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Research Article

Stationary Distribution and Periodic Solution of Stochastic Toxin-Producing Phytoplankton–Zooplankton Systems

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In this paper, we investigate the dynamics of autonomous and nonautonomous stochastic toxin-producing phytoplankton–zooplankton system. For the autonomous system, we establish the sufficient conditions for the existence of the globally positive solution as well as the solution of population extinction and persistence in the mean. Furthermore, by constructing some suitable Lyapunov functions, we also prove that there exists a single stationary distribution which is ergodic, what is more important is that Lyapunov function does not depend on existence and stability of equilibrium. For the nonautonomous periodic system, we prove that there exists at least one nontrivial positive periodic solution according to the theory of Khasminskii. Finally, some numerical simulations are introduced to illustrate our theoretical results. The results show that weaker white noise and/or toxicity will strengthen the stability of system, while stronger white noise and/or toxicity will result in the extinction of one or two populations.

1. Introduction

As well known, mathematical models describing the plankton dynamics have played an important role in understanding the various mechanisms involved in toxin-producing phytoplankton. There are many scientific works have been carried out to investigate the effects of toxin-producing phytoplankton on plankton ecosystems [1–7]. For example, Upadhyay and Craniopathy [1] proposed three species food chain model with different functional forms to describe the liberation of toxin. The obtained results show that the increase of toxic substances released by toxic-producing phytoplankton has a stabilizing effect. In particular, according to field observations, Chattopadhyay et al. [5] formulated the following toxin-producing phytoplankton–zooplankton model:

$$\begin{cases} \dfrac{dp}{dt} = rp\left(1 - \dfrac{p}{k}\right) - \alpha f(p)z, \\ \dfrac{dz}{dt} = \beta f(p)z - \mu z - \theta g(p)z, \end{cases} \tag{1}$$

where $p(t)$ and $z(t)$ denote the density of toxin-producing plankton (TPP) population and the zooplankton population at time t, respectively, subject to the nonnegative initial condition $p(0) = p_0 \geq 0$ and $z(0) = z_0 \geq 0$. r and k represent the intrinsic growth rate and the environmental carrying capacity of TPP population, respectively. α is the rate of predation of zooplankton on TPP population, β is the ratio of biomass consumed by zooplankton for its growth (satisfying the obvious restriction $0 < \beta < \alpha$), and μ denotes the mortality rate of zooplankton due to nature death, θ denotes the rate of toxin liberation by TPP population. $f(p)$ represents the predational response function and $g(p)$ describes the distribution of toxic substances. All parameters above are positive. As liberation of toxin reduces the growth of zooplankton, causes substantial mortality of zooplankton and in this period toxin-producing phytoplankton population is not easily accessible, hence a more common and intuitively obvious choice is of the saturation functional form to describe the grazing phenomena. For instance, Tapan et al. [8] studies the system (1) when $f(p) = g(p) = p/(\gamma + p)$ (where $\gamma > 0$ denotes the half-saturation constant). The obtained result indicates that there is a threshold limit of toxin liberation by the phytoplankton species below which the system does not have any excitable nature and above which the system shows excitability.

Clearly, these important and useful works on deterministic phytoplankton–zooplankton model provide a great insight into the dynamics of plankton ecosystems. However, in the real world, the dynamics of plankton ecosystems are inevitably perturbed by various types of environment noises. Development from the deterministic models to the stochastic models can give us new insights into the dynamics of plankton ecosystems [9]. May [10] pointed out that the birth rates, carrying capacity, competition coefficients and other parameters involved in the system can be affected by environmental noise. This may be especially true for plankton ecosystems due to unpredictability of photosynthetically active radiation, nutrient availability, water temperature, water depth, light, eutrophication, acidity, salinity, wind and many other physical factors embedded in aquatic ecosystems. Several scholars have studied the effect of environmental fluctuations on aquatic ecosystems [11–13]. Indeed, stochastic models could be more appropriate way of modeling in comparison with their deterministic counterparts, since they can provide some additional degree of realism. By introducing (stochastic) environmental noise, many investigators have studied stochastic epidemic models [14–23] and stochastic population models [24–42]. They focus on the effect of environmental fluctuations on the dynamic behavior of these models. For instance, Silva [11] investigated a stochastic model of phytoplankton–zooplankton interactions with toxin-producing phytoplankton. Theoretical results show that for certain values of the system parameters, the system posses asymptotic stability around the positive interior equilibrium which depicts the coexistence of all the species. Therefore, it is meaningful to further incorporate the environmental stochasticity into the model (1) with $f(p) = g(p) = p/(\gamma + p)$, which could provide us a deeper understanding for real aquatic ecosystems [43].

On the other hand, periodic behavior arises naturally in many real world problems, such as in biological, environmental and economic systems [44]. The phenomenon of periodic oscillations has been observed in the growth of populations, such as, a seasonal occurrence of *Hematodinium perezi* was reported by Newman and Johnson (1975) in the parasitic dinoflagellate in blue crabs on the east coast of the United States from late spring to early winter [45]. Seasonal changes are cyclic, largely predictable, and arguably represent the strongest and most ubiquitous source of external variation influencing human and natural systems [46]. However, to the best of our knowledge, there is little work on the existence of stochastic periodic solution for nonautonomous toxin-producing phytoplankton zooplankton model. Based on the aforementioned, we intend to study toxin-producing phytoplankton–zooplankton model with environmental fluctuations, and then we extend this model into a nonautonomous stochastic model by taking into account seasonal variation in the next section. Then in Section 3, we show the existence and uniqueness of the global positive solution. In Section 4, we obtain the sufficient conditions for the solution of population extinction and persistence in the mean. In Section 5 and Section 6, using Khasminskii's methods and Lyapunov functions, we derive sufficient conditions for the existence of the single ergodic stationary distribution of the autonomous system (2) and the existence of the nontrivial positive stochastic periodic solution of nonautonomous system (3). In Section 7, some numerical simulations are provided to demonstrate the analytical findings. Finally, some conclusions are given in Section 8.

2. Model Formulation

There are many kinds of approaches to introduce the white noise into the population models. For model (1), we assume that the growth rate r of phytoplankton and the death rate μ of zooplankton are subjected to the Gaussian white noise (we follow the way used in [38]), then we can obtain the following stochastic model:

$$\begin{cases} dp = \left[rp\left(1 - \dfrac{p}{k}\right) - \dfrac{\alpha p}{\gamma + p} z \right] dt + p\sigma_1 dB_1(t), \\ dz = \left[\dfrac{\beta p}{\gamma + p} z - \mu z - \dfrac{\theta p}{\gamma + p} z \right] dt + z\sigma_2 dB_2(t), \end{cases} \tag{2}$$

where $B_1(t), B_2(t)$ are independent Brownian motions defined on the complete probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t_{\geq 0}}, P\right)$ with a filtration $\{\mathcal{F}_t\}_{t_{\geq 0}}$ satisfying the usual conditions (i.e. it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets). σ_1^2, σ_2^2 represent the intensities of the white noise. Meanwhile, there are evidences suggesting that the toxic substances released by TPP do not remain constant but change over time, which is related to the seasonal changes. Therefore, for better understanding the toxin-producing phytoplankton–zooplankton sustained oscillatory patterns, we further consider the following periodic system with stochastic perturbation by the method of Khasminskii [47].

$$\begin{cases} dp = \left[r(t)p\left(1 - \dfrac{p}{k(t)}\right) - \dfrac{\alpha(t) p}{\gamma(t) + p} z \right] dt + p\sigma_1(t) dB_1(t), \\ dz = \left[\dfrac{\beta(t) p}{\gamma(t) + p} z - \mu(t) z - \dfrac{\theta(t) p}{\gamma(t) + p} z \right] dt + z\sigma_2(t) dB_2(t), \end{cases} \tag{3}$$

where $r(t), k(t), \alpha(t), \beta(t), \mu(t), \theta(t), \gamma(t)$ are all positive θ_1-periodic continuous functions. For biological significance, we always assume that $\beta(t) > \theta(t)$. Throughout this paper, let $R_+ = [0, \infty), R_+^2 = \{x = (x_1, x_2) \in R^2 : x_i > 0, \ i = 1, 2\}$, $u_x = \partial u/\partial x, u_{xx} = \partial^2 u/\partial x^2$, a.s. means almost surely.

3. Existence and Uniqueness of the Global Positive Solution

As we know, in order for a stochastic differential equation to have a single global solution (i.e. no explosion in a finite time) for any given initial value, the functions involved with stochastic system are generally required to satisfy the linear growth condition and local Lipschitz condition [48, 49]. However, the functions of system (2) do not satisfy the linear growth condition, so the solution of system (2) may explode at a finite time. In this section, we show that there exists a single positive local solution of system (2), then using the Lyapunov analysis method, we prove that this solution is global. Explanation for "explosion time" used in following lemma.

Lemma 1. *For $(p_0, z_0) \in R_+^2$, there exists a single positive local solution $(p(t), z(t))$ of system (2) for $t \in [0, \tau_e)$ almost surely, where τ_e is the explosion time.*

Proof. Using the transformation of variables $u(t) = \ln p(t), v(t) = \ln z(t)$ and applying Itô's formula, for system (2), we have.

$$\begin{cases} du(t) = \left(r - \frac{\sigma_1^2}{2} + \frac{r}{k}e^{u(t)} - \alpha\frac{e^{v(t)}}{\gamma + e^{u(t)}}\right)dt + \sigma_1 dB_1(t), \\ dv(t) = \left(\beta\frac{e^{u(t)}}{\gamma + e^{u(t)}} - \mu - \frac{\sigma_2^2}{2} - \theta\frac{e^{u(t)}}{\gamma + e^{u(t)}}\right)dt + \sigma_2 dB_2(t), \end{cases} \tag{4}$$

with the initial value $u(0) = \ln\ p_0, v(0) = \ln\ z_0$. The functions involved with drift part of above stochastic differential system satisfy the linear condition and locally Lipschitz condition. Hence there exists a single local solution $(u(t), v(t))$, for $t \in [0, \tau_e)$, where τ_e is any finite positive real number. Clearly, $(p(t), z(t)) = (e^{u(t)}, e^{v(t)})$ is the single positive local solution of stochastic differential system (2) starting from an interior point of the first quadrant.

Now we are in a position to show that this single solution is a global solution. To prove this we only need to show that $\tau_e = \infty$ a.s.

