(\{z\})$ implies $z \in \beta cl(\{y\})$. Assume that $z \notin \beta cl(\{y\})$, then there exists $H \in \beta O(Y, \sigma)$ such that $z \in H$ and $y \notin H$. Since (Y, σ) is β-R_0, $\beta cl(\{z\}) \subset H$ and $y \notin \beta cl(\{z\})$. Therefore, we have $\Lambda_\beta(\{y\}) \subset \beta cl(\{y\})$. Consequently, we obtain $\beta cl(\{x\}) = \Lambda_\beta(\{y\})$.

(c) $\Longrightarrow$ (b): let $J \in \beta O(Y, \sigma)$ and $y \in J$. Then, we have $\Lambda_\beta(\{y\}) \subset J$; by (b), $y \in \beta cl(\{y\}) \subset \Lambda_\beta(\{y\}) \subset G$, and hence, $\beta cl(\{y\}) \subset J$. This shows that (Y, σ) is β-R_0. □

3. Generalized Λ_β-Sets and Generalized V_β-Sets

Section 3 is devoted to the study of generalizing Λ_β-set ($g \cdot \Lambda_\beta$-sets) and V_β-set ($g \cdot \Lambda_\beta$-sets) development of Maki's work [2].

Definition 8. In a space Y which carries topology σ, a subset B is called a generalized Λ_β-set of $(Y, \sigma)(g \cdot \Lambda_\beta)$ if $\Lambda_\beta(B) \subseteq F$ whenever, $B \subseteq F$ and F is β-closed. A subset B is $g.V_\beta$-sets of (Y, σ) if Y-B is $g.\Lambda_\beta$-set of (Y, σ).

Now, E^{Λ_β} (resp., E^{V_β}) will be denoted by the set of all $g.\Lambda_\beta$-sets (resp., $g.V_\beta$) in (Y, σ).

Proposition 10. *Let (Y, σ) be a topological space and I be any index set. Then,*

(a) Every Λ_β-set is a $g \cdot \Lambda_\beta$-set

(b) Every V_β-set is a $g \cdot V_\beta$-set

(c) If $B_i \in E^{\Lambda_\beta}$ for all $i \in I$, then $\cup B_i \in E^{\Lambda_\beta}$

(d) If $B_i \in E^{V_\beta}$ for all $i \in I$, then $\cup B_i \in E^{V_\beta}$

Proof. (a) and (b) are proved by Definition 6, Proposition 1, and Definition 8. (c) and (d) are proved by (e) of Proposition 1 and Definition 8. □

Remark 6. The intersection of two $g.\Lambda_\beta$-sets is generally not a $g.\Lambda_\beta$-set. Also, the union of two $g.V_\beta$-sets in generally not a $g.V_\beta$-set as shown in the following example.

Example 6. Let $Y = \{1, 2, 3\}$ with $\sigma = \{Y, \phi, \{1\}, \{2\}, \{1, 2\}\}$; then, $\sigma^{\Lambda_\beta} = \{Y, \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$. If $B = \{1, 3\}$ and $C = \{2, 3\}$, then B and C are $g.\Lambda_\beta$-sets but $B \cap C = \{3\}$ is not $g.\Lambda_\beta$-set. If $B = \{1\}$ and $C = \{2\}$, then B and C are $g.V_\beta$-sets but $B \cup C = \{1, 2\}$ is not $g.V_\beta$-set.

Remark 7. The converse of (a) and (b) of Proposition 10 is not true in general as shown by the following example.

