

Research Article

Synchronization Analysis of Complex Dynamical Networks Subject to *Delayed Impulsive Disturbances*

Mingyue Li,¹ Huanzhen Chen ,¹ and Xiaodi Li ^{1,2}

¹School of Mathematics and Statistics, Shandong Normal University, Ji'nan 250014, China

²Shandong Province Key Laboratory of Medical Physics and Image Processing Technology, Shandong Normal University, Ji'nan 250014, China

Correspondence should be addressed to Xiaodi Li; sodymath@163.com

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This paper studies the problem of leader-following synchronization for complex networks subject to delayed impulsive disturbances, where two kinds of time delays considered exist in internal complex networks and impulsive disturbances. Some delay-dependent sufficient criteria are derived in terms of linear matrix inequalities (LMIs) by using the delayed impulsive differential inequality method. Moreover, a feedback controller is designed to realize desired synchronization via the established LMIs. Our proposed results show that the requirements of impulse intervals and impulse sizes are dropped, and delayed impulses and large scale impulses are allowed to coexist. Finally, some examples are given to show the effectiveness of the obtained results.

1. Introduction

Complex networks are composed of a large number of highly interconnected dynamical units and are used to describe various practical systems, such as social interacting species, transportation networks, biological and chemical systems, and neural networks [1–8]. As a type of complex networks, coupled neural networks have received great attention and a lot of previous studies mainly focused on stability and stabilization analysis [9–13]. Since synchronization is a widespread collective behavior in nature that has been broadly used in various domains, such as information science, signal processing, and communication system [14–20]. More and more investigators devote to investigating the various types of synchronization of complex networks: phase synchronization [21], complete synchronization [22], and cluster synchronization [23]. In actual operating process, time delays [24] and external disturbances [25] are inevitable due to the inherent communication time between connected nodes and sudden changes in the external environment, which directly affects the efficiency of any process running at the top of the complex networks. Thus, various efforts have been paid for synchronization of delayed complex networks with disturbances [26–29].

In many cases, real complex networks may encounter some abrupt changes at some moments, which are very short relative to the entire motion process. Therefore, the instantaneous change can be regarded as the form of impulse, which is called impulse phenomenon. In the past decades, impulse effect in dynamical systems has received considerable attention [26–35] owing to its widespread existences in nature and society. From the perspective of impulse effect, impulse can be divided into two categories: impulsive control and impulsive disturbance, where impulse control as a discontinuous control input is to achieve desired performance of the networks, whereas impulse disturbance is a robust analysis which means that the networks can maintain its performance under impulse disturbances. Similarly, for the synchronization of complex networks, impulse effects can be divided into two categories: synchronizing impulses and desynchronizing impulses. Synchronizing impulses mean that a complex network without impulses cannot achieve the desired synchronization, but it may possess synchronization via proper impulsive control. While desynchronizing impulses can be regarded as impulsive disturbances, which mean that complex networks without impulses can achieve the synchronization, and it can remain

the desired synchronization under impulsive disturbances. Obviously, the desynchronizing impulses may restrain the synchronizability of complex networks, and it is actually a class of robustness problem. Until now, most of the existing works mainly focus on synchronizing impulses. For example, under the impulsive control, the exponential synchronization of chaotic delayed neural networks could be achieved in [31]. On this basis, Kan et al. [34] investigated the exponential synchronization of master-slave time-varying delayed complex valued neural networks by a hybrid impulsive controller. In recent years, some results about desynchronizing impulses have also been proposed. For instance, Zhang et al. [27] dealt with robust stabilization problem of Takagi–Sugeno fuzzy time delay systems subject to impulsive disturbances. From impulsive disturbance and impulsive control point of view, respectively, Li et al. [29] proposed several criteria ensuring the uniform stability of impulsive functional differential equations with finite or infinite delay by establishing some new Razumikhin conditions.

Impulses involving time delays are sometimes called delayed impulses. Such kind of impulses describes a phenomenon, where impulsive transients depend on not only their current but also historical states of the system. Due to their potential applications in various domains, such as population dynamic systems, communication security systems, and electrical engineering systems, delayed impulses have received much more attention [36–39]. For example, a pinning delayed impulse control scheme was designed to achieve synchronization of complex dynamical networks in [37]. The input-to-state stability and integral input-to-state stability of nonlinear systems with delayed impulses were studied in [38], where the relationship between impulse frequency and the time delay existing in impulses was established. However, there are few results considering synchronization of complex networks under delayed impulsive disturbances. Thus, it is still an open problem and need further studies.

Motivated by the abovementioned discussions, in this paper, we aim to investigate the leader-following synchronization of complex networks subject to delayed impulsive disturbances. By constructing a suitable Lyapunov function, employing the impulsive delay differential inequality and several LMI-based conditions, the synchronization criteria of delayed complex networks which subject to delayed impulse disturbances are obtained. The rest of this paper is organized as follows. In Section 2, some necessary preliminary knowledge is introduced and the problem statement is present for synchronization of complex networks with delayed impulsive disturbances. In Section 3, some sufficient conditions are presented to ensure the leader-following synchronization. Examples are provided in Section 4. Finally, the paper is concluded in Section 5.

2. Preliminaries

Notations 1. Let \mathbb{R} denote the set of real numbers, \mathbb{Z}_+ the set of positive integers, and \mathbb{R}^n and $\mathbb{R}^{n \times m}$ the n -dimensional and $n \times m$ -dimensional real spaces equipped with the Euclidean norm $\|\cdot\|$, respectively. $A > 0$ or $A < 0$ denotes that the matrix

A is a symmetric and positive or negative definite matrix. The notation A^T and A^{-1} denote the transpose and the inverse of A , respectively. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalue of symmetric matrix A , respectively. I denotes the identity matrix with appropriate dimensions and $\Lambda = \{1, 2, \dots, N\}$. Let $\alpha \vee \beta$ and $\alpha \wedge \beta$ be the maximum and minimum value of α and β , respectively. $PC([t_0 - \tau, t_0], \mathbb{R}_+)$ denotes the class of piecewise continuous functions mapping $[t_0 - \tau, t_0]$ to \mathbb{R}_+ . $A \otimes B$ denotes the Kronecker product of matrices A and B .