Theorem 2. *For any initial value $(p(0), z(0)) \in R_+^2$, the system (2) has a single global positive solution $(p(t), z(t))$ for all $t > 0$, and the solution will remain in R_+^2 with probability one.*

Proof. By Lemma 1, we only need to prove that $\tau_e = \infty$ a.s. From the first equation of model (2), we have

$$dp(t) \le p(t)\left(r - \frac{r}{k}p(t)\right)dt + \sigma_1 p(t)dB_1(t). \tag{5}$$

Let

$$\Phi(t) = \frac{e^{(r-(\sigma_1^2/2))t+\sigma_1 B_1(t)}}{1p_0 + (r/k)\int_0^t e^{(r-(\sigma_1^2/2))s+\sigma_1 B_1(s)}ds}, \tag{6}$$

then $\Phi(t)$ is the single solution of the following auxiliary system:

$$\begin{cases} d\Phi(t) = \Phi(t)\left(r - \frac{r}{k}\Phi(t)\right)dt + \sigma_1\Phi(t)dB_1(t), \\ d\Phi(0) = P_0. \end{cases} \tag{7}$$

Then by the comparison theorem for stochastic equations, we have

$$p(t) \le \Phi(t), \quad t \in [0, \tau_e)\ a.s. \tag{8}$$

On the other hand, from the second equation of model (2), we have

$$dz(t) \le z(t)(\beta - \mu)dt + \sigma_1 z(t)dB_1(t). \tag{9}$$

Consider the following auxiliary system:

$$\begin{cases} dM(t) = M(t)(\beta - \mu)dt + \sigma_2 M(t)dB_2(t), \\ dM(0) = z_0, \end{cases} \tag{10}$$

Then $M(t) = z_0 e^{(\beta-\mu-(\sigma_2^2/2))t+\sigma_2 B_2(t)}$ is the single solution of above system. Similarly by the comparison theorem for stochastic differential equations, we have

$$z(t) \le M(t), \ \ t \in [0, \tau_e)\ a.s. \tag{11}$$

Similarly, we have

$$p(t) \ge \Psi(t), \ \ t \in [0, \tau_e)\ a.s., \tag{12}$$

where $\Psi(t) = e^{(r-(\sigma_1^2/2))t-(\alpha/\gamma)\int_0^s M(s)ds+\sigma_1 B_1(t)} / \left((1/p_0) + (r/k)\int_0^t e^{(r-(\sigma_1^2/2))s-(\alpha/\gamma)\int_0^s M(u)du+\sigma_1 B_1(s)}ds\right)$ is the solution of the following system:

$$\begin{cases} d\Psi(t) = \Psi(t)\left(r - \frac{r}{k}\Psi(t) - \frac{\alpha M(t)}{\gamma}\right)dt + \sigma_1\Psi(t)dB_1(t), \\ dM(0) = p_0, \end{cases} \tag{13}$$

By the same arguments as above, we have

$$\begin{aligned} dz &\ge z(t)\left(\beta - \frac{\beta\gamma}{\gamma + p(t)} - \mu - \theta\right)dt + \sigma_2 dB_2(t) \\ &\ge z(t)\left(\beta - \frac{\beta\gamma}{\gamma + \Psi(t)} - \mu - \theta\right)dt + \sigma_2 z(t)dB_2(t). \end{aligned} \tag{14}$$

Similarly, we can obtain

$$z(t) \ge W(t), \quad t \in [0, \tau_e)\ a.s., \tag{15}$$

where $W(t) = z_0 e^{(\beta-\mu-\theta-(\sigma_2^2/2))t-\int_0^t(\gamma\beta/(\gamma+\Psi(t))ds)+\sigma_2 B_2(t)}$ is the single solution of the following system:

$$\begin{cases} dW(t) = W(t)\left(\beta - \frac{\beta\gamma}{\gamma + \Psi(t)} - \mu - \theta\right)dt + \sigma_2 W(t)dB_2(t), \\ dW(0) = z_0, \end{cases} \tag{16}$$

By (8), (11), (12), (15), we can get that

$$\Psi(t) \le p(t) \le \Phi(t), \quad W(t) \le z(t) \le M(t), \quad t \ge 0\ a.s. \tag{17}$$

It follows from [34] that $\Psi(t), \Phi(t), W(t)$ and $M(t)$ will not be exploded at any finite time, then by the comparison theorem of stochastic differential equations, we can derive that $(p(t), z(t))$ will globally exist. This completes the proof.

Remark 3. For any initial value $(p(0), z(0)) \in R_+^2$, the system (3) has a single global positive solution $(p(t), z(t))$, for all $t > 0$, and the solution will remain in R_+^2 with probability one.

Proof. The proof is similar to Theorem 2, so we omit it.

Remark 4. When we choose $f(p) = p^n/(\gamma^n + p^n), g(p) = p^n/(\gamma^n + p^n)(n = 1, 2)$, then the stochastic system (2) has a single global positive solution $(p(t), z(t))$ for all $t > 0$, and the solution will remain in R_+^2 with probability one.

Proof. The proof is similar to Theorem 2, so we omit it.

4. Extinction and Persistence in Mean

In this section, we will investigate the persistence and extinction of the system (2) under certain conditions. We give the definition and lemma which can be used for our main results.

Definition 5 [38].

(1) System (2) is said to be extinct if $\lim_{t\to\infty} p(t) = 0 a.s.$;
(2) System (2) is said to be persistent in mean if $\lim_{t\to\infty}(1/t)\int_0^t p(s)ds > 0 a.s.$

Let $C(\Omega \times [0, \infty), R_+)$ be the family functions on Ω which are continuously differentiable with respect to $x \in \Omega$ and continuously differentiable with respect to $t \in [0, \infty)$.

Lemma 6 [35]. *Suppose that $y(t) \in C(\Omega \times [0,\infty), R_+)$, if there are positive constant $T, \mu_0, \mu > 0$ such that for $t \geq T$.*

(1) $\ln y(t) \leq \mu t - \mu_0 \int_0^t y(s)ds + \sum_{i=1}^n \beta_i B_i(t)$, *then* $\limsup_{t\to\infty} (1/t)\int_0^t y(s)ds \leq (\mu/\mu_0)$ a.s.;

(2) $\ln y(t) \geq \mu t - \mu_0 \int_0^t y(s)ds + \sum_{i=1}^n \beta_i B_i(t)$, *then* $\liminf_{t\to\infty} (1/t)\int_0^t y(s)ds \geq (\mu/\mu_0)$ a.s.,

where $\beta_i (1 \leq i \leq n)$ is a constant.

Theorem 7. *Let $(p(t), z(t))$ be the solution of the system (2), then the following statements hold:*

(i) *If* $\sigma_1^2 < 2r, \sigma_2^2 > 2\beta - 2\mu$, *then* $\lim_{t\to\infty} z(t) = 0, \lim_{t\to\infty}(1/t) \int_0^t p(s)ds = k\left(r - \left(\sigma_1^2/2\right)\right)/r$ a.s.;

(ii) *If* $\sigma_1^2 > 2r, \sigma_2^2 > 2\beta - 2\mu$, *then* $\lim_{t\to\infty} z(t) = 0$, $\lim_{t\to\infty} p(t) = 0$ a.s.

Proof. (i) By the positiveness of the solution $(p(t), z(t))$ of system (2), we have

$$\begin{aligned} dz &= z\left(\beta \frac{p}{\gamma + p} - \mu - \theta \frac{p}{\gamma + p}\right)dt + z\sigma_2 dB_2(t) \\ &\leq z(\beta - \mu)dt + z\sigma_2 dB_2(t). \end{aligned} \tag{18}$$

Consider the following system

$$\begin{cases} \mathrm{d}\Psi(t) = \Psi(\beta - \mu)dt + \Psi\sigma_2 dB_2(t), \\ \mathrm{d}\Psi(0) = z(0). \end{cases} \tag{19}$$

We have

$$d\ln\Psi(t) = \left(\beta - \mu - \frac{\sigma_2^2}{2}\right)dt + \sigma_2 dB_2(t). \tag{20}$$

By the comparison principle of the stochastic differential equation and the theory of diffusion process (see [33] and Lemma A.2 in [34]), we can easily check that

$$\begin{aligned} S(-\infty) &= \int_0^{-\infty} \exp\left\{-\int_0^v \frac{2\left(\beta - \mu - \left(\sigma_2^2/2\right)\right)}{\sigma_2^2} ds\right\} dv \\ &= \frac{\sigma_2^2}{2\left(\beta - \mu - \left(\sigma_2^2/2\right)\right)} > -\infty, \end{aligned} \tag{21}$$

$$S(+\infty) = \int_0^{+\infty} \exp\left\{-\int_0^v \frac{2\left(\beta - \mu - \left(\sigma_2^2/2\right)\right)}{\sigma_2^2} ds\right\} dv = +\infty.$$

Since $\sigma_2^2 > 2\beta - 2\mu$, we can obtain that $\lim_{t\to\infty} \ln z(t) = -\infty$, that is $\lim_{t\to\infty} z(t) = 0$ a.s. For any $\epsilon > 0$, there exisit $T > 0$ and a set Ω_ϵ such that $P(\Omega_\epsilon) > 1 - \epsilon$ and $z(t) < \epsilon$ for $t > T, \omega \in \Omega_\epsilon$.

By the first equation of system (2), we have

$$\begin{aligned} &\left(r - \frac{\sigma_1^2}{2} - \frac{\alpha\epsilon}{\gamma} - \frac{r}{k}p\right)dt + \sigma_1 dB_1(t) \leq d \\ &\quad \cdot \ln p \leq \left(r - \frac{\sigma_1^2}{2} - \frac{r}{k}p\right)dt + \sigma_1 dB_1(t). \end{aligned} \tag{22}$$

By Lemma 6, for $\forall\epsilon \to 0$, we have $\liminf_{t\to\infty}(1/t)\int_0^t p(s)ds \geq k\left(r - \left(\sigma_1^2/2\right)\right)/r$, and $\limsup_{t\to\infty} (1/t)\int_0^t p(s)ds \leq k\left(r - \left(\sigma_1^2/2\right)\right)/r$, that is to say

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t p(s)ds = \frac{k\left(r - \left(\sigma_1^2/2\right)\right)}{r} \text{ a.s.}. \tag{23}$$

(ii) Similar to the proof of (i), we obtain that $\lim_{t\to\infty} z(t) = 0$ a.s. for $\sigma_2^2 > 2\beta - 2\mu$. In addition, by the first equation of system (2) and $r - \left(\sigma_1^2/2\right) < 0$, we obtain $\lim_{t\to\infty} p(t) = 0$*a.s.* This completes the proof.

Remark 8. From Theorem 7, we can see, when σ_2 is a constant, if $\sigma_1^2 < 2r$, then the population $p(t)$ will be persistent in mean, but when we choose σ_1 large enough such that $\sigma_1^2 > 2r$, then the population $p(t)$ will go to extinct. From an ecological point of views, the intensity of white noise has a negative effect on the survival of $p(t)$ population, which imply that weaker white noise will strengthen the stability of the system, while stronger white noise will lead to population $p(t)$ extinct.

5. Stationary Distribution and Ergodicity

In this section, we shall consider whether there exists a single stationary distribution of system (2), which means that the zooplankton population can persist and not die out.

Let $X(t)$ be a regular time-homogeneous Markov process in R_+^d described by the following stochastic differential equation

$$dX(t) = f(X(t))dt + \sum_{r=1}^k \sigma_r(X(t))dB_r(t). \tag{24}$$

The diffusion matrix is defined as follows

$$A(X) = \left(a_{ij}(x)\right), \quad a_{ij}(x) = \sum_{r=1}^k \sigma_r^i(x)\sigma_r^j(x). \tag{25}$$

Next, we shall introduce a lemma which guarantees the existence and uniqueness of a stationary distribution and ergodicity (see Khasminskii [47]).

Let U be a given open set in the d-dimensional Euclidean space R^d and C^2 denotes the class of functions in R^d which are twice continuously differentiable with respect to x.

Lemma 9. *The Markov process $X(t)$ has a unique stationary distribution $\pi(\cdot)$, if there exists a bounded domain $U \subset R^d$ with regular boundary Γ such that its closure $\overline{U} \subset R^d$, having the following properties:*

(B.1) In the open domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(t)$ is bounded away from zero.

(B.2) If $x \in R^d \backslash U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E^x\tau < \infty$ for every compact subset $K \subset R^d$. Moreover, if $f(\cdot)$ is a function integrable with respect to the measure π, then

$$P\left(\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X^x(t))dt = \int_{R^d} f(x)\pi(dx)\right) = 1, \quad \forall x \in R^d. \tag{26}$$

Remark 10. To prove condition (B.1), it suffices to verify that F is uniformly elliptical in U, where $Fu = f(x)u_x + (1/2)Tr\left(A(x)u_{xx}\right)$, *i.e.*, there exists a positive number $\overline{M}$ *such that* $\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j > \overline{M}|\xi|^2, x \in U, \xi \in R^d$.

To verify condition (B.2), it is sufficient to prove that exists a nonnegative C^2–function $V(x)$ and a neighborhood U such that for some positive constant M, $LV(x) \le -M$ for any $x \in R^d \backslash U$.