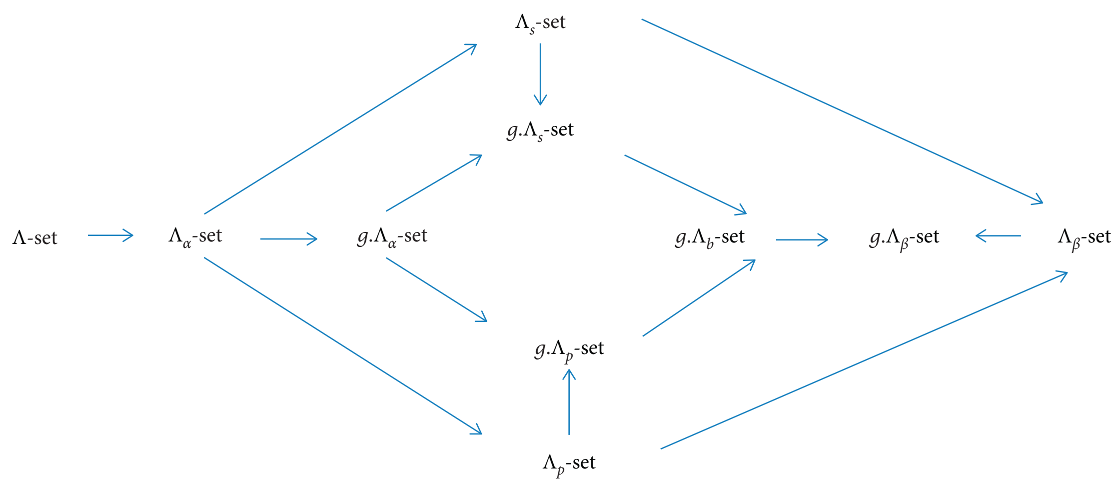

FIGURE 2: Relationship between some types of Λ-generalized closed sets.

Example 7. Let (Y, σ) be the space in Example 6; the subset $B = \{1, 3\}$ is $g.\Lambda_\beta$-set but it is not a Λ_β-set. Also, the subset $\{1\}$ is $g.V_\beta$-set but is not V_β-set.

Proposition 11. *Let (Y, σ) be a topological space. Then,*

(a) *Each $\{y\} \in Y$ is a β–open set or $Y - \{y\}$ is a $g.\Lambda_\beta$-set of (X, τ)*

(b) *Each $\{y\} \in Y$ is a β–open set or $Y - \{y\}$ is a $g.V_\beta$-set of (Y, σ)*

Proof

(a) Let $\{y\}$ be not a β-open set; then, only β-closed set F containing $Y - \{y\}$ is Y. Thus, $(Y - \{y\})^{\Lambda_\beta} \subseteq F = Y$ and $Y - \{y\}$ is a $g.\Lambda_\beta$-set of (Y, σ).

(b) It follows from (a) and Definition 8.

The known relationships between the types of Λ-generalized closed sets are summarized in Figure 2. □

Definition 9. A function $\xi\colon (Y, \sigma) \longrightarrow (Z, \eta)$ is said to be

(a) Strongly β-irresolute [19] if for each $y \in Y$ and each β-open set J of Z containing $\xi(y)$, there is an open set H containing y such that $\xi(H) \subseteq J$, and equivalently, if the inverse image of each β-open set is open.

(b) β-Irresolute [10] if for each $y \in Y$, and each β-open set J of Z containing $\xi(y)$, there is a β-open set H containing y such that $\xi(H) \subseteq J$, and equivalently, if the inverse image of each β-open set is β-open.

Theorem 3

(a) *If a function $\xi\colon (Y, \sigma) \longrightarrow (Z, \eta)$ is β-irresolute, then $\xi\colon (Y, \sigma^{\wedge\beta}) \longrightarrow (Z, \eta^{\wedge\beta})$ is continuous*

(b) *The identity function $id_y\colon (Y, \sigma) \longrightarrow (Z, \eta)$ is strongly β-irresolute*

