Research Article

The $d$-Shadowing Property and Average Shadowing Property for Iterated Function Systems

Jie Jiang,¹ Lidong Wang,² and Yingcui Zhao³

¹School of Mathematics and Information Science, North Minzu University, Yinchuan750021, China
²Zhuhai College of Jilin University, Zhuhai, Guangdong519041, China
³School of Mathematical Sciences, Dalian University of Technology, Dalian116024, China

Correspondence should be addressed to Lidong Wang; wld0707@126.com

Received 25 August 2019; Revised 4 December 2019; Accepted 25 January 2020; Published 29 April 2020

Academic Editor: Mojtaba Ahmadieh Khanesar

Copyright © 2020 Jie Jiang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we introduce the definitions of $d$-shadowing property, $d$-shadowing property, topological ergodicity, and strong ergodicity of iterated function systems IFS($f_0, f_1$). Then, we show the following: (1) if IFS($f_0, f_1$) has the $d$-shadowing property (respectively, $d$-shadowing property), then $F^k$ has the $d$-shadowing property (respectively, $d$-shadowing property) for any $k \in \mathbb{Z}$; (2) if $F^k$ has the $d$-shadowing property (respectively, $d$-shadowing property) for some $k \in \mathbb{Z}$, then IFS($f_0, f_1$) has the $d$-shadowing property (respectively, $d$-shadowing property); (3) if IFS($f_0, f_1$) has the $d$-shadowing property or $d$-shadowing property, and $f_0$ or $f_1$ is surjective, then IFS($f_0, f_1$) is chain mixing; (4) let $f_0, f_1$ be open maps. For IFS($f_0, f_1$) with the $d$-shadowing property (respectively, $d$-shadowing property), if $A \subset X$ is dense in $X$, and $s$ is a minimal point of $f_0$ or $f_1$ for any $s \in A$, then IFS($f_0, f_1$) is strongly ergodic, and hence, $F^k$ is strongly ergodic; and (5) for IFS($f_0, f_1$) with the average shadowing property, if $S \subset X$ is dense in $X$, and $s$ is a quasi-weakly almost periodic point of $f_0$ or $f_1$ for any $s \in S$, then IFS($f_0, f_1$) is ergodic.

1. Introduction

In this paper, let $N = \{0, 1, 2, \ldots \}$ and $Z^* = \{1, 2, 3, \ldots \}$. Suppose that $X$ is a compact metric space and $f : X \to X$ a continuous map. The set $F \subset N$ is a syndetic set if there is $N_0 \in Z^*$ such that $[n, n + N_0] \cap F \neq \emptyset$ for each $n \in N$. For any $x \in X$, $\varepsilon > 0$, let $B(x, \varepsilon)$ denote the $\varepsilon$-neighborhood of $x$. $x \in X$ is a minimal point of $f$ if for any neighborhood $U$ of $x$, the set $N(x, U) = \{n \in N : f^n(x) \in U\}$ is syndetic, and the set of all minimal points of $f$ is denoted by AP($f$). $x \in X$ is called a quasi-weakly almost periodic point of $f$ if for any neighborhood $U$ of $x$, the set $N(x, U) = \{n \in N : f^n(x) \in U\}$ has positive upper density.

The shadowing property is a very important notion in dynamical systems. Many researchers have found some relationship among various shadowing properties, chain transitivity, transitivity, and ergodicity. Gu [1] proved that if $(X, f)$ has the asymptotic average shadowing property, and $f$ is surjective, then $(X, f)$ is chain transitive. For more recent results about various shadowing properties, one can refer to [2–9] and references therein.

A $\delta$-ergodic-pseudo-orbit of $f$ is a sequence $\{x_i\}_{i \geq 0}$ such that for any $i \in N$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left( f^i(x_i), x_i+1 \right) = 1,$$

where $\|\cdot\|$ represents the cardinality.

$f$ is said to have the $d$-shadowing property if for any $\varepsilon > 0$, there is $\delta > 0$ such that every $\delta$-ergodic-pseudo-orbit $\{x_i\}_{i \geq 0}$ is $\varepsilon$-shadowed by a true orbit $\{f^i(z)\}_{i \geq 0}$ in a way such that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left( f^i(z), x_i \right) < \varepsilon > \frac{1}{2},$$

where $f$ is said to have the $d$-shadowing property if for any $\varepsilon > 0$, there is $\delta > 0$ such that every $\delta$-ergodic-pseudo-orbit $\{x_i\}_{i \geq 0}$ is $\varepsilon$-shadowed by a true orbit $\{f^i(z)\}_{i \geq 0}$ in a way such that
\[
\lim_{n \to \infty} \frac{1}{n} |\{i \in \mathbb{N} : 0 \leq i < n, d(f_i(z), x_i) < \varepsilon\}| > 0. \tag{3}
\]

Let \(X\) be a metric space and \(f_0, f_1\) be continuous maps on \(X\). The iterated function system \(\text{IFS}(f_0, f_1)\) is the action of the semigroup generated by \(\{f_0, f_1\}\) on \(X\). In this paper, we introduce the definitions of the \(d\)-shadowsing property and \(\bar{d}\)-shadowsing property for \(\text{IFS}(f_0, f_1)\).

An orbit of \(\text{IFS}(f_0, f_1)\) is a sequence \(\{f^i_w(x)\}_{i \geq 0}\) where \(\omega = \omega_0 \omega_1 \omega_2 \ldots \in \Sigma^2 = \{\alpha = \alpha_0 \alpha_1 \alpha_2 \ldots : \alpha_i \in \{0, 1\}\}\), and for any \(i \in \mathbb{N}\),
\[
f^i_w(x) = f^\omega_{\omega_i} \circ \ldots \circ f^\omega_{\omega_0}(x),
\]
\[
f^0_w(x) = x. \tag{4}
\]

A sequence \(\{\xi_i\}_{i \geq 0}\) is called a \(\delta\)-ergodic-pseudo-orbit for \(\text{IFS}(f_0, f_1)\) if there is \(\omega \in \Sigma^2\) such that
\[
\lim_{n \to \infty} \frac{1}{n} |\{i \in \mathbb{N} : 0 \leq i < n, d(f_i(\xi), \xi_i) < \delta\}| = 1. \tag{5}
\]