In this paper, we consider synchronization of a class of complex networks subject to impulse noise disturbances, where the complex network is coupled by a large number of delayed neural networks. The dynamic of the i th neural network is described by

$$\begin{aligned} \dot{x}_i(t) = & -Dx_i(t) + Af(x_i(t)) + Bf(x_i(t) - \tau(t)) \\ & + c \sum_{j=1}^N l_{ij} \Gamma x_j(t) + v_i(t), \quad i \in \Lambda, \end{aligned} \quad (1)$$

where $x_i(t) = (x_{i1}, x_{i2}, \dots, x_{in})^T \in \mathbb{R}^n$ is the state vector of the i th neural network at time t ; $D = \text{diag}\{d_1, d_2, \dots, d_n\}$ is a diagonal matrix with $d_i > 0$, $i \in \Lambda$, which denotes the rate with which the neuron i resets its potential to the resting when being isolated from other neurons and inputs; $f(x(\cdot)) = (f_1(x_1(\cdot)), f_2(x_2(\cdot)), \dots, f_n(x_n(\cdot)))^T$ is the activation function at time t ; $\tau(t)$ is the time-varying delay in the continuous protocol satisfying $0 \leq \tau(t) \leq \tau$ ($\tau > 0$ is a constant); and c is the coupling strength, which denotes a measure of the interdependence of the networks. $\Gamma = \text{diag}\{\gamma_1, \gamma_2, \dots, \gamma_n\} > 0$ represents the current state inner coupling matrix; $L = (l_{ij})_{n \times n}$ is the coupling configuration matrix representing the topological structure of the networks, which is defined as follows: if there is an edge from node j and i at time t , then $l_{ij} > 0$, otherwise $l_{ij} = 0$; the diagonal elements are defined as $l_{ii} = -\sum_{j=1, j \neq i}^N l_{ij}$; $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ are the connection weight matrix and the delayed connection matrix, respectively; and v_i is the actual controller.

The leader to complex dynamical network (1) is described by

$$\dot{s}(t) = -Ds(t) + Af(s(t)) + Bf(s(t) - \tau(t)), \quad (2)$$

where $s(t)$ may be a periodic orbit, a chaotic orbit, or an equilibrium point. Let $e_i(t) = x_i(t) - s(t)$. Then, we have

$$\begin{aligned} \dot{e}_i(t) = & -De_i(t) + Ag(e_i(t)) + Bg(e_i(t) - \tau(t)) \\ & + c \sum_{j=1}^N b_{ij} \Gamma e_j(t) + v_i, \quad i \in \Lambda, \end{aligned} \quad (3)$$

where $g(e_i(\cdot)) = f(e_i(\cdot) + s(\cdot)) - f(s(\cdot))$.

In real applications, the control input is usually affected by switching phenomena, frequency change or other sudden noise, which appears in the form of impulses. Therefore, it is natural to consider the actual control with impulse noise disturbances. The corresponding synchronization framework of neural network subjects to impulse noise disturbances is shown in Figure 1. In Figure 1, $u_i(t)$ is the feedback controller

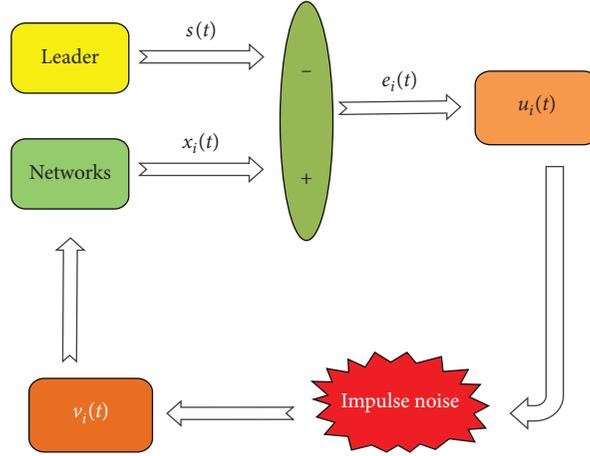


FIGURE 1: The synchronization framework with impulse noise disturbances.

and $v_i(t)$ is the actual controller including impulsive disturbances. We consider the controller $u_i(t)$ given by

$$u_i(t) = Ke_i(t), \quad i \in \Lambda, \quad (4)$$

where $K \in \mathbb{R}^{n \times n}$ is the control gain matrix to be determined. The actual control v_i is described by

$$v_i = u_i + \sum_{k=1}^{\infty} [G_k u_i(t - r_k) - e_i(t^-)] \delta(t - t_k), \quad (5)$$

where the second term on the right represents the delay impulsive disturbances, the impulse time instants $\{t_k\}$ satisfy $0 \leq t_0 < t_1 < \dots < t_k \rightarrow +\infty$ as $k \rightarrow +\infty$; r_k is the delay in impulsive disturbances and satisfies $0 \leq r_k \leq r$ ($r > 0$ is a constant); $\delta(\cdot)$ is the Dirac delta function; and $G_k \in \mathbb{R}^{n \times n}$ is the impulse matrix.

Then, the error system (3) can be expressed by

$$\begin{cases} \dot{e}_i(t) = (K - D)e_i(t) + Ag(e_i(t)) + Bg(e_i(t - \tau(t))) + c \sum_{j=1}^N l_{ij} \Gamma e_j(t), & t \neq t_k, \\ e_i(t_k) = G_k u_i(t_k^- - r_k), & t = t_k, i \in \Lambda. \end{cases} \quad (6)$$

The initial conditions of (6) are given by

$$e_i(s) = \phi_i(s), \quad t_0 - \zeta \leq s \leq t_0, \quad (7)$$

where $\phi_i \in PC([t_0 - \zeta, t_0], \mathbb{R}_+)$, $i \in \Lambda$, $\zeta = r \vee \tau$.

Assumption 1. For the nonlinear function $f(z) = (f_1(z), f_2(z), \dots, f_n(z))^T$, there exist nonnegative constants α_{ij} ($i, j = 1, 2, \dots, n$), for any $z_1 = (z_{11}, z_{12}, \dots, z_{1n})^T$, $z_2 = (z_{21}, z_{22}, \dots, z_{2n})^T \in \mathbb{R}^n$, such that

$$|f_i(z_{11}, \dots, z_{1n}) - f_i(z_{21}, \dots, z_{2n})| \leq \sum_{j=1}^n \alpha_{ij} |z_{1j} - z_{2j}|. \quad (8)$$

Definition 1. The neural network (1) is said to achieve synchronization with system (2) if for any $x_i(t_0), s(t_0) \in \mathbb{R}^n$, the error signal $e_i(t) = x_i(t) - s(t)$ converges to zero as $t \rightarrow \infty$, $i \in \Lambda$.