System (2) can be written into the following form:

$$d\begin{pmatrix} p(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} p\left(r - \dfrac{r}{k}p - \dfrac{\alpha z}{\gamma + p}\right) \\ z\left(\dfrac{\beta p}{\gamma + p} - \mu - \dfrac{\theta p}{\gamma + p}\right) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 p(t) \\ 0 \end{pmatrix} dB_1(t) + \begin{pmatrix} 0 \\ \sigma_2 z(t) \end{pmatrix} dB_2(t). \quad (27)$$

Suppose the following condition (H) holds:

$$(H)\lambda \triangleq \frac{\beta}{r(1 + (\gamma/k))}\left(r - \frac{\sigma_1^2}{2}\right) - \mu - \frac{\sigma_2^2}{2} - \theta > 0. \quad (28)$$

Theorem 11. *Assume the condition (H) holds and $\sigma_i^2 > 0 (i = 1, 2)$, then for any initial value $(p_0, z_0) \in R_+^2$, system (2) has a single stationary distribution and it has ergodic property.*

Proof. By Theorem 2, we have obtained that for any initial value $(p_0, z_0) \in R_+^2$, the system (2) has a single global positive solution $(p(t), z(t)) \in R_+^2$.

The following proof is motivated by Liu et al. [38], if we substitute $p(t) = e^{\xi(t)}, z(t) = e^{\eta(t)}$ into system (2) and then by Itô's formula, we get

$$\begin{cases} d\xi(t) = \left(r - \frac{\sigma_1^2}{2} - \frac{r}{k}e^{\xi(t)} - \alpha\frac{e^{\eta(t)}}{\gamma + e^{\xi(t)}}\right)dt + \sigma_1 dB_1(t), \\ d\eta(t) = \left(-\mu - \frac{\sigma_2^2}{2} + \beta\frac{e^{\xi(t)}}{\gamma + e^{\xi(t)}} - \theta\frac{e^{\xi(t)}}{\gamma + e^{\xi(t)}}\right)dt + \sigma_2 dB_2(t). \end{cases} \quad (29)$$

Define a C^2-function $V : R^2 \to R_+$

$$\begin{aligned} V(\xi, \eta) = M\Bigg\{ &-\eta(t) + \frac{\beta}{r(1 + (\gamma/k))}\Bigg[\ln\left(\gamma + e^{\xi(t)}\right) - \xi(t) \\ &+ \frac{(\alpha\beta/\gamma) + \theta(r + (r/k))}{\mu\beta}e^{\eta(t)}\Bigg]\Bigg\} \\ &+ \frac{\left(e^{\xi(t)} + me^{\eta(t)}\right)^{q+1}}{q + 1} = V_1 + V_2, \end{aligned} \quad (30)$$

where $m = \alpha/\beta$, q is a constant satisfying $0 < q < \min\{1, (2\mu/\sigma_2^2)\}$, $M = (2/\lambda)\max\{2, \sup_{(\xi,\eta)\in R^2}(-(r/4k)e^{(2+q)\xi} - (m^{1+q}(\mu - (\sigma_2^2 q/2)/4)e^{(1+q)\eta} + N))\}$, $N = \sup_{(\xi,\eta)\in R^2}\{-(r/2k)e^{(2+q)\xi} - (m^{1+q}(\mu - (\sigma_2^2 q)/2)/2)e^{(1+q)\eta} + r(e^\xi + me^\eta)e^\xi + (\sigma_1^2 q/2)e^{(1+q)\xi}\}$. Clearly, $(M\lambda/4) \ge 1$.

An application of the operator to $-\eta(t)$ and $\ln(\gamma + e^{\xi(t)}) - \xi(t) + ((\alpha\beta/\gamma) + \theta(r + (r/k))/\mu\beta)e^{\eta(t)}$ leads to

$$\begin{aligned} L(-\eta) &= \mu + \frac{\sigma_2^2}{2} + \theta\frac{e^\xi}{\gamma + e^\xi} - \beta\frac{e^\xi}{\gamma + e^\xi} \\ &= \mu + \frac{\sigma_2^2}{2} + \theta\frac{e^\xi}{\gamma + e^\xi} - \beta\frac{e^\xi}{\gamma + e^\xi} + \frac{\beta}{1 + (\gamma/k)} - \frac{\beta}{1 + (\gamma/k)} \\ &\le \mu + \frac{\sigma_2^2}{2} + \theta - \frac{\beta}{1 + (\gamma/k)} + \frac{\beta}{r(1 + (\gamma/k))} \cdot \frac{\gamma(r - (r/k)e^\xi)}{\gamma + e^\xi}, \end{aligned} \quad (31)$$

and

$$\begin{aligned} &L\left[\ln\left(\gamma + e^\xi\right) - \xi + \frac{(\alpha\beta/\gamma) + \theta(r + (r/k))}{\mu\beta}e^\eta\right] \\ &= \frac{e^\xi}{\gamma + e^\xi}\left(r - \frac{\sigma_1^2}{2} - \frac{r}{k}e^\xi - \alpha\frac{e^\eta}{\gamma + e^\xi}\right) \\ &\quad + \left(-r + \frac{\sigma_1^2}{2} + \frac{r}{k}e^\xi + \alpha\frac{e^\eta}{\gamma + e^\xi}\right) \\ &\quad - \frac{\gamma\sigma_1^2 e^\xi}{2\left(\gamma + e^\xi\right)^2} + \frac{((\alpha\beta/\gamma) + \theta(r + (r/k)))\sigma_2^2}{2\mu\beta}e^\eta \\ &\quad + \frac{(\alpha\beta/\gamma) + \theta(r + (r/k))}{\mu\beta}e^\eta\left(-\mu - \frac{\sigma_2^2}{2} + \beta\frac{e^\xi}{\gamma + e^\xi} - \theta\frac{e^\xi}{\gamma + e^\xi}\right) \\ &\le \left[\frac{e^\xi}{\gamma + e^\xi}\left(r - \frac{r}{k}e^\xi\right) - \frac{\alpha e^{\xi+\eta}}{\left(\gamma + e^\xi\right)^2} - \frac{\gamma\sigma_1^2 e^\xi}{2\left(\gamma + e^\xi\right)^2}\right] \\ &\quad + \left(-r + \frac{\sigma_1^2}{2} + \frac{r}{k}e^\xi + \alpha\frac{e^\eta}{\gamma + e^\xi}\right) \\ &\quad + \frac{(\alpha\beta/\gamma) + \theta(r + (r/k))}{\mu\beta}\left(-\mu e^\eta + \frac{\beta e^{\xi+\eta}}{\gamma + e^\xi}\right) \\ &\le \left[r - \frac{r}{k}e^\xi - \frac{\gamma\left(r - (r/k)e^\xi\right)}{\gamma + e^\xi}\right] + \left(-r + \frac{\sigma_1^2}{2} + \frac{r}{k}e^\xi + \frac{\alpha}{\gamma}e^\eta\right) \\ &\quad + \frac{(\alpha\beta/\gamma) + \theta(r + (r/k))}{\mu\beta}\left(-\mu e^\eta + \frac{\beta}{\gamma}e^{\xi+\eta}\right) \\ &\le \frac{\sigma_1^2}{2} - \frac{\gamma\left(r - (r/k)e^\xi\right)}{\gamma + e^\xi} + \frac{(\alpha\beta/\gamma) + \theta(r + (r/k))}{\gamma\mu} \cdot e^{\xi+\eta}. \end{aligned} \quad (32)$$

Using the above two inequations, LV_1 can be estimated as follows

$$\begin{aligned} LV_1 &\le M\left[\mu + \frac{\sigma_2^2}{2} + \theta - \frac{\beta}{r(1 + (\gamma/k))}\left(r - \frac{\sigma_1^2}{2}\right) + \frac{\beta}{r(1 + (\gamma/k))} \cdot \frac{(\alpha\beta/\gamma) + \theta(r + (r/k))}{\gamma\mu} \cdot e^{\xi+\eta}\right] \\ &= -M\lambda + Mhe^{\xi+\eta}, \end{aligned} \quad (33)$$

where $h = \beta/(r(1 + \gamma/k)) \cdot (\alpha\beta/\gamma + \theta(r + r/k))/\gamma\mu$.

$$\begin{aligned} LV_2 &= \left(e^\xi + me^\eta\right)^q\left[re^\xi - \frac{r}{k}e^{2\xi} - m\mu e^\eta - \frac{m\theta e^{\xi+\eta}}{\gamma + e^\xi} + \frac{(m\beta - \alpha)e^{\xi+\eta}}{\gamma + e^\xi}\right] \\ &\quad + \frac{q\left(e^\xi + me^\eta\right)^{q-1}\left(e^{2\xi}\sigma_1^2 + m^2 e^{2\eta}\sigma_2^2\right)}{2} \\ &\le r\left(e^\xi + me^\eta\right)^q e^\xi - \frac{r}{k}e^{(2+q)\xi} - m^{1+q}\mu e^{(1+q)\eta} \\ &\quad + \frac{q\left[e^{(1+q)\xi}\sigma_1^2 + m^{1+q}e^{(1+q)\eta}\sigma_2^2\right]}{2} \\ &\le r\left(e^\xi + me^\eta\right)^q e^\xi - \frac{r}{k}e^{(2+q)\xi} \\ &\quad - m^{1+q}\left(\mu - \frac{\sigma_2^2 q}{2}\right)e^{(1+q)\eta} + \frac{\sigma_1^2 q}{2}e^{(1+q)\xi} \\ &= -\frac{r}{2k}e^{(2+q)\xi} - \frac{m^{1+q}\left(\mu - \left(\sigma_2^2 q/2\right)\right)}{2}e^{(1+q)\eta} + N, \end{aligned} \quad (34)$$

where $N=\sup_{(\xi,\eta)\in R^2}\{-(\gamma/2k)e^{(2+q)\xi}-(m^{1+q}(\mu-\sigma_2^2q/2)/2)e^{(1+q)\eta}+r(e^{\xi}+me^{\eta})^qe^{\xi}+(\sigma_1^2q/2)e^{(1+q)\xi}\}$. Hence

$$LV=LV_1+LV_2\le -M\lambda+Mhe^{\xi+\eta}-\frac{r}{2k}e^{(2+q)\xi}-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{2}e^{(1+q)\eta}+N. \tag{35}$$

To confirm the condition (B.2) of Lemma 9, we consider the bounded open subset

$$U_\epsilon=\{(\xi,\eta):|\xi|\le\ln\epsilon^{-1},|\eta|\le\ln\epsilon^{-1},(\xi,\eta)\in R^2\}, \tag{36}$$

where $0<\epsilon<1$ is a sufficiently small number. In the set $U^c=R^2\backslash U$, let us choose sufficiently small ϵ such that

$$0<\epsilon<\frac{\lambda}{4h}, \tag{37}$$

$$0<\epsilon<\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{4Mh}, \tag{38}$$

$$0<\epsilon<\frac{r}{4kMh}, \tag{39}$$

$$-M\lambda-\frac{r}{4k\epsilon}+k_1\le -1, \tag{40}$$

$$-M\lambda-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{4\epsilon}+k_2\le -1. \tag{41}$$

For convenience, we divide U^c into four domains,

$$\begin{aligned}U_\epsilon^1&=\{(\xi,\eta)\in R^2:-\infty<\xi\le\ln\epsilon\},\\ U_\epsilon^2&=\{(\xi,\eta)\in R^2:-\infty<\eta\le\ln\epsilon\},\\ U_\epsilon^3&=\{(\xi,\eta)\in R^2:\xi\ge\ln\epsilon^{-1}\},\\ U_\epsilon^4&=\{(\xi,\eta)\in R^2:\eta\ge\ln\epsilon^{-1}\}.\end{aligned} \tag{42}$$

Clearly, $U^c=U_\epsilon^1\cup U_\epsilon^2\cup U_\epsilon^3\cup U_\epsilon^4$. Next, we will prove $LV(\xi,\eta)\le -1$ for any $(\xi,\eta)\in U^c$.