\(\text{IFS}(f_0, f_1)\) is said to have the \(d\)-shadowing property if for any \(\varepsilon > 0\), there is \(\delta > 0\) such that every \(\delta\)-ergodic-pseudo-orbit \(\{\xi_i\}_{i \geq 0}\) is \(\varepsilon\)-shadowed by a true orbit \(\{f_i(\omega)(z)\}_{i \geq 0}\) in a way such that
\[
\lim_{n \to \infty} \frac{1}{n} |\{i \in \mathbb{N} : 0 \leq i < n, d(f_i(\omega)(z), \xi_i) < \varepsilon\}| > \frac{1}{2}. \tag{6}
\]

\(\text{IFS}(f_0, f_1)\) is said to have the \(\bar{d}\)-shadowing property if for any \(\varepsilon > 0\), there is \(\delta > 0\) such that every \(\delta\)-ergodic-pseudo-orbit \(\{\xi_i\}_{i \geq 0}\) is \(\varepsilon\)-shadowed by a true orbit \(\{f_i(\omega)(z)\}_{i \geq 0}\) in a way such that
\[
\lim_{n \to \infty} \frac{1}{n} |\{i \in \mathbb{N} : 0 \leq i < n, d(f_i(\omega)(z), \xi_i) < \varepsilon\}| > 0. \tag{7}
\]

Denote \(\mathcal{F}^k = \{X : f_0_{\omega_k} \circ \ldots \circ f_0_{\omega_0} f_{\omega_0} : \omega_0, \omega_1, \ldots, \omega_{k-1} \in \{0, 1\}\}, \) where \(k \in \mathbb{Z}^+\).

In this paper, the definitions of the shadowing property and average shadowing property for \(\text{IFS}(f_0, f_1)\) are introduced by Bahabadi [10], and the definition of the asymptotic average shadowing property for \(\text{IFS}(f_0, f_1)\) is introduced by Nia [11]. Let \(f_0, f_1\) be open maps. For \(\text{IFS}(f_0, f_1)\) with the \(d\)-shadowsing property (respectively, \(\bar{d}\)-shadowsing property), if \(A \subset X\) is dense in \(X\), and \(s\) is a minimal point of \(f_0\) or \(f_1\) for any \(s \in A\), then \(\text{IFS}(f_0, f_1)\) is strongly ergodic. Under similar conditions, Niu [2] researched that \((X, f)\) has the average shadowing property and then the conclusion is true. Wang and Niu showed that, for \((X, f)\) with the average shadowing property, if \(S \subset X\) is dense in \(X\), and \(s\) is a quasi-weakly almost periodic point of \(f\) for any \(s \in S\), then \(f\) is transitivity (see [12]). However, under similar conditions, we will prove that \(\text{IFS}(f_0, f_1)\) is ergodic. Then, we will come up with a situation that \(\text{IFS}(f_0, f_1)\) does not have the asymptotic average shadowing property.

According to Bahabadi [10], a finite sequence \(\xi_0 = x, \ldots, \xi_k = y\) is called a \(\delta\)-chain of \(\text{IFS}(f_0, f_1)\) if for any \(i = 0, \ldots, k-1\), there is \(\omega_i \in \{0, 1\}\) such that \(d(f_\omega(\xi_i), \xi_{i+1}) < \delta\).

\[\text{Definition 1} \text{ (see [10])}. \text{ IFS}(f_0, f_1)\text{ is as follows:}
\]

(1) Chain transitive if for any \(x, y \in X\) and any \(\delta > 0\), there is a \(\delta\)-chain of \(\text{IFS}(f_0, f_1)\) from \(x\) to \(y\)

(2) Transitive if for any nonempty open sets \(U, V \subset X\), there are \(\omega \in \Sigma^2\) and \(n \in \mathbb{N}\) such that \(f^\omega_n(U) \cap V \neq \emptyset\)

(3) Mixing if for any nonempty open sets \(U, V \subset X\), there are \(\omega \in \Sigma^2\) and \(N \in \mathbb{N}\) such that \(f^\omega_n(U) \cap V \neq \emptyset\) for any \(n \geq N\)

(4) Chain mixing if for any \(x, y \in X\) and any \(\delta > 0\), there is \(N \in \mathbb{Z}^+\) such that for any \(n \geq N\), there is a \(\delta\)-chain of \(\text{IFS}(f_0, f_1)\) from \(x\) to \(y\) consisting of exactly \(n\) elements.

For any \(A \subset \mathbb{N}\), define the positive upper density of \(A\) by
\[
\bar{d}(A) = \limsup_{n \to \infty} \frac{1}{n} |\{0, 1, \ldots, n-1\}|. \tag{8}
\]

Define the lower density of \(A\) by
\[
d(A) = \liminf_{n \to \infty} \frac{1}{n} |\{0, 1, \ldots, n-1\}|. \tag{9}
\]

If \(A \subset \mathbb{N}\), \(A^c\) is the complementary set of \(A\).

\[\text{ω in the pseudo-orbit of the shadowing property}\ \bar{d}\text{-shadowing property, and } \bar{d}\text{-shadowing property for IFS}(f_0, f_1)\text{ is the same as the one chosen in the shadowing orbit, while the two } \omega\text{s in the definitions of the average shadowing property and asymptotic average shadowing property may be different.}

2. The \(\bar{d}\)-Shadowing Property and \(d\)-Shadowing Property for IFS \((f_0, f_1)\)

Bahabadi [10] introduced the definition of the shadowing property for IFS \((f_0, f_1)\). In this paper, we will introduce the definitions of the \(\bar{d}\)-shadowing property and \(d\)-shadowing property for IFS \((f_0, f_1)\) and give some results.