Proof

(a) Let J be any Λ_β-set of (Z, η), i.e., $J \in \sigma^{\Lambda_\beta}$. Then, $J = \Lambda_\beta(J) = \cap\{W\colon J \subseteq W$ and W is β-open in $(Z, \eta)\}$. Since ξ is β-irresolute, $\xi^{-1}(W)$ is β-open in (Y, σ) for each W. Hence, we have $\xi^{-1}(Q) \supseteq \cap\{\xi^{-1}(W)\colon \xi^{-1}(J) \subseteq \xi^{-1}(W)$, and W is β-open in $(Z, \eta)\} \supseteq \cap\{H\colon \xi^{-1}(J) \subseteq H$ and H is β-open in $(Y, \sigma)\} = \Lambda_\beta(\xi^{-1}(J))$. On the other hand, by the definition, $\xi^{-1}(J) \subseteq \Lambda_\beta(\xi^{-1}(J))$. Hence, we obtain $\xi^{-1}(J) = \Lambda_\beta(\xi^{-1}(J))$. Therefore, $\xi^{-1}(J) \in \tau^{\Lambda_\beta}$, and $\xi\colon (Y, \sigma^{\wedge\beta}) \longrightarrow (Z, \eta^{\wedge\beta})$ is continuous.

(b) Let J be any β-open set of (Y, σ). Since J is β-open, by Proposition 11, $(\mathrm{id}_y)^{-1}(J) = J \in \sigma^{\Lambda_\beta}$ and hence, id_x is strongly β-irresolute. □

4. Application

The digital line or so-called Khalimsky line (Z, k) is the set of all integers Z, equipped with the topology k generated by $\{\{2n-1, 2n, 2n+1\} \mid n \in Z\}$ (cf.[10], [28, p. 175]), [29, p. 905, 908]. The digital line or an interval as its subspace is a topological model of a 1-dimensional computer screen and points are the pixels. In (Z, k), each singleton $\{2n\}$ is closed and each singleton $\{2n+1\}$ is open (=regular open), where $n \in Z$.The digital n-space (e.g., [19], Definition 4) is the topological product of n-copies of digital line (Z, k) and this space is denoted by (Z^n, k^n). For $n = 2$, (Z^2, k^2) is called the digital plane. In (Z^2, k^2), singletons whose coordinates are odd and are open; those with coordinates even are closed. There are points which are neither open nor closed.

Theorem 4. *For the digital n-space (Z^n, k^n) where $n \geq 1$, we have the following properties:*

(1) *For any point x of (Z^n, k^n), β-$\Lambda(\{x\}) = \{x\}$*

(2) *For any subset E of (Z^n, k^n), β-$\Lambda(E) = E$*

Proof. (i) We first note that for the case where $n = 1$, $Z^n = (Z^n)_{k^n} \cup (Z^n)_{f^n}$ (disjoint union) holds, where $n = 1$, and for the case where $n \geq 2$, $Z^n = (Z^n)_{k^n} \cup (Z^n)_{f^n} \cup (\cup \{ (Z^n)_{\mathrm{mix}(r)} \mid 1 \leq r \leq n-1 \})$ (disjoint union) and $(Z^n)_{mix(r)} \neq \phi (1 \leq r \leq n-1\})$ hold, where $n \geq 2$. Let $x \in Z^n$. It is enough to consider the following three cases for the point $x \in Z^n$. □

Case 1. $x \in (Z^n)_{k^n}$: since $\{x\}$ is open in (Z^n, k^n), it is β-open. Then, it is obvious that $\beta - \Lambda(\{x\}) = \{x\}$ in (Z^n, k^n). We note this result is true for the case where $n \geq 1$.

Case 2. $x \in (Z^n)_{f^n}$: we put $x = (2s_1, 2s_2, \ldots, 2s_n$ where $s_i \in Z | 1 \leq i \leq n$. Note that, for the point $x \in (Z^n)_{f^n}, U^n(x) = \prod_{i=1}^n \{2s_i - 1, 2s_i, 2s_i + 1\}$ is the smallest open set containing x. Then, there exist 2^n β-open sets $\{x\} \cup \{p(x,u)\} | (1 \leq u \leq 2^n)$ containing the point $x \in (Z^n)_{f^n}$ such that $\{p(x,u) \mid 1 \leq u \leq 2^n\} = (U^n(x))_{k^n} = \{(2s_1 + i_1, 2s_2 + i_2, \ldots, 2s_n + i_n) \mid i_k \in \{+1, -1\} \mid (1 \leq k \leq n)\}$ and the family of $((U^n(x))_{k^n}) = 2^n$. Thus, we have $\beta\text{-}\Lambda(\{x\}) \subset \cap \{\{x\} \cup \{p(x,u)\} \mid 1 \leq u \leq 2^n\}$; moreover, $\cap \{\{x\} \cup \{p(x,u)\} \mid 1 \leq u \leq 2^n\} = \{x\}$, because $\cap \{\{p(x,u)\} \mid 1 \leq u \leq 2^n\} = \phi$. We conclude that $\beta\text{-}\Lambda$ $(\{x\}) = \{x\}$ holds for this case. We note the result above it is true for the case where $n \geq 1$.