Lemma 1 (see [30]). Let p, q, r_k , and τ denote nonnegative constants with $p > q$, $a_k \in \mathbb{R}$, $b_k \in \mathbb{R}$, and function $f \in PC(\mathbb{R}, \mathbb{R}_+)$ satisfies the scalar impulsive differential inequality:

$$\begin{cases} D^+ f(t) \leq -pf(t) + q \sup_{t-\tau \leq s \leq t} f(s), & t \neq t_k, t \geq t_0, \\ f(t_k) \leq a_k f(t_k^-) + b_k f(t_k - r_k), & k \in \mathbb{Z}_+. \end{cases} \quad (9)$$

If there exist constants $M > 0$, $\eta \in (0, \lambda)$ such that

$$\prod_{k=1}^n \max\{1, a_k + b_k e^{\lambda r_k}\} \leq M e^{\eta(t_n - t_0)}, \quad n \in \mathbb{Z}_+, \quad (10)$$

where $\lambda > 0$ satisfies $\lambda < p - qe^{\lambda \tau}$. Then,

$$f(t) \leq M \bar{f}(t_0) e^{-(\lambda - \eta)(t - t_0)}, \quad t \geq t_0, \quad (11)$$

where $\bar{f}(t_0) = \sup_{t_0 - \zeta \leq s \leq t_0} f(s)$.

3. Main Results

In this section, some sufficient conditions for synchronization of delayed neural networks subject to delayed impulse noise disturbances are established.

Theorem 1. *Suppose that Assumption 1 holds. Systems (1) and (2) are exponentially synchronized if there exist $n \times n$ real matrix $W > 0$, $n \times n$ diagonal matrixes $Q_1, Q_2 > 0$, $n \times n$ matrix Ψ , positive constants $M, \eta, \lambda, \alpha, \beta$ with $\alpha > \beta, \eta \in (0, \lambda)$, and $b_k \in \mathbb{R}$, such that*

$$\begin{pmatrix} \Phi & I_N \otimes AQ_1 & I_N \otimes BQ_2 & I_N \otimes W\Pi^T \\ * & -I_N \otimes Q_1 & 0 & 0 \\ * & * & -I_N \otimes Q_2 & 0 \\ * & * & * & -I_N \otimes Q_1 \end{pmatrix} < 0, \quad (12)$$

$$\begin{pmatrix} -\beta W & W\Pi^T \\ * & -Q_2 \end{pmatrix} < 0, \quad (13)$$

$$\begin{pmatrix} -b_k W & \Psi^T G_k^T \\ * & -W \end{pmatrix} < 0, \quad (14)$$

$$\prod_{k=1}^n \max\{1, b_k e^{\lambda r_k}\} \leq M e^{\eta(t_n - t_0)}, \quad n \in \mathbb{Z}_+, \quad (15)$$

where $\Phi = I_N \otimes (\Psi + \Psi^T - DW - WD^T + \alpha W) + 2c(L \otimes \Gamma W)$, and λ satisfies $\lambda < \alpha - \beta e^{\lambda \tau}$. Moreover, the control gain matrix K is designed by $K = \Psi W^{-1}$.

Proof. Consider the Lyapunov function as

$$V(t) = \sum_{i=1}^N e_i^T(t) W^{-1} e_i(t). \quad (16)$$

When $t \in [t_{k-1}, t_k], k \in \mathbb{Z}_+$, calculating the derivative of $V(t)$ along the trajectory of system (6), one may obtain that

$$\begin{aligned} \dot{V}(t) = & 2 \sum_{i=1}^N e_i^T(t) W^{-1} \left((K - D)e_i(t) + Ag(e_i(t)) \right. \\ & \left. + Bg(e_i(t - \tau(t))) + c \sum_{j=1}^N l_{ij} \Gamma e_j(t) \right). \end{aligned} \quad (17)$$

It then follows from Assumption 1 and Lemma 1 that

$$\begin{aligned} 2e_i^T W^{-1} Ag(e_i(t)) & \leq e_i^T(t) W^{-1} AQ_1 A^T W^{-1} e_i(t) \\ & \quad + g^T(e_i(t)) Q_1^{-1} g(e_i(t)) \\ & \leq e_i^T(t) W^{-1} AQ_1 A^T W^{-1} e_i(t) \\ & \quad + e_i^T(t) \Pi^T Q_1^{-1} \Pi e_i(t), \end{aligned} \quad (18)$$

$$\begin{aligned} 2e_i^T W^{-1} Bg(e_i(t - \tau(t))) & \leq e_i^T(t) W^{-1} BQ_2 B^T W^{-1} e_i(t) \\ & \quad + g^T(e_i(t - \tau(t))) Q_2^{-1} g(e_i(t - \tau(t))) \\ & \leq e_i^T(t) W^{-1} BQ_2 B^T W^{-1} e_i(t) \\ & \quad + e_i^T(t - \tau(t)) \Pi^T Q_2^{-1} \Pi e_i(t - \tau(t)). \end{aligned} \quad (19)$$

Note that $e(t) = (e_1^T(t), e_2^T(t), \dots, e_N^T(t))^T$, then the coupling term can be rewritten as follows:

$$2c \sum_{i=1}^N e_i^T(t) W^{-1} \sum_{j=1}^N l_{ij} \Gamma e_j(t) = 2ce^T(t) (L \otimes W^{-1} \Gamma) e(t). \quad (20)$$

Substituting (18) and (19) into (17), it holds that

$$\begin{aligned} \dot{V}(t) \leq & \sum_{i=1}^N e_i^T(t) (W^{-1} (K - D) + (K - D)^T W^{-1} \\ & + W^{-1} AQ_1 A^T W^{-1} + \Pi^T Q_1^{-1} \Pi + W^{-1} BQ_2 B^T W^{-1}) e_i(t) \\ & + \sum_{i=1}^N e_i^T(t - \tau(t)) \Pi^T Q_2^{-1} \Pi e_i(t - \tau(t)) \\ & + 2ce^T(t) (L \otimes W^{-1} \Gamma) e(t). \end{aligned} \quad (21)$$

It then follows from (12) and (13) that

$$\begin{aligned} \dot{V}(t) \leq & -\alpha \sum_{i=1}^N e_i^T(t) W^{-1} e_i(t) \\ & + \beta \sum_{i=1}^N e_i^T(t - \tau(t)) W^{-1} e_i(t - \tau(t)) \\ \leq & -\alpha V(t) + \beta V(t - \tau(t)). \end{aligned} \quad (22)$$

When $t = t_k$, from (14), we have

$$\begin{aligned} V(t_k^+) & = \sum_{i=1}^N e_i^T(t_k^+) W^{-1} e_i(t_k^+) \\ & = \sum_{i=1}^N e_i^T(t_k^- - r_k) K^T G_k^T W^{-1} G_k K e_i(t_k^- - r_k) \\ & \leq b_k V(t_k^- - r_k). \end{aligned} \quad (23)$$