\[\text{Definition 2}. \text{ A sequence } \{\xi_i\}_{i \geq 0} \text{ is called a } \delta\text{-pseudo-orbit for IFS}(f_0, f_1) \text{ if there is } \omega \in \Sigma^2 \text{ such that for any } i \in \mathbb{N},
\]
\[
d(f_\omega(\xi_i), \xi_{i+1}) < \delta. \tag{10}
\]

\[\text{Definition 3}. \text{ IFS}(f_0, f_1) \text{ has the shadowing property if for any } \varepsilon > 0, \text{ there is } \delta > 0 \text{ such that every } \delta\text{-pseudo-orbit } \{\xi_i\}_{i \geq 0} \text{ is } \varepsilon\text{-shadowed by a point } z \in X, \i.e., \text{ there is } z \in X \text{ such that for any } i \in \mathbb{N},
\]
\[
d(f_\omega(z), \xi_i) < \varepsilon. \tag{11}
\]

\[\text{Example 1}. \text{ Suppose that } f_0(x) = 0, f_1(x) = 1, x \in [0, 1]. \text{ For any } \varepsilon > 0, \text{ there is } \delta = \varepsilon > 0. \text{ Let } \{\xi_i\}_{i \geq 0} \text{ be a } \delta\text{-ergodic-pseudo-orbit for IFS}(f_0, f_1). \text{ Therefore, there is } \omega = \omega_0 \omega_1 \omega_2 \ldots \in \Sigma^2 \text{ such that}
\]
\[
\lim_{n \to \infty} \frac{1}{n} |\{i \in \mathbb{N} : 0 \leq i < n, d(f_\omega(\xi_i), \xi_{i+1}) < \delta\}| = 1. \tag{12}
\]

Then, there is \(z = \xi_0\) such that
Theorem 1. For any $k \in \mathbb{Z}^*$, and any $\epsilon > 0$, there is $0 < \delta < (\epsilon/k)$ such that there are any $\delta$-chain of length $k + 1$ for IFS $(f_0, f_1)\{\xi_0, \xi_1, \ldots, \xi_k\}$, i.e., there is $\omega \in \Sigma^2$ such that $d(f_{\omega_1}(\xi_i), \xi_{i+1}) < \delta$ satisfies
\[
\begin{align*}
\lim_{n \to \infty} \frac{1}{n} \left| \{i \in \mathbb{N} : 0 \leq i < n, d(f_{\omega_i}(z), x_i) < \epsilon \} \right| = 1.
\end{align*}
\]

Obviously, IFS $(f_0, f_1)$ has the $\mathcal{D}$-shadowing property and $d$-shadowing property.

Lemma 1. For any $k \in \mathbb{Z}^*$, and any $\epsilon > 0$, there is $0 < \delta < (\epsilon/k)$ such that there are any $\delta$-chain of length $k + 1$ for IFS $(f_0, f_1)\{\xi_0, \xi_1, \ldots, \xi_k\}$, i.e., there is $\omega \in \Sigma^2$ such that $d(f_{\omega_1}(\xi_i), \xi_{i+1}) < \delta$ satisfies
\[
\begin{align*}
\lim_{n \to \infty} \frac{1}{n} \left| \{i \in \mathbb{N} : 0 \leq i < n, d(f_{\omega_i}(z), x_i) < \epsilon \} \right| = 1.
\end{align*}
\]

Proof. Fix $k \in \mathbb{Z}^*$ and let $\epsilon > 0$. Since $f_0$ and $f_1$ are uniformly continuous, for any $\omega^1 = \omega_0, \omega_1, \ldots, \omega_{k-1}$, $f_{\omega^1}$ is also uniformly continuous, where $i = 0, 1, \ldots, k$. Then for ($\epsilon/k$)$> 0$, there is $0 < \delta < (\epsilon/k)$ such that $d(f^j_{\omega}(x), f^j_{\omega}(y)) < (\epsilon/k)$ for any $x, y \in X$ satisfies $d(x, y) < \delta$. Since $\xi_0, \xi_1, \ldots, \xi_k$ is a $\delta$-chain of IFS $(f_0, f_1)$, there is $\omega \in \Sigma^2$ such that
\[
\begin{align*}
&d(f_{\omega_1}(\xi_i), \xi_{i+1}) < \delta, \quad 0 \leq i \leq k - 1.
\end{align*}
\]

Put $\omega^1 = \omega$. Then,
\[
\begin{align*}
&d(f_{\omega^1}(\xi_0), \xi_1) \\
\leq & d(f_{\omega_1, \omega_0}(\xi_0), f_{\omega_1, \omega_0}(\xi_1)) + \ldots + \\
&d(f_{\omega_{j-1}, \omega_j}(\xi_{j-1}), \xi_j) \\
< & \frac{\epsilon}{k} + \frac{\epsilon}{k} + \ldots + \frac{\epsilon}{k} + \frac{\epsilon}{k} = \epsilon,
\end{align*}
\]
where $0 \leq j \leq k$.

Theorem 1

(1) If IFS $(f_0, f_1)$ has the $d$-shadowing property, then $\mathcal{F}^k$ has the $\mathcal{D}$-shadowing property for any $k \in \mathbb{Z}^*$

(2) If IFS $(f_0, f_1)$ has the $d$-shadowing property, then $\mathcal{F}^k$ has the $d$-shadowing property for any $k \in \mathbb{Z}^*$

Proof. Fix $k \in \mathbb{Z}^*$ and let $\epsilon > 0$. We can find a number $\delta > 0$ satisfying Lemma 1. We will show that (1) is true. We can find a number $\delta_1 > 0$ such that each $\delta_1$-ergodic-pseudo-orbit of IFS $(f_0, f_1)$ is $\delta_1$-shadowed by a true orbit along a set with positive lower density, and $\delta > \delta_1$. Let $\{x_i\}_{i \geq 0}$ be a $\delta_1$-ergodic-pseudo-orbit of $\mathcal{F}^k$. That is to say, there is $\omega \in \Sigma^2$ such that
\[
\begin{align*}
&\lim_{n \to \infty} \frac{1}{n} \left| \{i \in \mathbb{N} : 0 \leq i < n, d(f_{\omega_i}(x_i), x_{i+1}) < \delta_1 \} \right| = 1.
\end{align*}
\]

Let 
\[
\begin{align*}
&\{\xi_i\}_{i \geq 0} = \{x_0, f_{\omega_1}(x_0), \ldots, f_{\omega_{k-1}}(x_{k-1}), x_1, f_{\omega_2}(x_1), \ldots, f_{\omega_{2k-2}}(x_{2k-2}), \ldots, f_{\omega_{n-k+2}}(x_{n-k+2}), \ldots\}.
\end{align*}
\]
Claim 2. Let $M = \{i \in \mathbb{N} : 0 \leq i < n, d(f_{\omega}(\xi_i), \xi_{i+1}) < \delta\}$, then
\[ \lim_{n \to \infty} \frac{1}{n} |M| = 1. \quad (22) \]

Corollary 1. Let IFS $(f_0, f_1)$ be a iterated function system, then the following statements are equivalent:

1. IFS $(f_0, f_1)$ has the $\delta$-shadowing property (respectively, $\delta$-shadowng property).
2. $\mathcal{F}_k$ has the $\delta$-shadowing property (respectively, $\delta$-shadowng property) for some $k \in \mathbb{Z}^*$.
3. $\mathcal{F}_k$ has the $\delta$-shadowing property (respectively, $\delta$-shadowng property) for any $k \in \mathbb{Z}^*$.