Case 1. In domain U_ϵ^1, owing to $-\infty<\xi\le\ln\epsilon$, then $e^{\xi+\eta}\le\epsilon e^{\eta}\le\epsilon(1+e^{(1+q)\eta})$, we have

$$\begin{aligned}LV(\xi,\eta)\le{}&\frac{-M\lambda}{4}+\left(\frac{-M\lambda}{4}+\epsilon Mh\right)\\&+\left(-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{4}+\epsilon Mh\right)e^{(1+q)\eta}-\frac{r}{4k}e^{(2+q)\xi}\\&+\left(\frac{-M\lambda}{2}-\frac{r}{4k}e^{(2+q)\xi}-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{4}e^{(1+q)\eta}+N\right),\\ \le{}&\frac{-M\lambda}{4}+\left(\frac{-M\lambda}{4}+\epsilon Mh\right)\\&+\left(-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{4}+\epsilon Mh\right)e^{(1+q)\eta}-\frac{r}{4k}e^{(2+q)\xi}\\&+\left[\frac{-M\lambda}{2}+\sup_{(\xi,\eta)\in R^2}\right.\\&\left.\cdot\left(-\frac{r}{4k}e^{(2+q)\xi}-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{4}e^{(1+q)\eta}+N\right)\right],\end{aligned} \tag{43}$$

since $M\lambda/4\ge 1$, combing with the definition of M and Eqs. (37) and (38), we get

$$LV(\xi,\eta)\le\frac{-M\lambda}{4}-\frac{r}{4k}e^{(2+q)\xi}\le\frac{-M\lambda}{4}\le -1. \tag{44}$$

Case 2. Similarly, for any $(\xi,\eta)\in U_\epsilon^2$, owing to $e^{\xi+\eta}\le\epsilon e^{\xi}\le\epsilon(1+e^{(2+q)\xi})$, we have

$$\begin{aligned}LV(\xi,\eta)\le{}&\frac{-M\lambda}{4}+\left(\frac{-M\lambda}{4}+\epsilon Mh\right)\\&+\left(-\frac{r}{4k}+\epsilon Mh\right)e^{(2+q)\xi}-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{4}e^{(1+q)\eta}\\&+\left(\frac{-M\lambda}{2}-\frac{r}{4k}e^{(2+q)\xi}-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{4}e^{(1+q)\eta}+N\right),\\ \le{}&\frac{-M\lambda}{4}+\left(\frac{-M\lambda}{4}+\epsilon Mh\right)\\&+\left(-\frac{r}{4k}+\epsilon Mh\right)e^{(2+q)\xi}-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{4}e^{(1+q)\eta}\\&+\left[\frac{-M\lambda}{2}+\sup_{(\xi,\eta)\in R^2}\left(-\frac{r}{4k}e^{(2+q)\xi}\right.\right.\\&\left.\left.-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{4}e^{(1+q)\eta}+N\right)\right],\end{aligned} \tag{45}$$

by Eqs. (37) and (39), we also obtain

$$LV(\xi,\eta)\le\frac{-M\lambda}{4}-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{4}e^{(1+q)\eta}\le\frac{-M\lambda}{4}\le -1. \tag{46}$$

Case 3. In domain U_ϵ^3 we have

$$\begin{aligned}LV\le{}&-M\lambda+Mhe^{\xi+\eta}-\frac{r}{2k}e^{(2+q)\xi}\\&-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{2}e^{(1+q)\eta}+N\le -M\lambda-\frac{r}{4k\epsilon}+k_1,\end{aligned} \tag{47}$$

where $k_1=\sup_{(\xi,\eta)\in R^2}(-(r/4k)e^{(2+q)\xi}-(m^{1+q}(\mu-\sigma_2^2q/2)/2)e^{(1+q)\eta}+Mhe^{\xi+\eta}+N)$, which gives $LV\le -1$ in view of Eq. (40).

Case 4. In domain U_ϵ^4 we have

$$\begin{aligned}LV\le{}&-M\lambda+Mhe^{\xi+\eta}-\frac{r}{2k}e^{(2+q)\xi}-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{2}e^{(1+q)\eta}\\&+N\le -M\lambda-\frac{m^{1+q}(\mu-(\sigma_2^2q/2))}{4\epsilon}+k_2,\end{aligned} \tag{48}$$

where $k_2=\sup_{(\xi,\eta)\in R^2}(-(m^{1+q}(\mu-\sigma_2^2q/2)/4)e^{(1+q)\eta}-(r/2k)e^{(2+q)\xi}+Mhe^{\xi+\eta}+N)$, which gives $LV\le -1$ according to Eq. (41). Consequently, we can deduce that

$$LV(\xi,\eta)\le -1,\forall(\xi,\eta)\in U^c. \tag{49}$$

That is, the condition (B.2) holds.

On the other hand, there exists $\overline{M}=\min_{(p,z)\in\overline{U}}\{\sigma_1^2p^2,\sigma_2^2z^2\}>0$ such that

$$\sum_{i,j=1}^{2} a_{i,j}(p,z)\epsilon_i\epsilon_j = \sigma_1^2 p^2 \xi_1^2 + \sigma_2^2 z^2 \xi_2^2 \geq \overline{M}|\xi|^2, \quad \forall (p,z) \in \overline{U}, \xi = (\xi_1, \xi_2) \in R^2. \tag{50}$$

That is, the condition (B.1) is satisfied. Therefore, according to Lemma 9, we know that the system (2) has a single stationary distribution which is ergodic.

Remark 12. From Theorem 11, we can see that system (2) exists a single stationary distribution provided that the effects of both environment noise $\sigma_i (i = 1, 2)$ and the rate of release of toxic substances θ are not too large such that $\lambda > 0$; The ergodic property reflects the solution of system (2) converges to the single stationary distribution.

6. The Existence of Periodic Solution of Nonautonomous System

For convenience, we denote

$$f^u = \max_{t\in[0,\theta_1]} f(t), f^l = \min_{t\in[0,\theta_1]} f(t), \tag{51}$$

where $f(t)$ is a continuous θ_1-periodic function.

In this section, we will recall a basic definition and introduce a lemma which gives a criteria for the existence of a periodic Markov process (see Khasminskii [47]).

Definition 13 [47]. A stochastic process $\xi(t) = \xi(t,\omega)$ $(-\infty < t < +\infty)$ is said to be θ_1-periodic if for every finite sequence of numbers $t_1, t_2, \cdots, t_n$, the joint distribution of random variables $\xi(t_1 + h), \xi(t_2 + h), \cdots, \xi(t_n + h)$ $(h = k\theta_1 (k = 1, 2, \cdots))$ is independent of h.

Consider the integral equation

$$X(t) = X(t_0) + \int_{t_0}^{t} b(s, X(s))ds + \sum_{r=1}^{k} \int_{t_0}^{t} \sigma_r(s, X(s))d\xi_r(s), \tag{52}$$

where $b(s,x), \sigma_i(s,x) (i = 1, 2, \cdots, k)\left(s \in [t_0, T], x \in R^l\right)$ are continuous functions of (s,x) and for some constant B, the following conditions hold.

$$\begin{aligned} &|b(s,x) - b(s,y)| + \sum_{r=1}^{k} |\sigma_r(s,x), \sigma_r(s,y)| \leq B|x-y|, \\ &|b(s,x)| + \sum_{r=1}^{k} |\sigma_r(s,x)| \leq B(1 + |x|). \end{aligned} \tag{53}$$

Let U be a given open set in the d-dimensional Euclidean space R^d. $E = R^d \times [0,\infty)$, $C^{2,1}$ is the family functions on E which are twice continuously differentiable with respect to $x \in R^d$ and continuously differentiable with respect to $t \in [0,\infty)$.

Lemma 14 [47]. *Suppose that the coefficients of Equation (52) are θ_1-periodic in t and satisfy the conditions Equation (53) in every cylinder $U \times I$, and assume further there exists a function $V(x,t) \in C^{2,1}$, which is θ_1-periodic in t and satisfies:*

(B.1) $\inf_{|x|>\mathfrak{R}} V(x,t) \to \infty$ *as* $\mathfrak{R} \to \infty$,

(B.2) $LV(x,t) \leq -1$ *outside some compact set,*

then the system (52) exists at least a θ_1-periodic Markov process.

Theorem 15. *Assume the following conditions hold:*

(H.1) $\lambda_1 = (1/\theta_1)\int_0^{\theta_1}\left[\left(\frac{\beta^l}{r^u(1+(\gamma^u/k^l))}r(s) - \frac{\beta^u}{r^l(1+(\gamma^l/k^u))}\frac{\sigma_1^2(s)}{2}\right) - \left(\mu(s) + \theta(s) + \frac{\sigma_2^2(s)}{2}\right)\right]ds > 0,$

(H.2) $\lambda_2 = (1/\theta_1)\int_0^{\theta_1}\left(\mu(s) - \frac{q(s)\sigma_2^2(s)}{2}\right)ds > 0,$

then for any initial value $(p_0, z_0) \in R_+^2$, the system (3) has a positive θ_1-periodic solution.

Proof. By the same way as in Theorem 2, we can obtain that, for any initial value $(p_0, z_0) \in R_+^2$, the system 3 has a single global positive solution. Next, we only need to verify the conditions (B.1), (B.2) of Lemma 14.

Define a $C^{2,1}$-function $V : R^2 \times [0,\infty) \to R_+$ by

$$\begin{aligned} V(\xi,\eta,t) = {} & M_1\Bigg\{-\eta(t) + \frac{\beta^u}{r^l\left(1 + \left(\gamma^l/k^u\right)\right)}\Bigg[\ln\left(\gamma^u + e^{\xi(t)}\right) - \xi(t) \\ & + \frac{\left(\alpha^u\beta^u/\gamma^l\right) + \theta^u\left(r^u + \left(r^u/k^l\right)\right)}{\mu^l\beta^l}e^{\eta(t)}\Bigg] + \omega_1(t)\Bigg\} \\ & + e^{\omega_2(t)}\frac{\left(e^{\xi(t)} + m^u e^{\eta(t)}\right)^{q+1}}{q+1} \\ = {} & V_1 + V_1, \end{aligned} \tag{54}$$

where $m = \alpha(t)/\beta(t)$, $0 < q$ is a constant, $M_1 = (2/\lambda_1)\max\Big\{2, \sup_{(\xi,\eta)\in R^2}\Big(-(r(t)/4)k(t)e^{|\omega_2^u|}e^{(2+q)\xi} - \left(\lambda_2(m^u)^{1+q}/4\right)e^{|\omega_2^u|}e^{(1+q)\eta} + N_1\Big)\Big\}$, $N_1 = \sup_{(\xi,\eta)\in R^2}\Big\{\left(-(r^l/2)k^u e^{(2+q)\xi}\left(\lambda_2(m^u)^{1+q}/2\right)e^{(1+q)\eta} + r^u\left(e^\xi + m^u e^\eta\right)^q e^\xi + \left(\left(\sigma_1^2\right)^u q/2\right)e^{(1+q)\xi} + e^{(1+q)\xi}K\right)e^{|\omega_2^u|}\Big\}$. Clearly, $M_1\lambda_1/4 \geq 1$. Here, let

$$\begin{aligned} \omega_1'(t) = {} & -\lambda_1 + \left(\frac{\beta^l}{r^u\left(1 + \left(r^u/k^l\right)\right)}r(t) - \frac{\beta^u}{r^l\left(1 + \left(r^l/k^u\right)\right)}\frac{\sigma_1^2(t)}{2}\right) \\ & - \left(\mu(t) + \theta(t) + \frac{\sigma_2^2(t)}{2}\right), \\ \omega_2'(t) = {} & -\lambda_2 + \mu(t) - \frac{q\sigma_2^2(t)}{2}, \end{aligned} \tag{55}$$

which λ_1, λ_2 are defined by $(H.1), (H.2)$. It is not difficult to prove that $\omega_1(t), \omega_2(t)$ are all θ_1-periodic functions, and $\omega_2'(t)$ is a bounded function. That is, there exists $K > 0$ such that

$$|\omega_2'(t)| \leq K, \forall t > 0. \tag{56}$$

In order to confirm the condition (B.1) of Lemma 14, it is sufficient to verify that

$$\inf_{(\xi,\eta,t)\in[0,+\infty)\times R_+^2\setminus U_k} V(\xi,\eta,t) \to \infty, k \to \infty, \tag{57}$$

where $U_k = (1/k, k) \times (1/k, k)$, which is clearly established since the coefficients of the term $e^{(1+q)\xi(t)}, e^{(1+q)\eta(t)}$ of $V(\xi,\eta,t)$ are all positive. Next, we only verify the condition (B.2) of Lemma 14.