From (22), (23), and (15), using Lemma 1, we have

$$V(t) \leq M \bar{V}(t_0) e^{-(\lambda - \eta)(t - t_0)}, \quad (24)$$

where $M > 0, \eta > 0, \lambda > 0$, and $\lambda > \eta, \bar{V}(t_0) = \sup_{t_0 - \zeta \leq s \leq t_0} V(s)$. Then,

$$\lambda_{\min}(W^{-1}) \|e(t)\|^2 \leq V(t) \leq \lambda_{\max}(W^{-1}) M \|\phi\|^2 e^{-(\lambda - \eta)(t - t_0)}, \quad (25)$$

which implies that

$$\|e(t)\| \leq \sqrt{\bar{M}} \|\phi\| e^{-(\lambda - \eta)(t - t_0)/2}, \quad (26)$$

where $\tilde{M} = (\lambda_{\max}(W^{-1})/\lambda_{\min}(W^{-1}))M$. It can be concluded that synchronization is achieved. The proof is completed. \square

Corollary 1. Under Theorem 1, systems (1) and (2) are exponentially synchronized if condition (15) is replaced by

$$\prod_{k=1}^n \max\{1, b_k e^{\lambda r_k}\} \leq \infty, \quad k \in \mathbb{Z}_+. \quad (27)$$

Remark 1. Different from the results in [40], in which the problem of Leader-following synchronization was obtained by the impulsive controller, this paper studied the Leader-following synchronization of delayed neural networks from another point of view, that is, the impulsive disturbances involved in actual control input v_i . Actually, besides impulsive control, impulsive disturbance is also inevitable in real applications. Recently, many interesting results on impulsive disturbances have been reported in the literature [27, 29, 41], such as Zhang et al. [27] studied robust stabilization problem of Takagi–Sugeno fuzzy time delay systems

subject to impulsive disturbances, and the stability problem of impulsive functional differential equations with finite or infinite delay was considered in [29]. However, the time delay was considered only in systems, but the possible time delay existing in impulse was not addressed. In addition, the upper and lower bounds of the consecutive impulse intervals were enforced. In this paper, time delays in impulsive disturbances are fully considered, and moreover there are no restrictions on the upper and lower bounds of the consecutive impulse intervals. In addition, our results allow the coexistence of delayed impulse and large scale impulses, which are less conservative than recent research studies.

On the other hand, we will consider impulsive disturbances that are associated only with the controller with delay. Then, we have the actual control of the following form:

$$v_i = u_i + \sum_{k=1}^{\infty} G_k u_i(t - r_k) \delta(t - t_k). \quad (28)$$

Then, the error dynamic can be described by

$$\begin{cases} \dot{e}_i(t) = (K - D)e_i(t) + Ag(e_i(t)) + Bg(e_i(t - \tau(t))) + c \sum_{j=1}^N l_{ij} \Gamma e_j(t), & t \neq t_k, \\ e_i(t_k) = G_k u_i(t_k - r_k) + e_i(t_k^-), & t = t_k, i \in \Lambda. \end{cases} \quad (29)$$

The notations and conditions are the same as those in system (6).

Theorem 2. Suppose that Assumption 1 holds. Systems (1) and (2) are exponentially synchronized if there exist $n \times n$ real matrix $W > 0$, $n \times n$ diagonal matrixes $Q_1, Q_2 > 0$, $n \times n$ matrix Ψ , positive constants M, η, α, β with $\alpha > \beta, \eta \in (0, \lambda)$, and constants $a_k > 1, b_k \in \mathbb{R}$, such that

$$\begin{pmatrix} \Phi & I_N \otimes A Q_1 & I_N \otimes B Q_2 & I_N \otimes W \Pi^T \\ * & -I_N \otimes Q_1 & 0 & 0 \\ * & * & -I_N \otimes Q_2 & 0 \\ * & * & * & -I_N \otimes Q_1 \end{pmatrix} < 0, \quad (30)$$

$$\begin{pmatrix} -\beta W & W \Pi^T \\ * W & -Q_2 \end{pmatrix} < 0, \quad (31)$$

$$\begin{pmatrix} W - a_k W & G_k \Psi & 0 \\ * & -b_k W & \Psi^T G_k^T \\ * & * & -W \end{pmatrix} < 0, \quad (32)$$

$$\prod_{k=1}^n \max\{1, a_k + b_k e^{\lambda r_k}\} \leq M e^{\eta(t_n - t_0)}, \quad n \in \mathbb{Z}_+, \quad (33)$$

where $\Phi = I_N \otimes (\Psi + \Psi^T - DW - WD^T + \alpha W) + 2c(L \otimes \Gamma W)$ and λ satisfies $\lambda < \alpha - \beta e^{\lambda \tau}$. Moreover, the control gain matrix K is designed by $K = \Psi W^{-1}$.

Proof. Following from Theorem 1, the proof at continuous time is omitted here. When $t = t_k$, from (33), we have

$$\begin{aligned} V(t_k^+) &= \sum_{i=1}^N e_i^T(t_k^+) W^{-1} e_i(t_k^+) \\ &= \sum_{i=1}^N [e_i(t_k^-) + G_k K e_i(t_k - r_k)]^T W^{-1} \\ &\quad \cdot [e_i(t_k^-) + G_k K e_i(t_k - r_k)] \\ &= \sum_{i=1}^N [e_i^T(t_k^-), e_i^T(t_k - r_k)] \\ &\quad \cdot \begin{bmatrix} W^{-1} & W^{-1} G_k K \\ * & K^T G_k^T W^{-1} G_k K \end{bmatrix} \begin{bmatrix} e_i(t_k^-) \\ e_i(t_k - r_k) \end{bmatrix} \\ &\leq a_k V(t_k^-) + b_k V(t_k - r_k). \end{aligned} \quad (34)$$

From Lemma 1, it can be concluded that synchronization of system (1) is achieved. The proof is completed. \square

Corollary 2. Under Theorem 2, systems (1) and (2) are exponentially synchronized if condition (35) is replaced by

$$\prod_{k=1}^n \max\{1, a_k + b_k e^{\lambda r_k}\} \leq \infty, \quad k \in \mathbb{Z}_+. \quad (35)$$

Especially, if $a_k + b_k e^{\lambda r_k} > 1$, then the following results can be designed.

Corollary 3. Assume that (30)–(32) in Theorem 2 hold, and there exist positive constants η , α , and β with $\alpha > \beta$, $\eta \in (0, \lambda)$, and $a_k > 1$, $b_k \in \mathbb{R}$ with $a_k + b_k e^{\lambda r_k} > 1$, such that

$$\frac{\ln \theta}{\eta} \leq \frac{t_n - t_0}{n}, \quad n \in \mathbb{Z}_+, \quad (36)$$

where $\theta = \sup_{k \in \mathbb{Z}_+} \{a_k + b_k e^{\lambda r_k}\}$, λ satisfies $\lambda < \alpha - \beta e^{\lambda \tau}$. Moreover, the control gain matrix K is designed by $K = \Psi W^{-1}$. Then, systems (1) and (2) are exponentially synchronized.