Lemma 2. Let $A, B \subset \mathbb{N}$; (1) if $d(A) = \alpha$ and $\overline{d}(B) > 1 - \alpha$, then $\overline{d}(A \cap B) > 0$; (2) if $\overline{d}(A) = \alpha$ and $d(B) > 1 - \alpha$, then $\overline{d}(A \cap B) > 0$. Here, $0 \leq \alpha \leq 1$.

Proof. Firstly, we will show (1). Suppose on the contrary that $\overline{d}(A \cap B) = 0$. Then, $\overline{d}(B) \leq \overline{d}(A^c)$. Therefore, we can see $d(A) + d(A^c) \geq d(A) + d(B) > \alpha + 1 - \alpha = 1$. But $\overline{d}(A) + \overline{d}(A^c) = 1$, which contradicts the hypothesis. Hence, $\overline{d}(A \cap B) > 0$. The proof for the second case is the same as for (1).

Theorem 3. $f_0$ or $f_1$ is surjective.

1. If IFS $(f_0, f_1)$ has the $\delta$-shadowing property, then IFS $(f_0, f_1)$ is chain transitive.
2. If IFS $(f_0, f_1)$ has the $\delta$-shadowing property, then IFS $(f_0, f_1)$ is chain transitive.

Proof. Without loss of generality, suppose that $f_1$ is surjective. Then, there is a sequence $\{y_j\}_{j \geq 0} - \{y_0 = y, y_1, y_2, \ldots\}$ such that for any $i \geq 0$, $f_i(y_j) = y_{j+i}$. Let $\epsilon > 0$ be arbitrary, and $x, y \in X$. Firstly, we will prove that (1) is true. We can find a number $\delta > 0$ as in the definition of the $\delta$-shadowing property for IFS $(f_0, f_1)$. Put $a_i = 2^n, a_i \to -n$. Construct a sequence as follows: $\{\xi_i\}_{i \geq 0} = \{x, f_0(x), f_0^2(x), \ldots, f_0^n(x), \ldots\}$. Then, $\{\xi_i\}_{i \geq 0}$ is a $\delta$-ergodic-pseudo-orbit for IFS $(f_0, f_1)$, i.e., there is $\omega = \omega_0\omega_1\omega_2 \cdots \in \Sigma^\omega$ such that
\[ \lim_{n \to \infty} \frac{1}{n} |i \in \mathbb{N} : 0 \leq i < n, d(f_{\omega}(\xi_i), \xi_{i+1}) < \delta| = 1, \quad (24) \]

where
\[ \omega_i = \begin{cases} 0, & d(f_0(\xi_i), \xi_{i+1}) < \delta, \\ 1, & d(f_1(\xi_i), \xi_{i+1}) \geq \delta. \end{cases} \quad (25) \]

Put $A_1 = \{i \in \mathbb{N} : \xi_i \in \{f_0^j(x)\}_{j \geq 0}\} \text{ and } A_2 = \{i \in \mathbb{N} : \xi_i \notin \{f_0^j(y)\}_{j \geq 0}\}$. Then, $\overline{d}(A_1) = 1, \overline{d}(A_2) = 1$. Since IFS $(f_0, f_1)$ has the $\delta$-shadowing property, there is $z \in X$ such that
\[ \lim_{n \to \infty} \frac{1}{n} |i \in \mathbb{N} : 0 \leq i < n, d(f_{\omega}(z), \xi_i) < \delta| = 0. \quad (26) \]

According to Lemma 2, there are $i_0, j_0, l, s \in \mathbb{N}, l < s$ such that
\[ d(f_0^j(z), \xi_l) < \epsilon, d(f_0^s(z), \xi_l) < \epsilon, \xi_l = f_0^s(x), \xi_i = y_{j_0}. \quad (27) \]

So $\{x, f_0(x), f_0^2(x), \ldots, f_0^{j_0-1}(x), f_0^j(z), \ldots, f_0^{s-1}(z), y_{j_0}, y_{j_0+1}, \ldots, y_s\}$ is an $\epsilon$-chain from $x$ to $y$. Hence, IFS $(f_0, f_1)$ is chain transitive.

Then, we will prove that (2) is true. Construct a sequence as follows:
\[ \{\xi_i\}_{i \geq 0} = \{x, y, x, f_0(x), y_1, y, x, f_0(x), f_0^2(x), y_2, y, \ldots\}. \quad (28) \]

Then, $\{\xi_i\}_{i \geq 0}$ is a $\delta$-ergodic-pseudo-orbit for IFS $(f_0, f_1)$, i.e., there is $\omega = \omega_0\omega_1\omega_2 \cdots \in \Sigma^\omega$ such that
If IFS \( (f_0, f_1) \) has the \( \overline{d} \)-shadowing property and \( f_0 \) or \( f_1 \) is surjective, then IFS \( (f_0, f_1) \) is chain mixing.

Proof. The result is obtained by Corollary 1, Theorem 3, and Theorem 2.3 in [13].

Lemma 3 (see [10]). IFS \( (f_0, f_1) \) is chain transitive and has the shadowing property, and then IFS \( (f_0, f_1) \) is transitive.

Corollary 2. IFS \( (f_0, f_1) \) has the \( \overline{d} \)-shadowing property (respectively, \( \overline{d} \)-shadowing property) and the shadowing property, one of \( f_0, f_1 \) is surjective, and then IFS \( (f_0, f_1) \) is mixing.