By Ito's formula, we have

$$\begin{aligned}
L(-\eta) &= \mu(t) + \frac{\sigma_2^2(t)}{2} + \theta(t)\frac{e^\xi}{\gamma(t)+e^\xi} - \beta(t)\frac{e^\xi}{\gamma(t)+e^\xi} \\
&= \mu(t) + \frac{\sigma_2^2(t)}{2} + \theta(t)\frac{e^\xi}{\gamma(t)+e^\xi} - \beta(t)\frac{e^\xi}{\gamma(t)+e^\xi} \\
&\quad + \frac{\beta(t)}{1+(\gamma(t)/k(t))} - \frac{\beta(t)}{1+(\gamma(t)/k(t))} \\
&\le \mu(t) + \frac{\sigma_2^2(t)}{2} + \theta(t) - \frac{\beta(t)}{1+(\gamma(t)/k(t))} \\
&\quad + \frac{\beta(t)}{r(t)(1+(\gamma(t)/k(t)))}\cdot\frac{\gamma(t)\left(r(t)-(r(t)/k(t))e^\xi\right)}{\gamma(t)+e^\xi},
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
&L\left(\ln\left(\gamma^u+e^\xi\right) - \xi + \frac{\left(\alpha^u\beta^u/\gamma^l\right)+\theta^u\left(r^l+\left(r^l/k^u\right)\right)}{\mu^l\beta^l}e^\eta\right) \\
&= \frac{e^\xi}{\gamma^u+e^\xi}\left(r(t) - \frac{\sigma_1^2(t)}{2} - \frac{r(t)}{k(t)}e^\xi - \alpha(t)\frac{e^\eta}{\gamma(t)+e^\xi}\right) \\
&\quad + \left(-r(t) + \frac{\sigma_1^2(t)}{2} + \frac{r(t)}{k(t)}e^\xi + \alpha(t)\frac{e^\eta}{\gamma(t)+e^\xi}\right) \\
&\quad - \frac{\gamma^u\sigma_1^2(t)e^\xi}{2\left(\gamma^u+e^\xi\right)^2} + \frac{\left(\alpha^u\beta^u/\gamma^l\right)+\theta^u\left(r^u+\left(r^u/k^l\right)\right)}{\mu^l\beta^l}e^{\eta(t)} \\
&\quad \cdot\left(-\mu(t) - \frac{\sigma_2^2(t)}{2} + \beta(t)\frac{e^\xi}{\gamma(t)+e^\xi} - \theta(t)\frac{e^\xi}{\gamma(t)+e^\xi}\right) \\
&\quad + \frac{\left(\left(\alpha^u\beta^u/\gamma^l\right)+\theta^u\left(r^u+\left(r^u/k^l\right)\right)\right)\sigma_2^2(t)}{2\mu^l\beta^l}e^\eta \\
&\le \left[\frac{e^\xi}{\gamma^u+e^\xi}\left(r(t) - \frac{r(t)}{k(t)}e^\xi\right) - \frac{\alpha(t)e^{\xi+\eta}}{\left(\gamma^u+e^\xi\right)^2} - \frac{\gamma^u\sigma_1^2e^\xi}{2\left(\gamma^u+e^\xi\right)^2}\right] \\
&\quad + \left[-r(t) + \frac{\sigma_1^2(t)}{2} + \frac{r(t)}{k(t)}e^\xi + \alpha(t)\frac{e^\eta}{\gamma(t)+e^\xi}\right] \\
&\quad + \frac{\left(\alpha^u\beta^u/\gamma^l\right)+\theta^u\left(r^u+\left(r^u/k^l\right)\right)}{\mu^l\beta^l}\left[-\mu(t)e^\eta + \frac{\beta(t)e^{\xi+\eta}}{\gamma(t)+e^\xi}\right] \\
&\le \left[r(t) - \frac{r(t)}{k(t)}e^\xi - \frac{\gamma(t)\left(r(t)-(r(t)/k(t))e^\xi\right)}{\gamma(t)+e^\xi}\right] \\
&\quad + \left[-r(t) + \frac{\sigma_1^2(t)}{2} + \frac{r(t)}{k(t)}e^\xi + \frac{\alpha(t)}{\gamma(t)}e^\eta\right] \\
&\quad + \frac{\left(\alpha^u\beta^u/\gamma^l\right)+\theta^u\left(r^u+\left(r^u/k^l\right)\right)}{\mu^l\beta^l}\left[-\mu(t)e^\eta + \frac{\beta(t)}{\gamma(t)}e^{\xi+\eta}\right] \\
&\le \frac{\sigma_1^2(t)}{2} - \frac{\gamma(t)\left(r(t)-(r(t)/k(t))e^\xi\right)}{\gamma(t)+e^\xi} - \left(\frac{\alpha^u}{\gamma^l} - \frac{\alpha(t)}{\gamma(t)}\right)e^\eta \\
&\quad + \frac{\left(\alpha^u\left(\beta^u\right)^2/\gamma^l\right)+\beta^u\theta^u\left(r^u+\left(r^u/k^l\right)\right)}{\mu^l\beta^l\gamma^l}\cdot e^{\xi+\eta} \\
&\le \frac{\sigma_1^2(t)}{2} - \frac{\gamma(t)\left(r(t)-(r(t)/k(t))e^\xi\right)}{\gamma(t)+e^\xi} \\
&\quad + \frac{\left(\alpha^u\left(\beta^u\right)^2/\gamma^l\right)+\beta^u\theta^u\left(r^u+\left(r^u/k^l\right)\right)}{\mu^l\beta^l\gamma^l}\cdot e^{\xi+\eta}.
\end{aligned} \tag{59}$$

Using the above two inequations, LV_1 can be estimated as follows:

$$\begin{aligned}
LV_1 &\le M_1\left[\mu(t) + \frac{\sigma_2^2(t)}{2} + \theta(t) - \left(\frac{\beta^l}{r^u\left(1+\left(\gamma^u/k^l\right)\right)}r(t) - \frac{\beta^u}{r^l\left(1+\left(\gamma^l/k^u\right)\right)}\frac{\sigma_1^2(t)}{2}\right)\right. \\
&\quad + \frac{\beta^u}{r^l\left(1+\left(\gamma^l/k^u\right)\right)} \\
&\quad \left.\cdot\frac{\left(\alpha^u\left(\beta^u\right)^2/\gamma^l\right)+\beta^u\theta^u\left(r^u+\left(r^u/k^l\right)\right)}{\mu^l\beta^l\gamma^l}\cdot e^{\xi+\eta} + \omega_1'(t)\right] \\
&= -M_1\lambda_1 + M_1h_1e^{\xi+\eta},
\end{aligned} \tag{60}$$

where $h_1 = \beta^u/\left(r^l\left(1+\gamma^l/k^u\right)\right)\cdot\left(\alpha^u\left(\beta^u\right)^2/\gamma^l + \beta^u\theta^u\left(r^u+r^u/k^l\right)\right)/\mu^l\beta^l\gamma^l$, and

$$\begin{aligned}
LV_2 &= e^{\omega_2(t)}\left[\frac{\left(e^\xi+m^ue^\eta\right)^{q+1}}{q+1}\cdot\omega_2'(t) + \left(e^\xi+m^ue^\eta\right)^q\right. \\
&\quad \cdot\left(r(t)e^\xi - \frac{r(t)}{k(t)}e^{2\xi} - m^u\mu(t)e^\eta\right. \\
&\quad \left. - \frac{m^u\theta(t)e^{\xi+\eta}}{\gamma(t)+e^\xi} + \frac{\left(m^u\beta(t)-\alpha(t)\right)e^{\xi+\eta}}{\gamma(t)+e^\xi}\right) \\
&\quad \left. + \frac{q\left(e^\xi+m^ue^\eta\right)^{q-1}\left(e^{2\xi}\sigma_1^2(t)+\left(m^u\right)^2e^{2\eta}\sigma_2^2(t)\right)}{2}\right] \\
&\le e^{\omega_2(t)}\left[\left(e^{(1+q)\xi} + \left(m^u\right)^{1+q}e^{(1+q)\eta}\right)\cdot\omega_2'(t)\right. \\
&\quad + r^u\left(e^\xi+m^ue^\eta\right)^qe^\xi - \frac{r^l}{k^u}e^{(2+q)\xi} - \left(m^u\right)^{1+q}\mu(t)e^{(1+q)\eta} \\
&\quad \left. + \frac{q\left[e^{(1+q)\xi}\sigma_1^2(t) + \left(m^u\right)^{1+q}e^{(1+q)\eta}\sigma_2^2(t)\right]}{2}\right] \\
&\le e^{\omega_2(t)}\left[e^{(1+q)\xi}\cdot\omega_2'(t) + \left(m^u\right)^{1+q}e^{(1+q)\eta}\cdot\omega_2'(t)\right. \\
&\quad + r^u\left(e^\xi+m^ue^\eta\right)^qe^\xi - \frac{r^l}{k^u}e^{(2+q)\xi} - \left(m^u\right)^{1+q}\mu(t)e^{(1+q)\eta} \\
&\quad \left. + \frac{q\left[e^{(1+q)\xi}\sigma_1^2(t) + \left(m^u\right)^{1+q}e^{(1+q)\eta}\sigma_2^2(t)\right]}{2}\right] \\
&\le e^{\left|\omega_2^u\right|}\left[e^{(1+q)\xi}K + r^u\left(e^\xi+m^ue^\eta\right)^qe^\xi - \frac{r^l}{k^u}e^{(2+q)\xi}\right. \\
&\quad \left. + \frac{\left(\sigma_1^2\right)^uq}{2}e^{(1+q)\xi} - \lambda_2\left(m^u\right)^{1+q}e^{(1+q)\eta}\right] \\
&= -e^{\left|\omega_2^u\right|}\left(\frac{r^l}{2k^u}e^{(2+q)\xi} + \frac{\lambda_2\left(m^u\right)^{1+q}}{2}e^{(1+q)\eta}\right) + N_1,
\end{aligned} \tag{61}$$

where $N_1 = \sup_{(\xi,\eta)\in R^2}\left\{\left(-\left(r^l/2k^u\right)e^{(2+q)\xi} - \left(\lambda_2\left(m^u\right)^{1+q}/2\right)e^{(1+q)\eta} + r^u\left(e^\xi+m^ue^\eta\right)^qe^\xi + \left(\left(\sigma_1^2\right)^uq/2\right)e^{(1+q)\xi} + e^{(1+q)\xi}K\right)e^{\left|\omega_2^u\right|}\right\}$.

Hence

$$LV = LV_1 + LV_2 \le -M_1\lambda_1 + M_1 h_1 e^{\xi+\eta} - e^{|\omega_2^u|}\left(\frac{r^l}{2k^u}e^{(2+q)\xi} + \frac{\lambda_2 (m^u)^{1+q}}{2}e^{(1+q)\eta}\right) + N_1. \tag{62}$$

To confirm the condition *(B.2)* of Lemma 14, we choose small enough $\epsilon > 0$ such that

$$0 < \epsilon \le \frac{1}{4h_1}\min\left\{\lambda_1, \frac{\lambda_2 (m^u)^{1+q} e^{|\omega_2^u|}}{M_1}, \frac{r^l e^{|\omega_2^u|}}{k^u M_1}\right\}, \tag{63}$$

$$-M_1\lambda_1 + k_i' + 1 \le e^{|\omega_2^u|}\min\left\{\frac{r^l}{4k^u\epsilon}, \frac{\lambda_2 (m^u)^{1+q}}{4\epsilon}\right\} (i = 1, 2). \tag{64}$$

Consider the bounded open subset

$$U_\epsilon = \left\{(\xi,\eta) : |\xi| \le \ln\epsilon^{-1}, |\eta| \le \ln\epsilon^{-1}, (\xi,\eta) \in R^2\right\}. \tag{65}$$

Denote

$$\begin{aligned} U_\epsilon^1 &= \left\{(\xi,\eta) \in R^2 : -\infty < \xi \le \ln\epsilon\right\}, \\ U_\epsilon^2 &= \left\{(\xi,\eta) \in R^2 : -\infty < \eta \le \ln\epsilon\right\}, \\ U_\epsilon^3 &= \left\{(\xi,\eta) \in R^2 : \xi \ge \ln\epsilon^{-1}\right\}, \\ U_\epsilon^4 &= \left\{(\xi,\eta) \in R^2 : \eta \ge \ln\epsilon^{-1}\right\}. \end{aligned} \tag{66}$$

Clearly, $U^c = U_\epsilon^1 \cup U_\epsilon^2 \cup U_\epsilon^3 \cup U_\epsilon^4$. Next, we will prove $LV \le -1$ for any $(\xi,\eta) \in U^c$, respectively.