Corollary 4. Assume that (30)–(32) in Theorem 2 hold, and there exist positive constants m_k , η , α , and β with $\alpha > \beta$, $\eta \in (0, \lambda)$, and $a_k > 1$, $b_k \in \mathbb{R}$ with $a_k + b_k e^{\lambda r_k} > 1$, such that

$$a_k + b_k e^{\lambda r_k} \leq m_k e^{\eta(t_k - t_{k-1})}, \quad k \in \mathbb{Z}_+, \quad (37)$$

where λ satisfies $\lambda < \alpha - \beta e^{\lambda \tau}$. Moreover, the control gain matrix K is designed by $K = \Psi W^{-1}$. Then, systems (1) and (2) are exponentially synchronized.

Remark 2. Note that Corollary 3 and Corollary 4 both establish the relationship between the impulse strength and the size of impulse intervals. Corollary 3 shows that the impulsive disturbance in our feedback controller could occur in high frequency during a certain period of time, which provided that the impulse strength satisfies certain conditions. Corollary 4 shows that the large scale impulses could be allowed in our control schemes provided that the strength of the impulsive disturbance is limited by the size of impulse intervals. Hence, they are complementary to each other and can be applied to different cases in real applications.

4. Examples

In the section, examples and their simulations are given to illustrate the effectiveness of our proposed theoretical results.

Example 1. Consider the following cellular neural network [11] as the leader:

$$\begin{aligned} \begin{pmatrix} \dot{s}_1(t) \\ \dot{s}_2(t) \end{pmatrix} &= - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_1(t) \\ s_2(t) \end{pmatrix} \\ &+ \begin{pmatrix} 2 & -0.1 \\ -5 & 3.0 \end{pmatrix} \begin{pmatrix} f_1(s_1(t)) \\ f_2(s_2(t)) \end{pmatrix} \\ &+ \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{pmatrix} \begin{pmatrix} f_1(s_1(t-1)) \\ f_2(s_2(t-1)) \end{pmatrix}, \end{aligned} \quad (38)$$

where $f(s) = (\tanh(s_1), \tanh(s_2))^T$. Figure 2 shows the phase plot of the leader with initial values $s_0 = (0.4, 0.6)^T$.

A network consisting of 4 cellular neural networks is considered. The corresponding coupling matrix is obtained as follows:

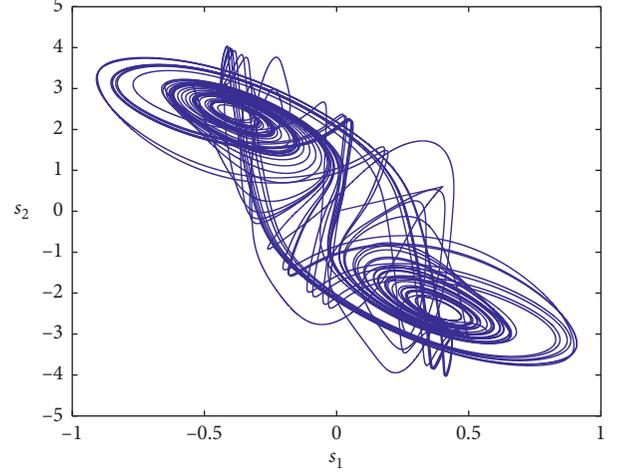


FIGURE 2: The phase plot of the leader for Example 1.

$$L = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \quad (39)$$

Assume that $\Gamma = \text{diag}\{1, 1\}$, $\Pi = \text{diag}\{1, 1\}$, $\tau(t) = 1$, $r(t) = 1$, $\alpha = 29$, $\beta = 0.6$, $b_k = (1 + 10/k^2)e^{-1}$, $c = 0.2$, $t_k = 0.3k$, and $G_k = \sqrt{((1 + 10)/k^2)e^{-1}}I$. Using Matlab LMI toolbox [42], we can obtain the following feasible solutions to LMIs in Theorem 1:

$$\begin{aligned} \Psi &= \begin{pmatrix} 0.0531 & 0.0233 \\ 0.0233 & -0.0713 \end{pmatrix}, \\ W &= \begin{pmatrix} 0.1760 & 0 \\ 0 & 0.1760 \end{pmatrix}, \\ Q_1 &= \begin{pmatrix} 0.0748 & 0 \\ 0 & 0.0748 \end{pmatrix}, \\ Q_2 &= \begin{pmatrix} 0.3754 & 0 \\ 0 & 0.3754 \end{pmatrix}. \end{aligned} \quad (40)$$

Then, the controller gain matrix K is designed as follows:

$$K = \Psi W^{-1} = \begin{pmatrix} 0.3019 & 0.1323 \\ 0.1323 & -0.4053 \end{pmatrix}. \quad (41)$$

Choose $\lambda = 1$ such that $\lambda < 29 - 0.6e^\lambda$. Then, we obtain

$$b_k e^{\lambda r_k} = \left(1 + \frac{10}{k^2}\right) e^{-1} e = \left(1 + \frac{10}{k^2}\right). \quad (42)$$

Thus, there is

$$\prod_{k=1}^n \max\left\{1, 1 + \frac{10}{k^2}\right\} < \infty. \quad (43)$$

Therefore, from Corollary 1, we obtain that the coupled neural networks subject to impulsive disturbances reach the leader-following synchronization. The numerical simulations in Figure 3 have testified the validity of the result. Figures 3(a) and 3(b) show that the state trajectories of the neural network cannot realize synchronization with the leader $s(t)$ without control input. One