Proof. The result is obtained by Theorem 4 and Lemma 3. IFS \( (f_0, f_1) \) is ergodic if for any pair of nonempty open subsets \( U, V \subset X \), there is \( \omega \in \Sigma^2 \) such that

\[ N(U, V) = \{ n \in \mathbb{N} : f_n^0(U) \cap V \neq \emptyset \} \]

has the positive upper density. IFS \( (f_0, f_1) \) is strongly ergodic if for any pair of nonempty open subsets \( U, V \subset X \), there is \( \omega \in \Sigma^2 \) such that

\[ N(U, V) = \{ n \in \mathbb{N} : f_n^0(U) \cap V \neq \emptyset \} \]

is a syndetic set.

A map \( f : X \to X \) is an open map if for any open set \( U \subset X \), \( f(U) \) is also an open set.

Theorem 5. Let \( f_0, f_1 \) be open maps. For IFS \( (f_0, f_1) \) with the \( \overline{d} \)-shadowing property (respectively, \( \overline{d} \)-shadowing property), if \( A \subset X \) is dense in \( X \), and \( s \) is a minimal point of \( f_0 \) or \( f_1 \) for any \( s \in A \), then IFS \( (f_0, f_1) \) is strongly ergodic.

Proof. Suppose that \( U \) and \( V \) are two nonempty open subsets of \( X \), then there are \( x \in U \cap A, y \in V \cap A \), and \( d \epsilon > 0 \) such that

\[ B(x, \epsilon) \subset U, B(y, \epsilon) \subset V \]

and \( f^p_0, f^p_1, f^q_0, f^q_1 \) are syndetic sets, where

\[ f^p_0 = \{ i \in N : f^p_i(0) \in B(x, (\epsilon/2))^p \}, f^q_0 = \{ i \in N : f^q_i(0) \in B(y, (\epsilon/2))^q \} \]

are syndetic sets, and we can find \( N_0, N_1 \subset \mathbb{Z} \) such that

\[ f^p_0\cap[n, n + N_0] \neq \emptyset, f^q_0\cap[n, n + N_0] \neq \emptyset \] for any \( n \in N \). Firstly, we will prove if IFS \( (f_0, f_1) \) has the \( \overline{d} \)-shadowing property, then the conclusion is true. Put \( N_0 = \max\{N_0, N_1\} \}

where

\[ \omega_i = \begin{cases} 0, & d(f_0(x_i), x_i) < \delta, \\ 1, & d(f_1(x_i), x_i) \geq \delta \end{cases} \]

Then, \( \overline{d}(A_1) = (1/2), \overline{d}(A_2) = (1/2) \). Since IFS \( (f_0, f_1) \) has the \( \overline{d} \)-shadowing property, there is \( z \in X \) such that

\[ \lim \sup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta(f_0(x_i), x_i) < \epsilon \]

Then, the proof of this case is the same as for (1).

Theorem 4. If IFS \( (f_0, f_1) \) has the \( \overline{d} \)-shadowing property or \( \overline{d} \)-shadowing property, and \( f_0 \) or \( f_1 \) is surjective, then IFS \( (f_0, f_1) \) is chain mixing.

Proof. The result is obtained by Corollary 1, Theorem 3, and Theorem 2.3 in [13].
which is in contrast with \( d(B) > 0 \). Another proof can be obtained similarly. Since \( n_{0} \in B \cap E \) and \( m_{0} \in B \cap F \), \( d(f_{u}^{n_{0}}(x), e_{i}) \) and \( d(f_{u}^{m_{0}}(x), e_{i}) < \epsilon_{1} \), where \( n_{0} = f_{p}^{-E_{2n_{0}}}(x) \) and \( m_{0} = f_{q}^{-E_{2m_{0}}}(y) \). Owing to \( x, y \in A \), there are \( 0 \leq n' \leq N_{0} \) such that \( d(f_{p}^{-E_{2n'}}, x) \) and \( d(f_{q}^{-E_{2m'}}, y) \) \( < \epsilon/2 \). Therefore, \( d(f_{u}^{n'}, x), e_{i}, e_{j} \) and \( d(f_{u}^{m'}, y), e_{i}, e_{j} \). Obviously, \( \{ f_{u}^{n'}(z_{1}), \ldots, f_{u}^{n'-(N_{0})}(z_{1}), \ldots, f_{u}^{n'}(z_{N}), \ldots, f_{u}^{n'}(z_{N}) \} \) and \( \{ f_{u}^{m'}(z_{1}), f_{u}^{m'-(N_{0})}(z_{1}), \ldots, f_{u}^{m'}(z_{N}), \ldots, f_{u}^{m'}(z_{N}) \} \) are \( e_{i} \)-chains. According to Lemma 1, \( d(f_{u}^{n'}, x) < \epsilon/2 \) and \( d(f_{u}^{m'}, y) < \epsilon \). Then, \( d(f_{u}^{n'}, x) < \epsilon \) and \( d(f_{u}^{m'}, y) < \epsilon \). Consequently, the conclusion is true.

\[\begin{align*}
E &= \bigcup_{n=0}^{\infty} \{E_{2n}, E_{2n+1}, \ldots, E_{2n+1}+1, \ldots, E_{2n+2}+1 \}, \\
F &= \bigcup_{n=0}^{\infty} \{E_{2n+1}, E_{2n+2}+1, \ldots, E_{2n+2}+2 \}.
\end{align*}\]

We can see that \( \bar{d}(E) = 1/2 \) and \( \bar{d}(F) = 1/2 \). Then, the proof of this case is the same as case for one.

**Theorem 6.** Let \( f_{0}, f_{1} \) be open maps. For IFS \( (f_{0}, f_{1}) \) with the \( d \)-shadowing property (respectively, \( \tilde{d} \)-shadowing property), if \( A \subset X \) is dense in \( X \), and \( s \) is a minimal point of \( f_{0} \) or \( f_{1} \) for any \( s \in A \), then \( \mathcal{F}^{k} \) is strongly ergodic.