Case 1. In domain U_ϵ^1, owing to $-\infty < \xi \le \ln\epsilon$ $e^{\xi+\eta} \le \epsilon e^{\eta} \le \epsilon\left(1 + e^{(1+q)\eta}\right)$, we have

$$\begin{aligned} LV &\le \frac{-M_1\lambda_1}{4} + \left(\frac{-M_1\lambda_1}{4} + \epsilon M_1 h_1\right) + \left(-\frac{\lambda_2 (m^u)^{1+q}}{4}e^{|\omega_2^u|} + \epsilon M_1 h_1\right)e^{(1+q)\eta} - \frac{r^l}{4k^u}e^{|\omega_2^u|}e^{(2+q)\xi} \\ &\quad + \left(\frac{-M_1\lambda_1}{2} - \frac{r^l}{4k^u}e^{|\omega_2^u|}e^{(2+q)\xi} - \frac{\lambda_2 (m^u)^{1+q}}{4}e^{|\omega_2^u|}e^{(1+q)\eta} + N_1\right), \\ &\le \frac{-M_1\lambda_1}{4} + \left(\frac{-M_1\lambda_1}{4} + \epsilon M_1 h_1\right) + \left(-\frac{\lambda_2 (m^u)^{1+q}}{4}e^{|\omega_2^u|} + \epsilon M_1 h_1\right)e^{(1+q)\eta} - \frac{r^l}{4k^u}e^{|\omega_2^u|}e^{(2+q)\xi} \\ &\quad + \left[\frac{-M_1\lambda_1}{2} + \sup_{(\xi,\eta)\in R^2}\left(-\frac{r^l}{4k^u}e^{|\omega_2^u|}e^{(2+q)\xi} - \frac{\lambda_2 (m^u)^{1+q}}{4}e^{|\omega_2^u|}e^{(1+q)\eta} + N_1\right)\right], \end{aligned} \tag{67}$$

since $M_1\lambda_1/4 \ge 1$, combined with the definition of M_1 and Eq. (63), we get

$$LV \le \frac{-M_1\lambda_1}{4} - \frac{r^l}{4k^u}e^{|\omega_2^u|}e^{(2+q)\xi} \le \frac{-M_1\lambda_1}{4} \le -1. \tag{68}$$

Case 2. Similarly, for any $(\xi,\eta) \in U_\epsilon^2$, owing to $e^{\xi+\eta} \le \epsilon e^{\xi} \le \epsilon\left(1 + e^{(2+q)\xi}\right)$, we have

$$\begin{aligned} LV &\le \frac{-M_1\lambda_1}{4} + \left(\frac{-M_1\lambda_1}{4} + \epsilon M_1 h_1\right) + \left(-\frac{r^l}{4k^u}e^{|\omega_2^u|} + \epsilon M_1 h_1\right)e^{(2+q)\xi} - \frac{\lambda_2 (m^u)^{1+q}}{4}e^{|\omega_2^u|}e^{(1+q)\eta} \\ &\quad + \left(\frac{-M_1\lambda_1}{2} - \frac{r^l}{4k^u}e^{|\omega_2^u|}e^{(2+q)\xi} - \frac{\lambda_2 (m^u)^{1+q}}{4}e^{|\omega_2^u|}e^{(1+q)\eta} + N_1\right), \\ &\le \frac{-M_1\lambda_1}{4} + \left(\frac{-M_1\lambda_1}{4} + \epsilon M_1 h_1\right) + \left(\frac{r^l}{4k^u}e^{|\omega_2^u|} + \epsilon M_1 h_1\right)e^{(2+q)\xi} - \frac{\lambda_2 (m^u)^{1+q}}{4}e^{|\omega_2^u|}e^{(1+q)\eta} \\ &\quad + \left[\frac{-M_1\lambda_1}{2} + \sup_{(\xi,\eta)\in R^2}\left(-\frac{r^l}{4k^u}e^{|\omega_2^u|}e^{(2+q)\xi} - \frac{\lambda_2 (m^u)^{1+q}}{4}e^{|\omega_2^u|}e^{(1+q)\eta} + N_1\right)\right], \end{aligned} \tag{69}$$

by Eq. (63), we also obtain

$$LV \le \frac{-M_1\lambda_1}{4} - \frac{\lambda_2 (m^u)^{1+q} e^{|\omega_2^u|}}{4}e^{(1+q)\eta} \le \frac{-M_1\lambda_1}{4} \le -1. \tag{70}$$

Case 3. In domain U_ϵ^3 we have

$$\begin{aligned} LV &\le -M_1\lambda_1 + M_1 h_1 e^{\xi+\eta} - e^{|\omega_2^u|}\left(\frac{r^l}{2k^u}e^{(2+q)\xi} + \frac{\lambda_2 (m^u)^{1+q}}{2}e^{(1+q)\eta}\right) \\ &\quad + N_1 \le -M_1\lambda_1 - \frac{r^l e^{|\omega_2^u|}}{4k^u\epsilon} + k_1', \end{aligned} \tag{71}$$

where $k_1' = \sup_{(\xi,\eta)\in R^2}\left(-\left(r^l e^{|\omega_2^u|}/4k^u\right)e^{(2+q)\xi} - \left(\lambda_2 (m^u)^{1+q} e^{|\omega_2^u|}/2\right) e^{(1+q)\eta} + M_1 h_1 e^{\xi+\eta} + N_1\right)$, which gives $LV \le -1$ in domain U_ϵ^3 in view of Eq. (64).

Case 4. In domain U_ϵ^4 we have

$$\begin{aligned} LV &\le -M_1\lambda_1 + M_1 h_1 e^{\xi+\eta} - e^{|\omega_2^u|}\left(\frac{r^l}{2k^u}e^{(2+q)\xi} + \frac{\lambda_2 (m^u)^{1+q}}{2}e^{(1+q)\eta}\right) \\ &\quad + N_1 \le -M_1\lambda_1 - \frac{\lambda_2 (m^u)^{1+q} e^{|\omega_2^u|}}{4\epsilon} + k_2', \end{aligned} \tag{72}$$

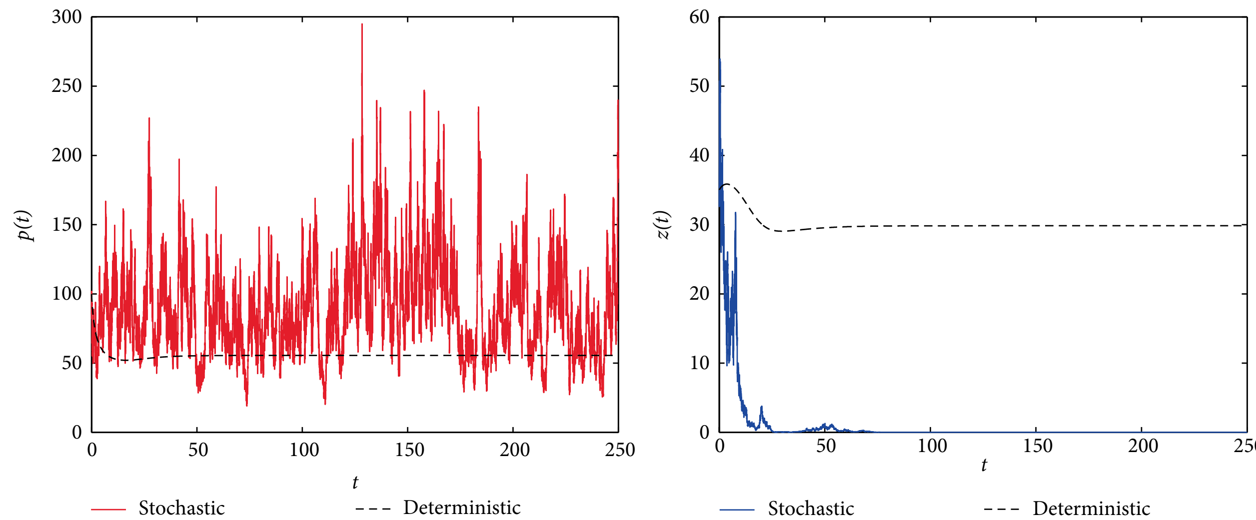

FIGURE 1: The solutions of system (2) and its corresponding deterministic system with $\sigma_1 = 0.5$, $\sigma_2 = 0.5$ and other parameters are taken as Eq. (76).

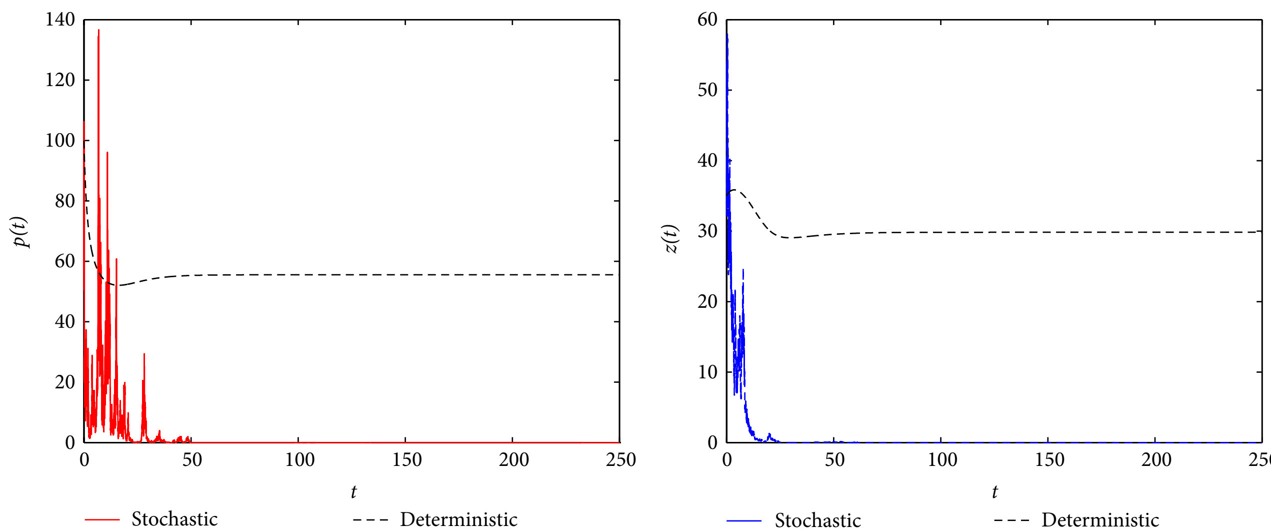

FIGURE 2: The solutions of system (2) and its corresponding deterministic system with $\sigma_1 = 1.5$, $\sigma_2 = 0.5$ and other parameters are taken as Eq. (76).

where $k_2' = \sup_{(\xi,\eta)\in R^2}\Big(-\big(\lambda_2(m^u)^{1+q}e^{|\omega_2^u|}/4\big)e^{(1+q)\eta} - \big(r^l e^{|\omega_2^u|}/2k^u\big) e^{(2+q)\xi} + M_1 h_1 e^{\xi+\eta} + N_1\Big)$, which gives $LV \leq -1$ in domain U_ϵ^4 in view of Eq. (64). Consequently, we can deduce that

$$LV \leq -1, \forall(\xi,\eta) \in U^c. \tag{73}$$

Thus, the condition (B.2) of Lemma 14 is satisfied. By Lemma 14, the system (3) exists a periodic Markov process. This completes the proof.