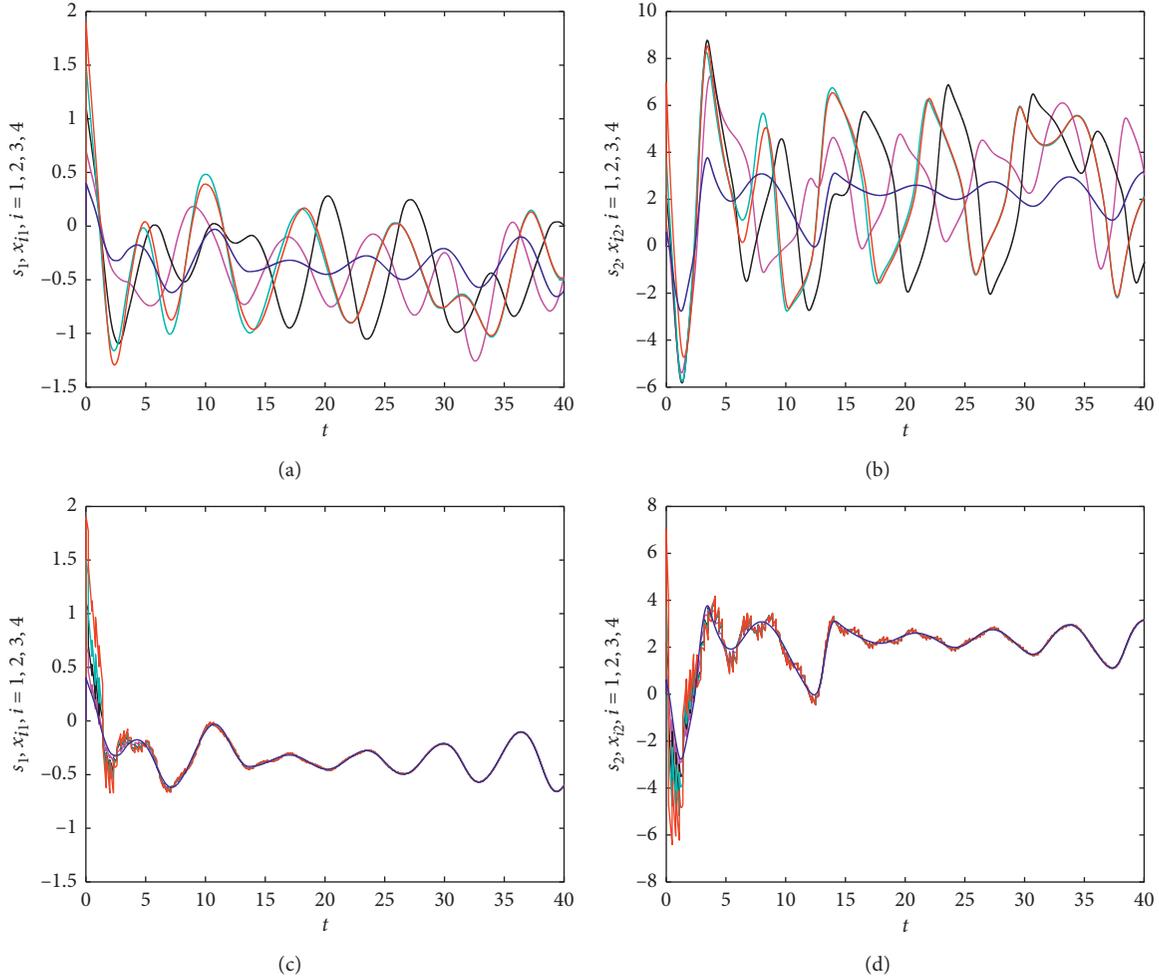


FIGURE 3: State trajectories for Example 1.

can see that synchronization is achieved with a fast rate of convergence by designing the feedback control to overcome the problem of delayed impulsive disturbances, see Figures 3(c) and 3(d).

Remark 3. Note that the coupling strength c is a measure of the interdependence of networks, which can affect the synchronization time of the networks [43]. In the present paper, the strong coupling strength will delay the synchronization time. For example, consider the second state of x_i in Example 1. Under Theorem 1, Figures 4(a)–4(c) show the state trajectories of the neuron x_{i2} with different coupling strength $c = 0.001, 0.2,$ and $1,$ respectively. One may observe that the stronger the coupling, the longer the synchronization time.

Example 2. Consider the leader system:

$$\begin{aligned}
 D &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
 A &= \begin{pmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1 \end{pmatrix}, \\
 B &= \begin{pmatrix} 6.3 & -8.5 & -3 \\ -3 & 1.2 & -5.5 \\ -3.2 & 4.5 & -2.3 \end{pmatrix}, \\
 f(s) &= \tanh(s), \tau(t) = 1.
 \end{aligned} \tag{44}$$

Figure 5 shows that delayed neural networks with initial values $s_0 = (-8.5, -5, 3)^T$. The coupling matrix of neural networks is obtained as follows:

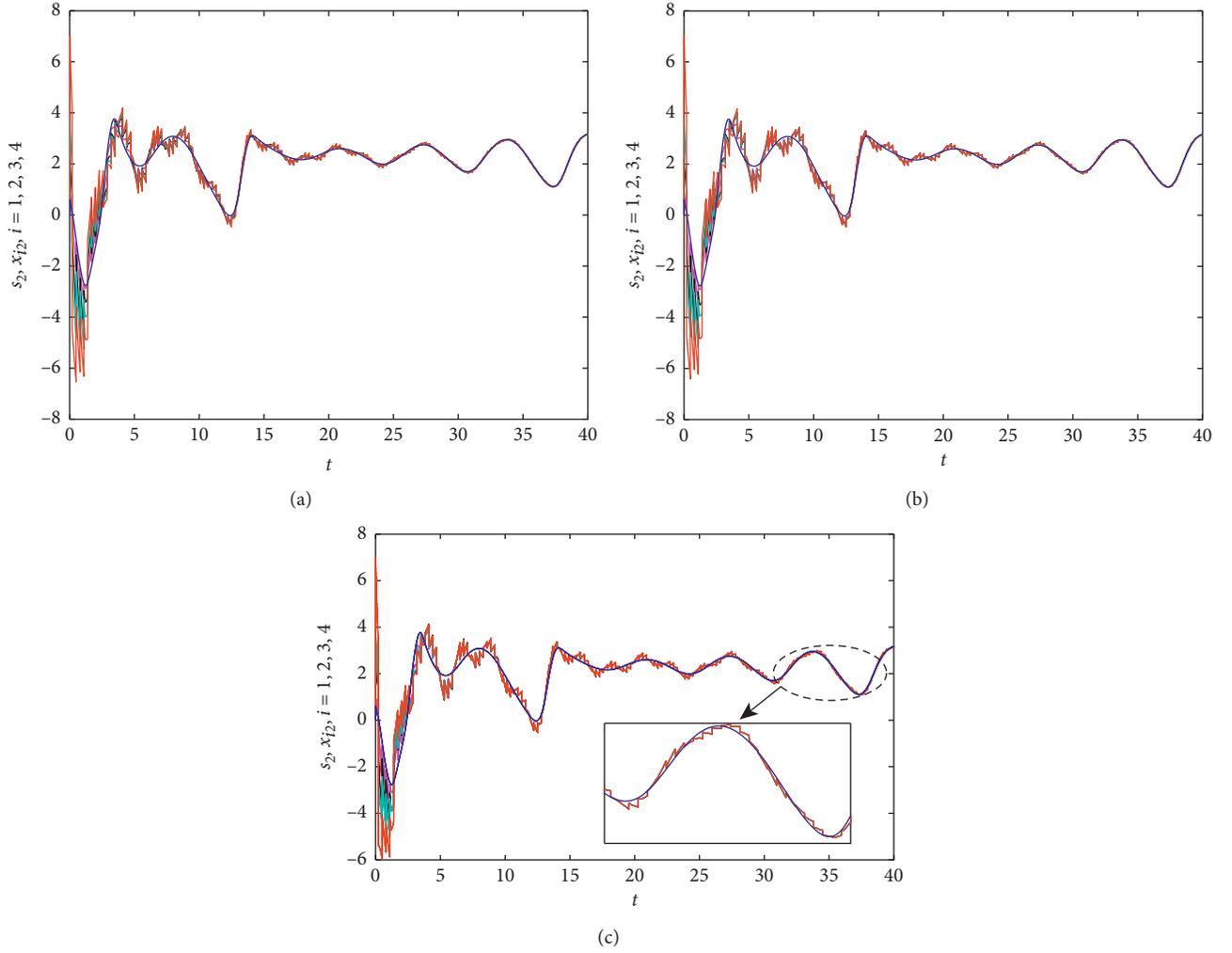


FIGURE 4: State trajectories with different coupling strength c for Example 1. (a) $c = 0.001$. (b) $c = 0.2$. (c) $c = 1$.

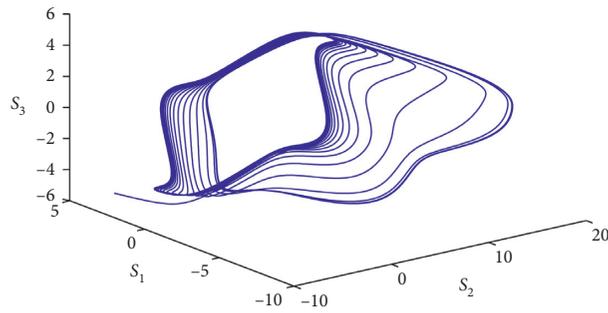


FIGURE 5: The phase plot of the leader for Example 2.

$$L = \begin{pmatrix} -2 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & -2 \end{pmatrix}, \quad (45)$$

and the impulse matrix is considered as $G_k = \text{diag}\{1.2, 1.2, 1.2\}$, $k \in \mathbb{Z}_+$. Let $\alpha = 0.8$, $\beta = 0.01$, $a_k = 1.2$, and $b_k = 0.3$. Using Matlab LMI toolbox [42], we can obtain the following feasible solutions to LMIs in Theorem 2.

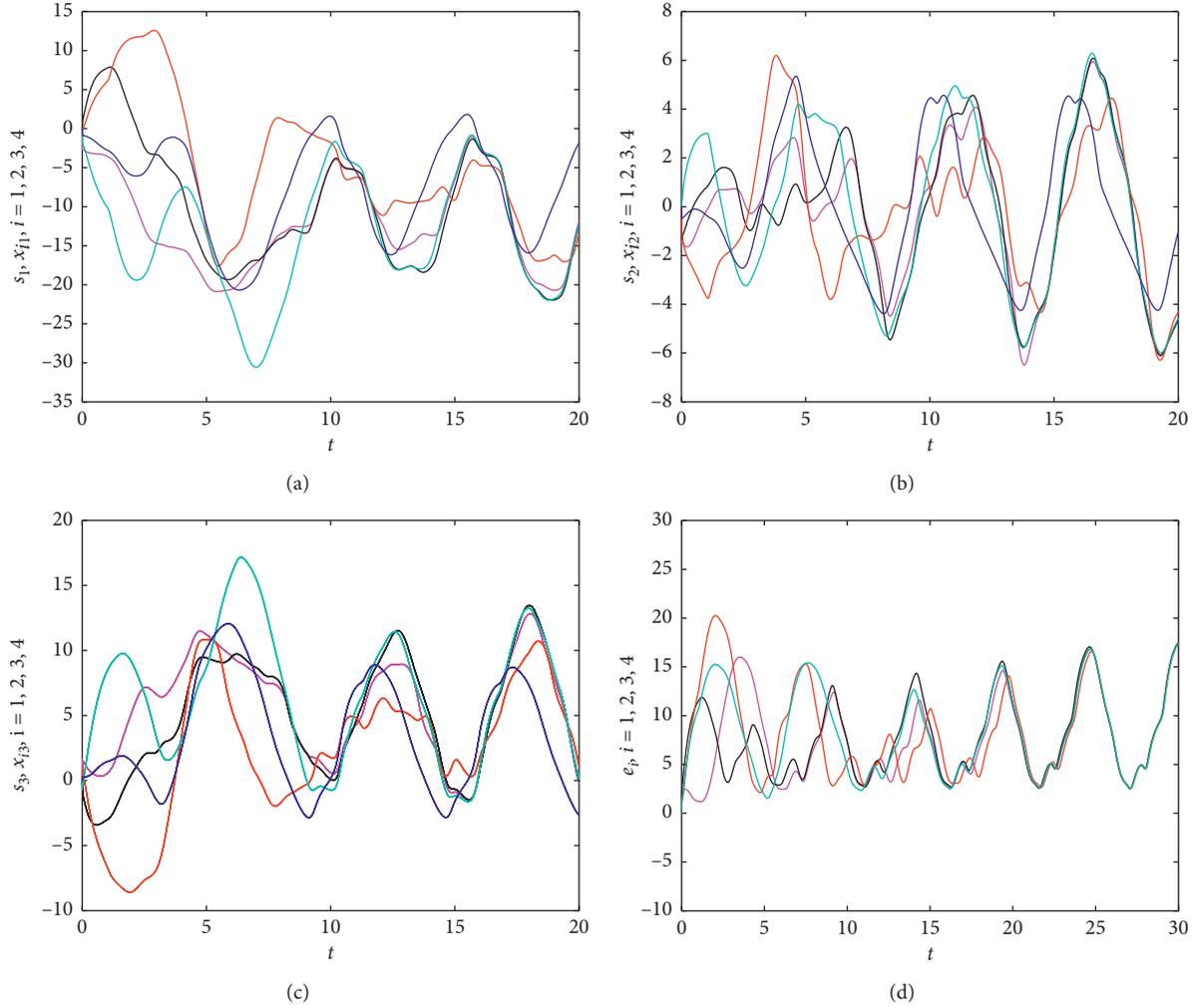


FIGURE 6: Trajectories without control input for Example 2.