**Proof.** According to the condition, for any \( s \in S \), \( s \) is a minimal point of \( f_{0} \) or \( f_{1} \). For any \( k \in \mathbb{Z}^{+} \), it is well known that \( AP(f) = AP(f') \) and \( s \) is a minimal point of \( f_{0}^{k} \) or \( f_{1}^{k} \). We can combine Corollary 1 and Theorem 5. Then, \( \mathcal{F}^{k} \) is strongly ergodic.

A point \( x \in X \) is called a stable point of IFS \( (f_{0}, f_{1}) \) if for any \( \epsilon > 0 \), there is \( \delta > 0 \) satisfying \( d(f_{0}^{i}(x), f_{1}^{i}(x)) < \epsilon \) for any \( y \in X \) with \( d(x, y) < \delta \) and any \( \omega = \omega_{2} \in \Sigma \), \( n \in N \). IFS \( (f_{0}, f_{1}) \) is called Lyapunov stable if any point of \( X \) is a stable point of IFS \( (f_{0}, f_{1}) \).

**Theorem 7.** For IFS \( (f_{0}, f_{1}) \) with the \( d \)-shadowing property (respectively, \( \tilde{d} \)-shadowing property), if IFS \( (f_{0}, f_{1}) \) is Lyapunov stable, and \( f_{0} \) or \( f_{1} \) is surjective, then IFS \( (f_{0}, f_{1}) \) is transitive. Hence, \( \mathcal{F}^{k} \) is transitive for any \( k \in \mathbb{Z}^{+} \).

**Proof.** Suppose that \( U \) and \( V \) are two nonempty open subsets of \( X \), then there are \( x \in U, y \in V \), \( \epsilon > 0 \) such that \( B(x, \epsilon) \subset U \) and \( B(y, \epsilon) \subset V \), where \( x, x \in X \) are stable points of IFS \( (f_{0}, f_{1}) \). Then, \( \epsilon_{i} > 0 \) such that for any \( u, v \in X \), \( d(u, v) < \epsilon_{i} \), then \( d(f_{0}^{n}(u), f_{0}^{n}(v)) < \epsilon_{i} \) for all \( n = 1, 2, \ldots \).

Without loss of generality, suppose that \( f_{1} \) is surjective. Then, there is a sequence \( \{ \gamma_{j} \}_{j=0}^{\infty} \) such that \( y_{j} = f_{j}(y_{j-1}) \) for all \( j \in \mathbb{Z}^{+} \) and \( y_{0} = y \). Firstly, we will prove that IFS \( (f_{0}, f_{1}) \) has the \( \tilde{d} \)-shadowing property, then the conclusion is true. Put \( a_{0} = 0, a_{1} = 1, a_{2} = 2n_{0}, \ldots, a_{n} = 2n_{0}^{2} + n_{1}, \ldots, n_{1}, a_{n} = a_{0} + a_{1} + \cdots + a_{n} \). Then, we can see that \( \bar{d}(E) = 1 \) and \( \bar{d}(F) = 1 \). For \( \epsilon_{1} > 0 \), we can find a number \( \delta > 0 \) as in the definition of the \( \tilde{d} \)-shadowing property for IFS \( (f_{0}, f_{1}) \). Construct a sequence \( \{ \xi_{i} \}_{i=0}^{\infty} \) as follows:

\[\xi_{i} = \begin{cases} f_{0}^{i-e_{2n_{0}}}(x), & e_{2n_{0}} \leq i < e_{2n_{1}}, \\
y_{i-e_{2n_{1}}}, & e_{2n_{1}} \leq i < e_{2n_{2}}. \end{cases}\]

That is to say, \( \{ \xi_{i} \}_{i=0}^{\infty} = \{ x, y_{1}, x, \ldots, y_{1}^{-1}(x), y_{1}^{-1} \ldots, y_{1}^{-1} \ldots \} \). Then, \( \{ \xi_{i} \}_{i=0}^{\infty} \) is a \( \delta \)-ergodic pseudo-orbit for IFS \( (f_{0}, f_{1}) \), i.e., there is \( \omega = \omega_{0} \omega_{1} \in \Sigma \) such that

\[\lim_{n \to \infty} \frac{1}{n} \left| \{ i \in N : 0 \leq i < n, d(f_{0}^{i}(\xi_{i}), \xi_{i+1}) < \delta \} \right| = 1,\]

where \( \omega_{0} = \begin{cases} 0, & \xi_{i} = f_{0}^{i-e_{2n_{0}}}(x), \\
1, & \xi_{i} = y_{i-e_{2n_{1}}}. \end{cases} \)

Since IFS \( (f_{0}, f_{1}) \) has the \( \tilde{d} \)-shadowing property, there is \( z_{0} \in X \) such that

\[\liminf_{n \to \infty} \frac{1}{n} \left| \{ i \in N : 0 \leq i < n, d(f_{0}^{i}(z_{0}), \xi_{i}) < \epsilon_{i} \} \right| > 0.\]
Suppose that \( f \) is a point in the set \( X \), then the complexity of \( f \) is defined as the number of \( \epsilon \)-balls needed to cover \( f \) without overlapping. We can find a \( k \in \mathbb{N} \) such that \( \delta(f_k) = \alpha > (4/k) \), and \( \delta(f_{k+1}) = \beta > (4/k+1) > 0 \), where \( \alpha = \max(k,1,k_0) \). Since the IFS \( f_{k+1} \) has an asymptotic property, there is \( \delta > 0 \) such that every \( \epsilon \)-average-pseudo-orbit of \( f_{k+1} \) is \( (\epsilon/k+1) \)-shadowed on average by a point in \( X \).

According to Corollary 1, the \( \delta \)-property is transitive for any \( k \in \mathbb{Z}^+ \).

Then, we will prove that \( f_{k+1} \) has the \( \delta \)-shadowing property, then the conclusion is true. Put \( a_0 = 0, a_1 = 1, a_2 = 2, \ldots, a_n = n \), and \( E_n = a_0 + a_1 + \cdots + a_n \),

\[
E = \bigcup_{n=0}^{\infty} \{E_{2n}, E_{2n+1}, E_{2n+2}, -\cdots\},
\]

\[
F = \bigcup_{n=0}^{\infty} \{E_{2n+1}, E_{2n+2}, E_{2n+3}, -\cdots\}.
\]

We can see that \( d(E) = (1/2) \) and \( d(F) = (1/2) \). Then, the proof of this case is the same as for case one.