Considering the corresponding deterministic system of system (3)

$$\begin{cases} \mathrm{d}p = p\left(r(t) - \dfrac{r(t)}{k(t)}p - \alpha(t)\dfrac{z}{\gamma(t)+p}\right)dt, \\ \mathrm{d}z = z\left(\beta(t)\dfrac{p}{\gamma(t)+p} - \mu(t) - \theta(t)\dfrac{p}{\gamma(t)+p}\right)dt. \end{cases} \tag{74}$$

We can obtain the following result according to Theorem 15.

Corollary 16. *Assume* $r(t), k(t), \alpha(t), \beta(t), \gamma(t), \mu(t), \theta(t)$ *are all positive* θ_1*-periodic continuous functions and* $\int_0^{\theta_1}\big[\big(\beta^l/r^u\big(1+\gamma^u/k^l\big)\big)r(s) - \big(\mu(s)+\theta(s)\big)\big]ds > 0$*, then for any initial value* $(p_0, z_0) \in R_+^2$*, the system (74) exists a positive* θ_1*-periodic solution.*

7. Numerical Simulation

In this section, we present some numerical simulations to illustrate our theoretical results obtained in previous sections. To this end, based on the method mentioned in Higham [50], we consider the following discretization equations

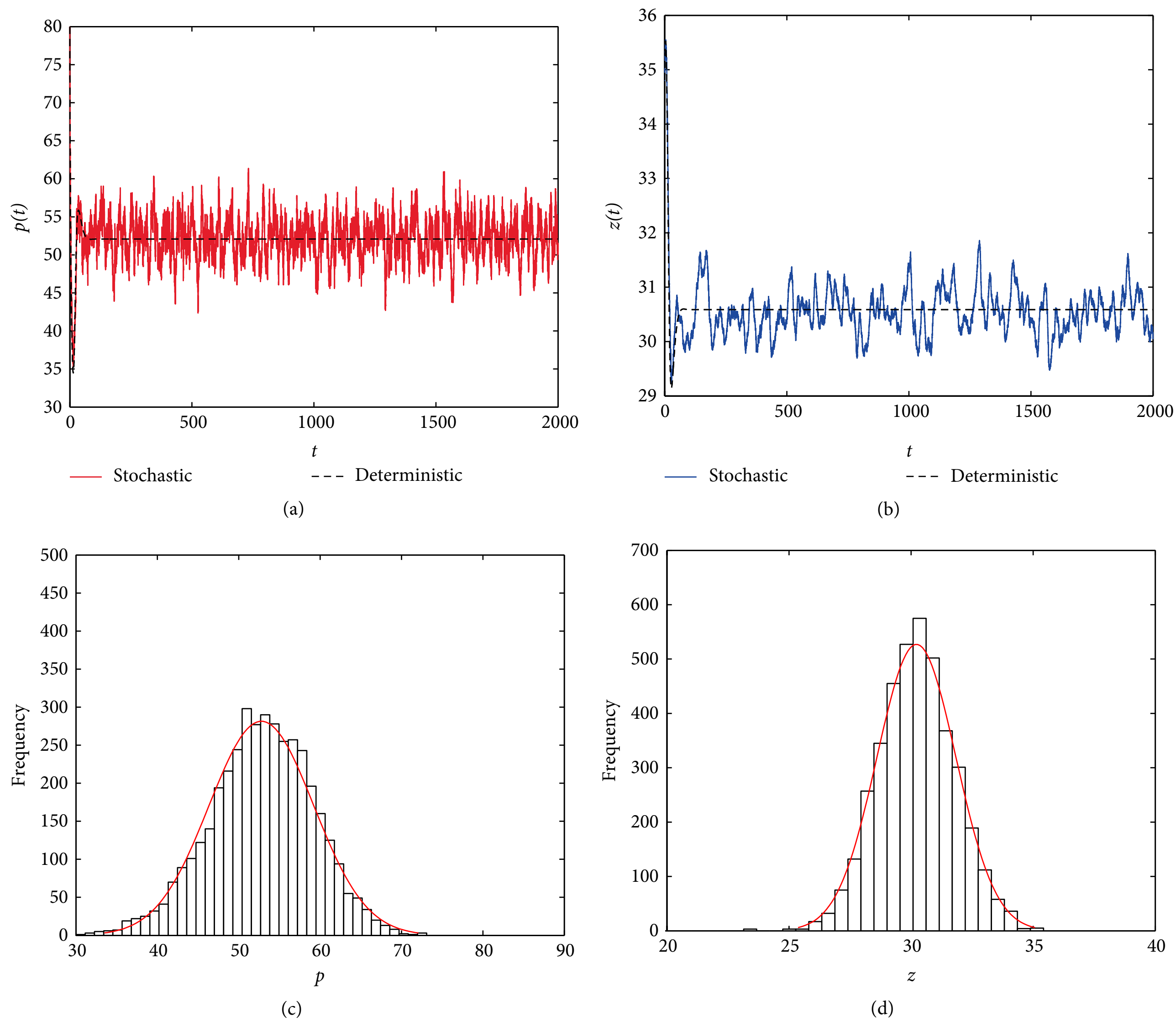

FIGURE 3: (a) and (b) are the solutions of system (2) and their corresponding deterministic system, respectively; (c) and (d) are the density function diagrams of $(p(t), z(t))$, where $\sigma_1 = 0.03, \sigma_2 = 0.0015$ and other parameters are taken as Eq. (76).

$$\begin{cases} p_{k+1} = p_k + p_k\left[r - \dfrac{r}{k}p_k - \alpha\dfrac{z_k}{\gamma + p_k}\right]\Delta t \\ \qquad +\sigma_1 p_k\sqrt{\Delta t}\xi_k + \dfrac{\sigma_1^2}{2}p_k\left(\xi_k^2 - 1\right)\Delta t, \\ z_{k+1} = z_k + z_k\left[\beta\dfrac{p_k}{\gamma + p_k} - \theta\dfrac{p_k}{\gamma + p_k} - \mu\right]\Delta t \\ \qquad +\sigma_2 z_k\sqrt{\Delta t}\eta_k + \dfrac{\sigma_2^2}{2}z_k\left(\eta_k^2 - 1\right)\Delta t, \end{cases} \tag{75}$$

where ξ_k and $\eta_k (k = 1, 2..., n)$ are independent Gaussian random variables which follow standard normal distribution $N(0, 1)$.

We choose the parameters by

$$r = 1, k = 100, \beta = 0.22, \alpha = 1.2, \mu = 0.1, \theta = 0.072, \gamma = 25, \tag{76}$$

with the initial value $(p_0, z_0) = (80, 35)$.

Next we use different values of σ_1, σ_2 to see the effect of the noise strength on the dynamics of the system (2).

(i) Fix $\sigma_2 = 0.5$, and let σ_1 vary to see the effect of noise on the dynamics of the system (2). We firstly take $\sigma_1 = 0.5$, then $\sigma_1^2 < 2r, \sigma_2^2 > 2\beta - 2\mu$, by Theorem 7 (i) the population $p(t)$ will be persistent in mean and population $z(t)$ will be extinct, see Figure 1; Now let σ_1 vary from $\sigma_1 = 0.5$ to $\sigma_1 = 1.5$, then $\sigma_1^2 > 2r, \sigma_2^2 > 2\beta - 2\mu$, by Theorem 7 (ii) the population $p(t)$ and $z(t)$ will be extinct, see Figure 2. From Figure 1 and Figure 2, we can see that under small noise, system (2) preserves some stability and large noise may lead to the extinction of population.

(ii) Let $\sigma_1 = 0.03, \sigma_2 = 0.0015$, then $\lambda \triangleq \beta/r(1 + \gamma/k)\left(r - \sigma_1^2/2\right) - \mu - \sigma_1^2/2 - \theta = 0.00391 > 0$, by Theorem 11, the system (2) exists a single ergodic stationary distribution, see Figure 3. This implies that system

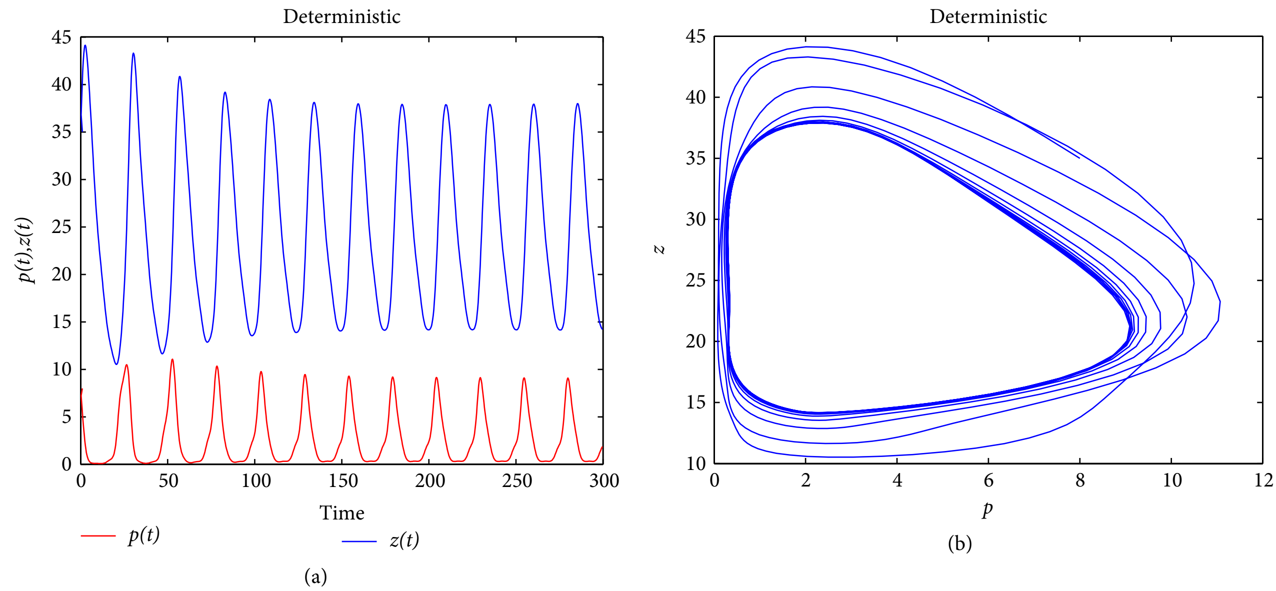

FIGURE 4: The solutions and phase diagram of the deterministic periodic system (74) with $\sigma_1(t) = 0$, $\sigma_2(t) = 0$ and other parameters are taken as Eq. (77).