Case 3. $x \in (Z^n)_{\mathrm{mix}(r)}$, where $1 \leq r \leq n-1 (n \geq 2)$: for the point x, we set $x = (x_1, x_2, \ldots, x_n$; then by definition, $r = \{i \mid x_i \text{is an even integer } (1 \leq i \leq n)\})$. We recall the following subsets $I_r(x)$ and $J_{n-r}(x)$ as follows: $I_r(x) = \{k \mid x_k \text{is even}\} = \{e(1), e(2), \ldots, e(r)\} (e(1) < e(2) < \cdots < e(r))$; $J_{n-r}(x) = \{j \mid x_j \text{is odd}\} = \{o(1), o(2), \ldots, o(n-r)\}$ $(o(1) < o(2) < \cdots < o(n-r))$; and $\{1, 2, \ldots, n\} = I_r(x) \cup J_{n-r}(x)$ (disjoint union), $I_r(x) \neq \phi, J_{n-r}(x) \neq \phi$.

For the point $x \in (Z^n)_{\mathrm{mix}(r)}, U^u(x) = \prod_{i=1}^n U(x_i)$ is the smallest open set containing x, where $U(x_{e(k)}) = \{x_{e(k)} - 1, x_{e(k)} + 1\} (1 \leq k \leq r)$ and $U(x_{o(j)}) = \{x_{o(j)}\}$ $(1 \leq j \leq n-r)$. Then, there exist 2^r β-open sets $\{x\} \cup \{p(x,u)\} (1 \leq u \leq 2^r$ containing the point $x \in (Z^n)_{\mathrm{mix}(r)}$ such that $\{p(x,u)\} (1 \leq u \leq 2^r = (U^n(x)_{k^n} = \{(z_1, z_2, \ldots, z_n) \mid z_{e(k)} \in \{x_{e(k)} + 1, x_{e(k)} - 1\} (1 \leq kr), z_{o(j)} (1 \leq j \leq n-r)\}$ and the family of $((U^n(x))_{k^n}) = 2^r$. Thus, it is shown that $\beta\text{-}\Lambda(\{x\}) \subset \cap \{\{x\} \cup \{p(x,u)\} (1 \leq u \leq 2^r\} = \{x\} \cup (\cap \{\{px,u)\} (1 \leq u \leq 2^r\}\}) = \{x\}$, because $\cap \{\{p(x,u)\} (1 \leq u \leq 2^r\} = \phi$. Then, we show that $\beta\text{-}\Lambda(\{x\}) = \{x\}$ holds for this case. Therefore, for all cases above, we have proved that $\beta\text{-}\Lambda(\{x\}) = \{x\}$ holds in $(Z^n, K^n), n \geq 1$.

(2) Since $E = \cup \{\{x\} \mid x \in E\}$, by Proposition 1, it is shown that $\text{-}\beta - \Lambda(E) = \cup \{\beta - \Lambda(\{x\}) \mid x \in E\} = \cup \{\{x\} \mid x \in E\} = E$.

5. Conclusion

Recently, the use of set and functions for topological spaces has significantly evolved in data analysis, information systems, digital topology, and others. By researching generalizations of closed sets, Λ_β-sets, and V_β-sets, some new separation axioms have been found, and they turn out to be useful in the study of digital topology. The notion of a kernel of a set has applications in computer science [10, 30]. This notion is used in most of this paper. Moreover, it also has applications in some crucial fields of science and technology.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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