### 3. The Average Shadowing Property for IFS \((f_0, f_1)\)

Bahabadi [10] introduced the definition of the average shadowing property for IFS \((f_0, f_1)\). In this section, we will study the relationship among the average shadowing property, ergodicity, and strong ergodicity.

**Definition 4.** A sequence \( \{\xi_1, \xi_2, \ldots \} \) is called a \( \delta \)-average-pseudo-orbit for IFS \((f_0, f_1)\) if there is \( \omega \in \Sigma^2 \) and \( N \in \mathbb{Z}^+ \). Then, \( N \) such that for every \( n \geq N \),

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(f(\omega_i, \xi_{i+1})) < \delta.
\]

**Definition 5.** IFS \((f_0, f_1)\) has the average shadowing property (ASP) if for any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that every \( \delta \)-average-pseudo-orbit \( \{\xi_1, \xi_2, \ldots \} \) is \( \epsilon \)-shadowed on average by a point \( z \in X \), i.e., there is \( \omega \in \Sigma^2 \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(\omega_i, \xi_{i+1})) < \epsilon.
\]

**Theorem 8.** For IFS \((f_0, f_1)\) with ASP, if \( C \subset X \) is dense in \( X \), and \( s \) is a quasi-weakly almost periodic point of \( f_0 \) or \( f_1 \) for any \( s \in S \), then IFS \((f_0, f_1)\) is ergodic.

**Proof.** Suppose that \( U, V \subset X \) are two nonempty open sets. We can find two points \( u \in U \), \( v \in V \), and \( \epsilon > 0 \) such that \( B(u, \epsilon) \) is a \( \delta \)-shadowed on average by a point in \( X \), i.e., there is \( \omega \in \Sigma^2 \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(\omega_i, \xi_{i+1})) < \epsilon.
\]

which is in contrast with (50). So \( \overline{d}(F) < (1/k) \) and \( d(F^c) = 1 - d(F) \geq 1 - (1/k) = (k-1/k) \). According to Lemma 2, \( \overline{d}(W_0^c \cap F^c) > 0 \) and \( d(W_0^c \cap F^c) > 0 \). There are \( i_0, f_{i_0}, s \) such that \( f_{i_0}^s(x) \in B(u, \epsilon/2) \), \( f_{i_0}^s(y) \in B(v, \epsilon/2) \), and \( f_{i_0}^s(z) \in B(v, \epsilon/2) \), where \( \xi_0 = f_{i_0}^s(x) \), \( \xi_1 = f_{i_0}^s(y) \). Then, \( d(f_{i_0}^s(z), u) < \epsilon, d(f_{i_0}^s(z), v) < \epsilon \). Therefore, \( f_{i_0}^s(z) \in B(u, \epsilon) \), and \( f_{i_0}^s(z) \in B(v, \epsilon/2) \). Let \( n_0 = s - l \), and \( w^s = \omega_0 \omega_1 \cdots w_0 w_1 \cdots \in \Sigma^2 \), then \( N(U, V) = \{n_0 \in \mathbb{Z}_+ : f_{i_0}^s(U \cap V, \omega^s \in \Sigma^2) \neq \emptyset \} \). Obviously,
\[
\limsup_{n \to \infty} \left\{ n_0: n_0 = s - l, s > l, l \in W^s_f \cap F^s, s \in W^s_f \cap F^s \right\} \cap \{0, \ldots, n - 1\} > 0. \tag{52}
\]

Therefore, \( \bar{d}(N(U, V)) > 0 \). As \( U \) and \( V \) are arbitrary, IFS \((f_0, f_1)\) is ergodic.

**Theorem 9.** Let \( f_0, f_1 \) be open maps. For IFS \((f_0, f_1)\) with the ASP, if \( S \subset X \) is dense in \( X \), and \( s \) is a minimal point of \( f_0 \) or \( f_1 \) for any \( s \in S \), then IFS \((f_0, f_1)\) is strongly ergodic.

**Proof.** Suppose that \( U \) and \( V \) are two nonempty open subsets of \( X \). Obviously, the sydetic set has the positive upper density. By Theorem 8, there are \( n_0 \in \mathbb{N}, \omega^3 = \omega_0^3 \omega_1^3 \cdots \omega_{n_0 - 1}^3 \in \Sigma^2 \) such that \( f^{n_0}_\omega(U) \cap V \neq \emptyset \). We write \( W = f^{n_0}_\omega(U) \cap V \neq \emptyset \). As \( f_0, f_1 \) are open sets, \( f^{n_0}_\omega(U) \) is an open set. Therefore, \( W \) is an open set. Then, there are \( z^* \in W \cap S, \omega^3 \in \Sigma^2 \) such that \( J = \{ n \in \mathbb{N}: f^n_\omega(z^*) \in W \} \) is a sydetic set. If \( m \in J \), then \( f^{m_0}_\omega(W) \cap V \neq \emptyset \). There is \( \omega^4 = \omega_0^4 \omega_1^4 \cdots \omega_2^4 \omega_{m_0 - 1}^4 \in \Sigma^2 \) such that \( \emptyset \neq f^{m_1}_\omega(W) \cap V = f^{m_1}_\omega(U) \cap V \). Since \( n_0, m: m \in J \subset N(U, V) = \{ n \in \mathbb{N}: f^n_\omega(U) \cap V \neq \emptyset \} \), \( N(U, V) \) is a sydetic set. As \( U \) and \( V \) are arbitrary, IFS \((f_0, f_1)\) is strongly ergodic.

**Lemma 4** (Theorem 3.1 in [13]). If IFS \((f_0, f_1)\) has the ASP, then \( \mathbb{F}^k \) has the ASP for any \( k \in \mathbb{Z}^* \).

**Theorem 10.** Let \( f_0, f_1 \) be open maps. For IFS \((f_0, f_1)\) with the ASP, if \( S \subset X \) is dense in \( X \), and \( s \) is a minimal point of \( f_0 \) or \( f_1 \) for any \( s \in S \), then \( \mathbb{F}^k \) is strongly ergodic.