$$\begin{aligned}
 \Psi &= \begin{pmatrix} -0.05 & 0.002 & 0.04 \\ 0.002 & -0.07 & -0.02 \\ 0.04 & -0.02 & -0.01 \end{pmatrix}, \\
 W &= \begin{pmatrix} 0.0083 & -0.0004 & 0.0060 \\ -0.0004 & 0.0051 & -0.0025 \\ 0.0060 & -0.0025 & 0.0155 \end{pmatrix}, \\
 Q_1 &= \begin{pmatrix} 0.0032 & 0 & 0 \\ 0 & 0.0032 & 0 \\ 0 & 0 & 0.0032 \end{pmatrix}, \\
 Q_2 &= \begin{pmatrix} 0.0012 & 0 & 0 \\ 0 & 0.0012 & 0 \\ 0 & 0 & 0.0012 \end{pmatrix},
 \end{aligned} \tag{46}$$

Then, the controller matrix K is designed as follows:

$$K = \Psi W^{-1} = \begin{pmatrix} -11.2010 & 3.0495 & 7.4029 \\ 3.0518 & -15.7831 & -4.9943 \\ 7.8114 & -5.4465 & -4.5399 \end{pmatrix}. \tag{47}$$

Choose $\lambda = 0.71$ such that $\lambda < 0.8 - 0.01e^\lambda$. Note that the impulse sequences are $t_k = 0.98k$ and the impulse delay $r(t) = 1$. Let $\eta = 0.7$, $m_k = 1$, then we have

$$a_k + b_k e^{\lambda r_k} = 1.811 \leq m_k e^{\eta(t_k - t_{k-1})} = 1.986. \tag{48}$$

By Corollary 4, we obtain that the coupled delayed neural networks reach the leader-following synchronization. When there is no control input, the state trajectories and error trajectories of the leader $s(t)$ with their four followers cannot realize synchronization, see Figures 6(a)–6(d). Considering the feedback control involving impulsive disturbances into account, one can see that synchronization is achieved with a fast rate of convergence, see Figures 7(a)–7(d). Note that those synchronization criteria in [14, 18, 19, 31] are infeasible for the abovementioned example due to the existence of delayed impulsive disturbance. Here, our proposed

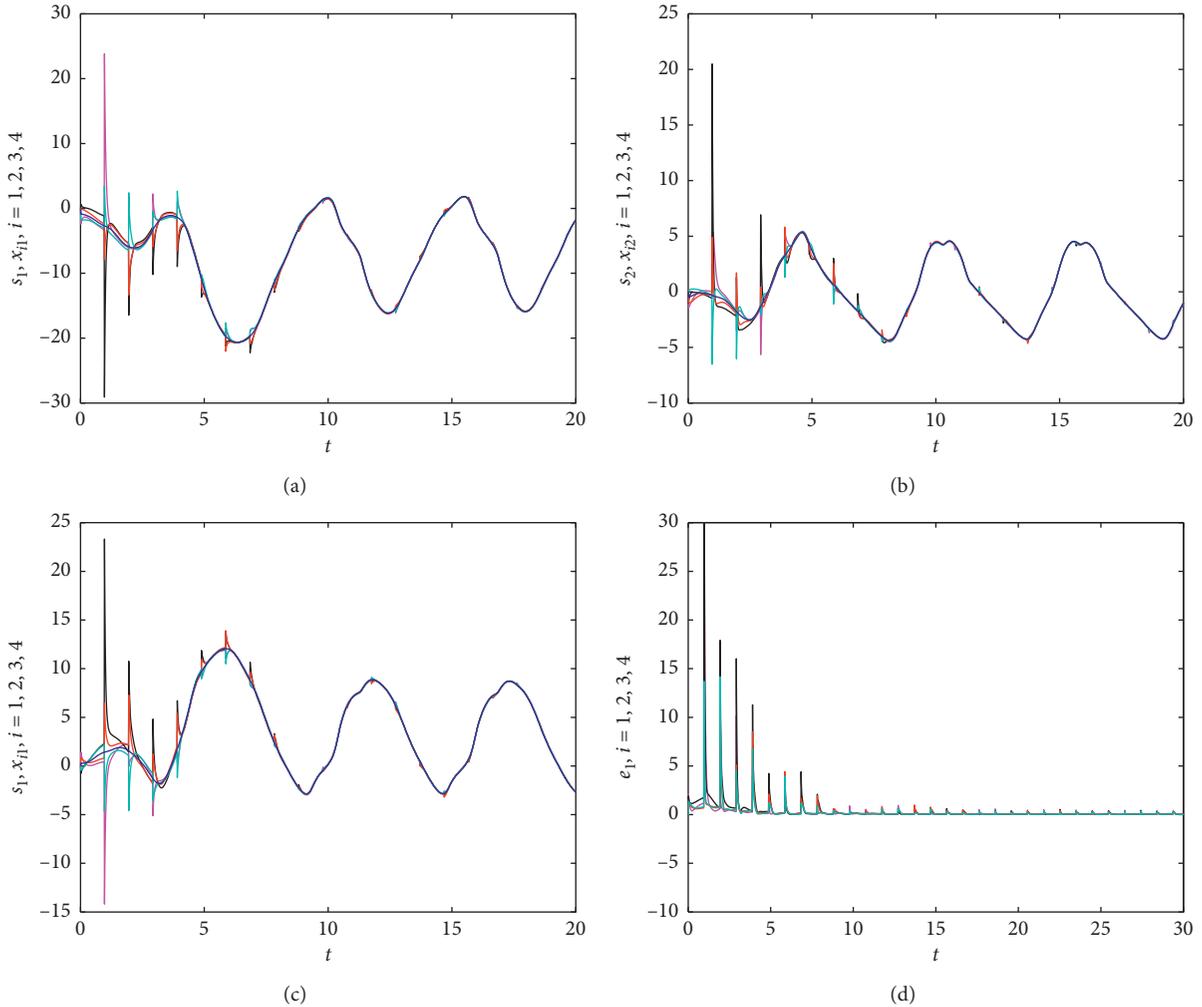


FIGURE 7: Trajectories with control input for Example 2.

theoretical result has wider applications than those existing results.

5. Conclusion

In this paper, some new criteria ensuring the leader-following synchronization of neural networks subject to delayed impulsive disturbances are presented. The results show that network synchronization can still be realized if the large scale impulsive disturbances meet certain conditions, in which the existence of large impulsive disturbances are allowed and the requirements of impulse interval are relaxed. Finally, the illustrated examples with their simulations have been given to demonstrate the effectiveness of the theory results. This paper considers the effect of delayed impulsive disturbances on the system, and perhaps we can extend it to other forms of disturbances. In addition, results in this paper can only be applied to delayed complex neural networks. Hence, the problem of synchronization of some other complex networks deserves further study in future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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