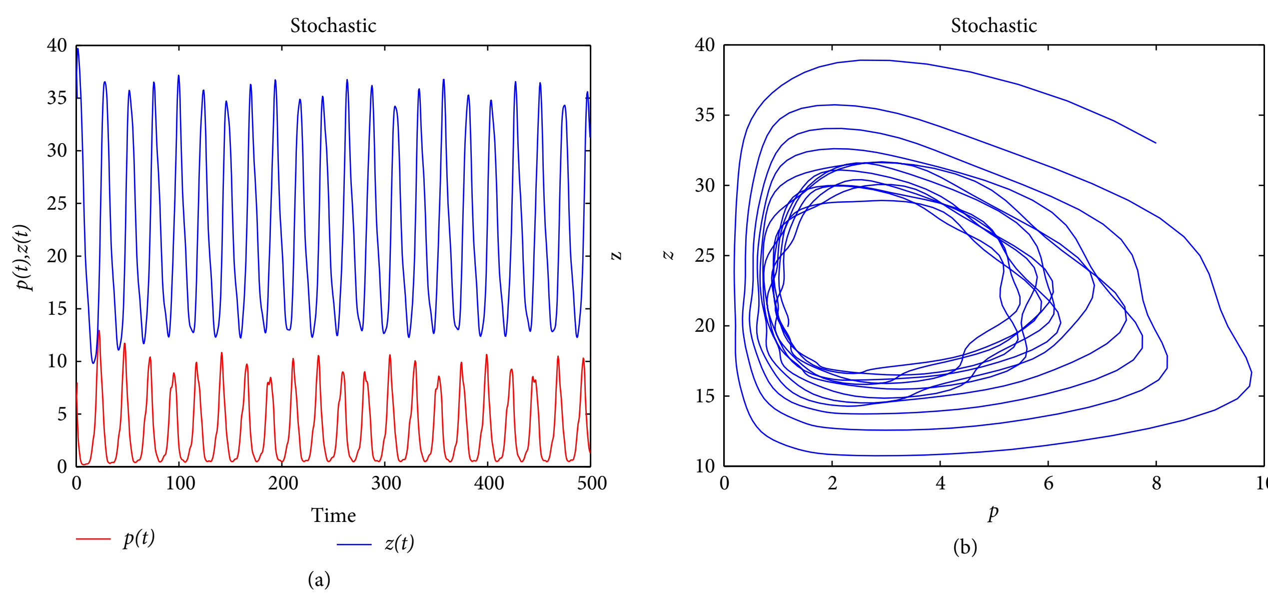

FIGURE 5: The solutions and phase diagram of the stochastic periodic system (3) with $\sigma_1(t) = 0.2 + 0.5\sin t$, $\sigma_2(t) = 0.12 + 0.1\sin t$ and other parameters are taken as Eq. (77).

(2) still keep some stability and exhibits oscillation around the interior equilibrium of the corresponding deterministic system.

Next, we choose the 2π-periodic functions

$$\begin{aligned} r(t) &= 1 + 0.3 \sin t, \\ k(t) &= 23 + 0.1 \sin t, \\ \beta(t) &= 1.2 + 0.05 \sin t, \\ \alpha(t) &= 1.6 + 0.1 \sin t, \\ \mu(t) &= 0.1 + 0.01 \sin t, \\ \gamma(t) &= 20 + 0.1 \sin t, \end{aligned} \tag{77}$$

with the initial value $(p_0, z_0) = (8, 33)$. Then we use different values of $\theta, \sigma_1, \sigma_2$ to explore how the environmental noise and toxicity affect the dynamics of system (3).

(iii) We firstly adopt $\theta(t) = 0.25 + 0.005\sin t$ and let $\sigma_1(t) = \sigma_2(t) = 0$. It is easy to verify that the condition of Corollary 16 holds and deterministic nonautonomous system (74) also exists a periodic solution, shown in the Figure 4(a), and the corresponding solutions of system (74) shown in the Figure 4(b). Next, we increase strengths of environmental forcing to $\sigma_1(t) = 0.2 + 0.5 \sin t, \sigma_2(t) = 0.12 + 0.1 \sin t$, then

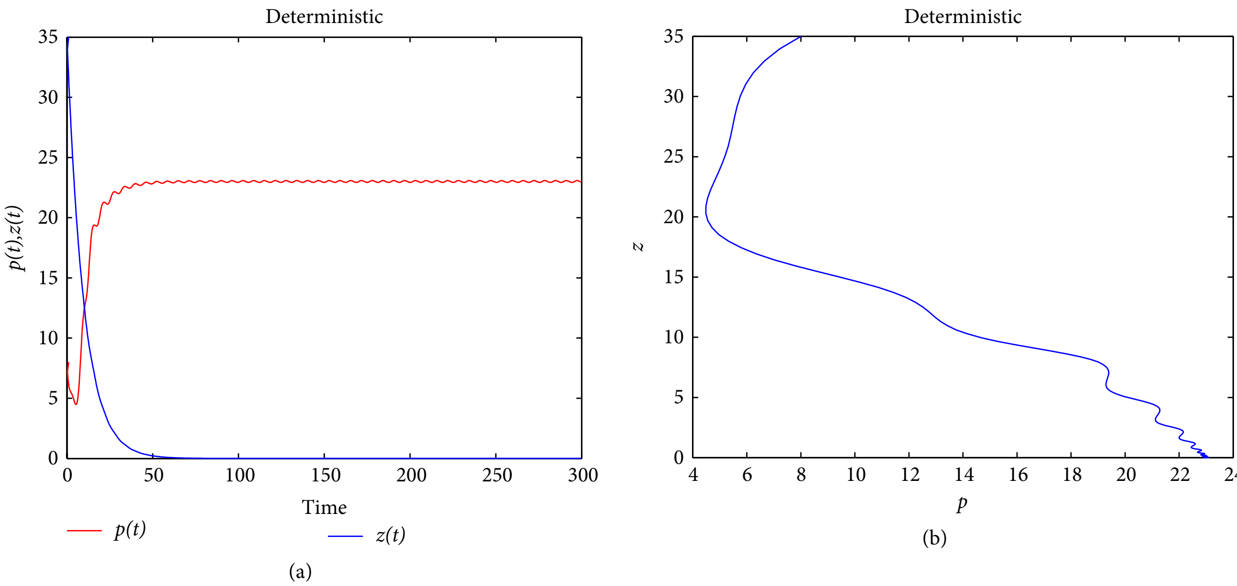

Figure 6: The solutions and phase diagram of the deterministic periodic system (72) with $\theta(t) = 1.2 + 0.005\sin t$, $\sigma_1(t) = 0, \sigma_2(t) = 0$ and other parameters are taken as Eq. (13).

the conditions (H.1), (H.2) hold. By Theorem 15, we know the system (3) exists a positive 2π-periodic Markov process, shown in the Figure 5(a), and the corresponding solutions of system (3) shown in the Figure 5(b).

(iv) Now, we select $\theta(t) = 1.2 + 0.005\sin t, \sigma_1(t) = \sigma_2(t) = 0$, then the population $z(t)$ will be extinct, shown in Figure 6. By comparing Figure 4 with Figure 6, one can observe that the stronger toxicity will result in the extinction of zooplankton population.

8. Conclusion

The field studies and laboratory studies point out that the toxic substance plays one of the important role on the growth of the zooplankton population and have a great impact on phytoplankton–zooplankton intersection. It is well established that large number of phytoplankton species produce toxic, such as *Gymnodinium breve*, *Gymodinium catenatum*, *Pyrodinium bahamense*, *Pfiesteria piscicida*, *Chrysochromulina polylepis*, *Prymnesium patelliferum*, *Noctiluca scintillans* and so on. Zooplankton species such as Paracalanus will be greatly affected by harmful phytoplankton species [1–5]. In this paper, we propose autonomous and nonautonomous stochastic toxin-producing phytoplankton–zooplankton model with Holling II functional response and made an attempt to reveal the effects of toxic intensity and environmental fluctuations on the plankton ecosystems. For the autonomous system, we establish sufficient conditions for the existence of the globally positive solution, and obtain sufficient conditions for the solution of population extinction and persistence in the mean. Furthermore, by using Khasminskii's method and technique of Lyapunov functions, we also prove that there exists a single ergodic stationary distribution, what is important is that Lyapunov function does not depend on existence and stability of equilibrium. For the nonautonomous periodic system, we mainly study the existence of positive periodic solution.

The theoretical results and numerical simulations show that the population would extinct as the enhancing of noise and/or toxic intensity, while the reductive speed of the population would slow down as the weakening of noise and/or toxic intensity, and the population would be persistent. That is, toxic intensity and environmental fluctuations have great influence on plankton ecosystems. From Theorems 7 and 11, we know that the survival of plankton can be significantly affected by the white noise densities $\sigma_i (i = 1, 2)$ and the release rate of toxic substances θ. That is, if σ_1 is sufficiently large, the phytoplankton $p(t)$ may suffer the danger of extinction. On the contrary, if the intensities of environment noise $\sigma_i (i = 1, 2)$ and the rate of release of toxic substances θ are not too large such that $\lambda > 0$, then the system (2) exists a single ergodic stationary distribution. The obtained results also implies that the TPP may provide a possible biological strategy to control the plankton ecosystems.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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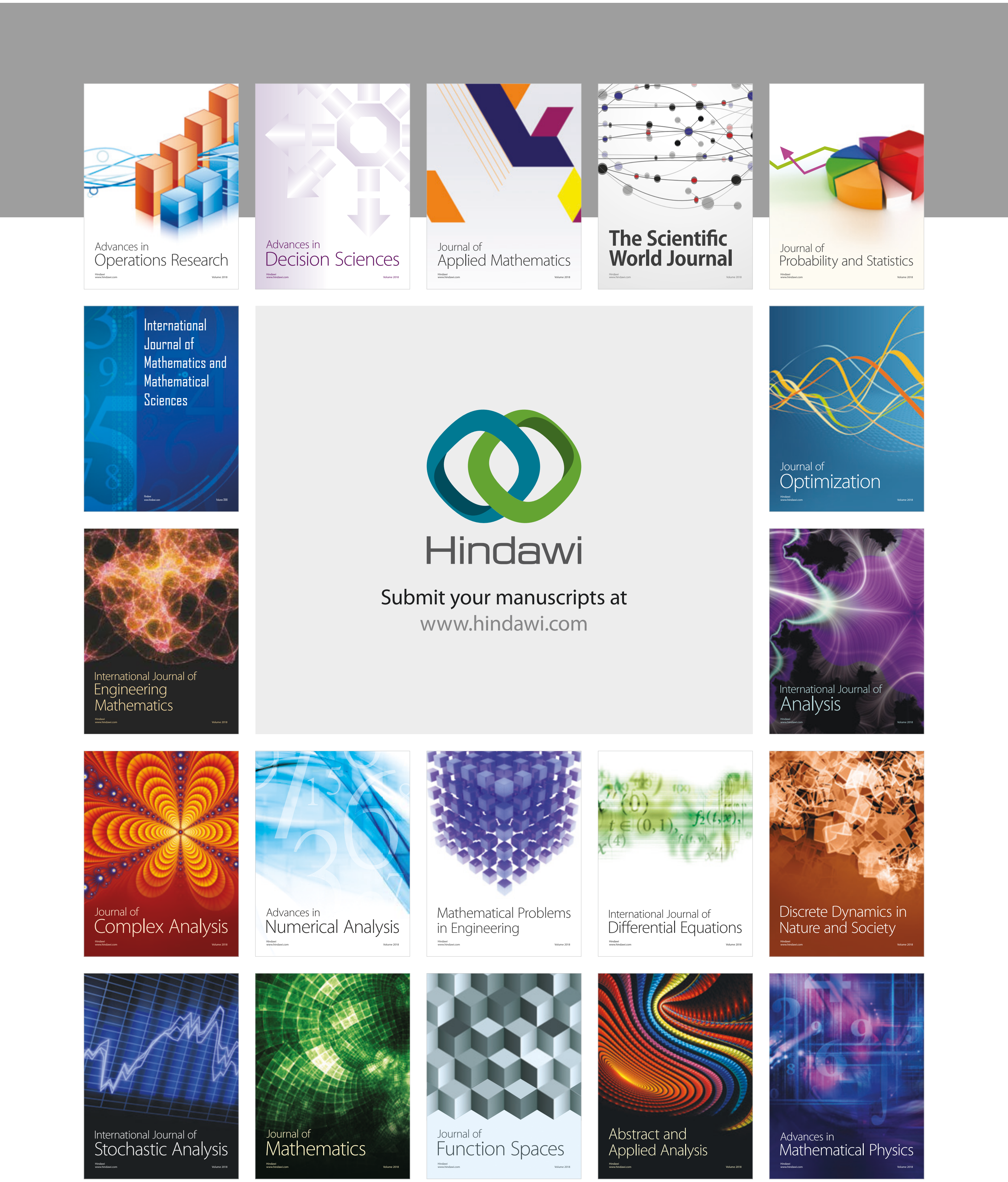
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