**Proof.** According to the condition, for any \( s \in S \), \( s \) is a minimal point of \( f_0 \) or \( f_1 \). For any \( k \in \mathbb{Z}^* \), it is well known that \( AP(f) = AP(f^k) \) and then \( s \) is a minimal point of \( f_0^k \) or \( f_1^k \). We can combine Lemma 4 and Theorem 9. Then, \( \mathbb{F}^k \) is strongly ergodic.

## 4. A Remark on the Asymptotic Average Shadowing Property for IFS \((f_0, f_1)\)

The definition of the asymptotic average shadowing property for IFS \((f_0, f_1)\) is introduced by Nia [11]. Here, we will come up with a situation to show that an example does not have the asymptotic average shadowing property.

**Definition 6.** A sequence \( \{\xi_i\}_{i \geq 0} \) is called an asymptotic average pseudo-orbit for IFS \((f_0, f_1)\) if there is \( \omega \in \Sigma^2 \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f_{\omega_i}(\xi_i), \xi_{i+1}) = 0. \tag{53}
\]

**Definition 7.** IFS \((f_0, f_1)\) has the asymptotic average shadowing property (AASP) if every asymptotic average pseudo-orbit is shadowed on average by a point \( z \in X \), i.e., there is \( \omega \in \Sigma^2 \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f_{\omega_i}(z), \xi_i) = 0. \tag{54}
\]

For any \( A, B \subset X \),

\[
d(A, B) = \min_{a \in A, b \in B} d(a, b). \tag{55}
\]

**Theorem 11.** If \( U \subset X \) is an open set, then \( U \) is invariant under \( f_i(t = 0, 1)(f_i(U) \subset U, i = 0, 1) \). Assume that there is \( x \in X \) such that \( \{ f_0(x) \}_{i \geq 0} \subset U \) or \( \{ f_1(x) \}_{i \geq 0} \subset U \). Suppose that \( y \in X \) is a point whose orbit is metrically separated from \( U \) (for any \( \omega \in \Sigma^2 \), \( d(\{ f_{\omega_i}(y) \}_{i \geq 0}, U) > 0 \)). Then, IFS \((f_0, f_1)\) does not have the AASP.

**Proof.** Without loss of generality, suppose \( \{ f_0(x) \}_{i \geq 0} \subset U \). Construct \( \{ \xi_i \}_{i \geq 0} \) as follows:\( \{ \xi_i \}_{i \geq 0} = \{ x, y, x, f_0(x), y, f_1(x), x, f_0(x), f_0(x), f_0(x), f_0(x), y, f_1(y), f_1(y), f_1(y), y, f_0(x), \cdots \}. \)

For any \( k \in \mathbb{N} \), we can find \( l \in \mathbb{N} \) with

\[
\sum_{i=1}^{l} 2^i \leq k \leq \sum_{i=1}^{l+1} 2^i. \tag{56}
\]

We can choose

\[
\omega = \begin{cases} 0, & d(f_0(\xi_i), \xi_{i+1}) = 0, \\ 1, & d(f_0(\xi_i), \xi_{i+1}) > 0. \end{cases} \tag{57}
\]

\[
\omega = \omega_0 \omega_1 \omega_2 \cdots \in \Sigma^2.
\]

We can see that

\[
\sum_{i=0}^{k-1} d(f_{\omega_i}(\xi_i), \xi_{i+1}) \leq 3l \cdot D, D = \text{diam}(X). \tag{58}
\]

This implies that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f_{\omega_i}(\xi_i), \xi_{i+1}) \leq \lim_{l \to \infty} \frac{3l \cdot D}{2^l} = 0. \tag{59}
\]

So \( \{ \xi_i \}_{i \geq 0} \) is an asymptotic average pseudo-orbit of IFS \((f_0, f_1)\). We will prove that IFS \((f_0, f_1)\) does not have the AASP. Suppose that IFS \((f_0, f_1)\) has the AASP. For asymptotic average pseudo-orbit \( \{ \xi_i \}_{i \geq 0} \) there are \( z \in X, \omega \in \Sigma^2 \) such that \( \{ \xi_i \}_{i \geq 0} \) is asymptotically shadowed on average by the point \( z \). Meanwhile, the orbit of \( z \) has to enter \( U \) at some point. Otherwise,

\[
\frac{1}{2^n} \sum_{i=0}^{2^n-1} d(f_{\omega_i}(z), \xi_i) \geq \frac{1}{2} d(\{ f_{\omega_i}(z) \}_{j \geq 0} \subset U) > 0, \quad (\forall n \in \mathbb{N}). \tag{60}
\]
So IFS \((f_0, f_1)\) does not have the AASP. Therefore, there is \(N_0 \in \mathbb{N}\) such that \(f_{n+1}^n(z) \in U\). Then, \(f_{n+2}^n(z) \in U\) for any \(n \geq N_0\). Therefore, for \(n\) that is large enough,
\[
\frac{1}{2^n} \sum_{i=0}^{2^n-1} d(f_i(z), \xi_i) \geq \frac{1}{4} d\left(\left\{f_1(y)\right\}_{y \in U}\right) > 0, \quad (61)
\]
which contradicts with the hypothesis. IFS \((f_0, f_1)\) does not have the AASP.

The following example comes from the study in [13]. It is a vitally important example of IFS \((f_0, f_1)\). It is controversial whether it has the AASP. We can use the above theorem to prove that it does not have the AASP.

**Example 2.** Let \(f_0, f_1: [0, 1] \rightarrow [0, 1]\) be two continuous maps such that \(f_1(x) > f_0(x) > x\) if and only if \(x \in [0, (1/2)]\) and \(f_0(0) = f_1(0) = (1/4)\). Obviously, we can see that \(f_t(1/8)\) \(\forall t \in [0, (1/2)]\) \(f_t(U) \subset U\), where \(t = 0, 1\). And for any \(\omega \in \Sigma^2\), \(d\left(\left\{f_\omega(7/8)\right\}, U\right) > 0\).

According to Theorem 11, IFS \((f_0, f_1)\) does not have the AASP.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

This work was supported by the Foundation of Zhubai College of Jilin University, Autonomous Research Foundation for Colleges and Universities of the Party Central Committee, National Young Science Foundation of China (no. 11701066), and Key Natural Science Foundation of Universities in Guangdong Province (no. 2019KZDXM027).

**References**


