Research Article

Existence of Positive Weak Solutions for a New Class of (p, q) Laplacian Nonlinear Elliptic System with Sign-Changing Weights

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In this paper, by using subsuper solutions method, we study the existence of weak positive solutions for a new class of (p, q) Laplacian nonlinear elliptic system in bounded domains, when $a(x)$, $b(x)$, $\alpha(x)$, and $\beta(x)$ are sign-changing functions that maybe negative near the boundary, without assuming sign conditions on $f(0)$, $g(0)$, $h(0)$, and $\gamma(0)$.

1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)$ -growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [1–15]). Many existence results have been obtained on this kind of problems, see, for example [10, 12, 16–18] and [19–30]. In [31, 32], Fan et al. studied the regularity of solutions for differential equations with nonstandard $p(x)$ -growth conditions.

In this article, we consider the following system:

$$\begin{cases} -\Delta_p u - |u|^{p-2}u = \lambda_1 a(x)f(v) + \mu_1 \alpha(x)h(u) & \text{in } \Omega, \\ -\Delta_q v - |v|^{q-2}v = \lambda_2 b(x)g(u) + \mu_2 \beta(x)\gamma(v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Delta_s z = \operatorname{div}(|\nabla z|^{s-2}\nabla z)$, $s > 1$, $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, $a(x)$, $b(x)$, $\alpha(x)$, $\beta(x) \in C(\overline{\Omega})$, and $\lambda_1, \lambda_2, \mu_1$, and μ_2 are nonnegative parameters.

In fact, we study the existence of positive solutions to system (1) with sign-changing weight functions $a(x)$, $b(x)$, $\alpha(x)$, and $\beta(x)$. Due to these weight functions, the extensions are challenging and nontrivial.

These problems arise in some physical models and are interesting in applications at combustion, mathematical biology, and chemical reactions. Our approach is based on the method of sub- and supersolutions (see [6, 23–25, 27, 30]).

We make the following assumptions:

(H1) $f, g, h, \gamma \in C^1([0, \infty))$ are nondecreasing functions such that $\lim_{s \rightarrow +\infty} f(s) = \lim_{s \rightarrow +\infty} g(s) = \lim_{s \rightarrow +\infty} h(s) = \lim_{s \rightarrow +\infty} \gamma(s) = +\infty$

(H2) $\lim_{s \rightarrow +\infty} ((f(M(g(s)))^{(1/q-1)})/s^{p-1}) = 0, \forall M > 0$

(H3) $\lim_{s \rightarrow +\infty} (h(s)/s^{p-1}) = \lim_{s \rightarrow +\infty} (\gamma(s)/s^{q-1}) = 0$

Let σ_r be the first eigenvalue of $-\Delta_r$ with Dirichlet boundary conditions and ϕ_r be the corresponding eigenfunction with $\phi_r > 0$ in Ω and $\|\phi_r\| = 1$ for $r = p, q$. Let $m, \eta, \delta > 0$ be such that $(|\nabla \phi_r|^r - \sigma_r \phi_r \geq m)$ on

$\overline{\Omega_\delta} = \{x \in \Omega, d(x, \partial\Omega) \leq \delta\}$ and $\phi_r \geq \eta$ on $(\Omega/\overline{\Omega_\delta})$ for $r = p, q$.

Here, we assume that the weights $a(x)$, $b(x)$, $\alpha(x)$, and $\beta(x)$ take negative values in $\overline{\Omega_\delta}$ but require $a(x)$, $b(x)$, $\alpha(x)$, and $\beta(x)$ to be strictly positive in $(\Omega/\overline{\Omega_\delta})$. To be precise, we assume that there exist positive constants $a_0, a_1, b_0, b_1, \alpha_0, \alpha_1, \beta_0$, and β_1 such that

$$\begin{aligned} a(x) &\geq -a_0, \\ b(x) &\geq -b_0, \\ \alpha(x) &\geq -\alpha_0, \\ \beta(x) &\geq -\beta_0, \\ x &\in \overline{\Omega_\delta}, \\ a(x) &\geq a_1, \\ b(x) &\geq b_1, \\ \alpha(x) &\geq \alpha_1, \\ \beta(x) &\geq \beta_1, \\ x &\in (\Omega/\overline{\Omega_\delta}). \end{aligned} \quad (2)$$

Also, let $s_0 \geq 0$ be such that $f(s_0), g(s_0), h(s_0), \gamma(s_0) > 0$, and

$$\begin{aligned} K_1 &= \left(\frac{p-1}{p}\right) \eta^{p/(p-1)}, \\ K_2 &= \left(\frac{q-1}{q}\right) \eta^{q/(q-1)}, \\ \theta_0 &= \max \left\{ \left(\frac{s_0}{K_1}\right)^{1/(p-1)}, \left(\frac{s_0}{K_2}\right)^{q/(q-1)} \right\}. \end{aligned} \quad (3)$$

For $\theta > \theta_0$, we define

$$\begin{aligned} \lambda_{1*}(\theta) &:= \frac{\theta \sigma_p}{d a_1 f(\theta^{1/(q-1)} K_2)}, \\ \lambda_1^*(\theta) &:= \frac{\theta m}{d a_0 f(\theta^{1/(q-1)})}, \\ \mu_{1*}(\theta) &:= \frac{\theta \sigma_p}{d' \alpha_1 h(\theta^{1/(p-1)} K_1)}, \\ \mu_1^*(\theta) &:= \frac{\theta m}{d' \alpha_0 h(\theta^{1/(q-1)})}, \\ \lambda_{2*}(\theta) &:= \frac{\theta \sigma_q}{d b_1 g(\theta^{1/(p-1)} K_1)}, \\ \lambda_2^*(\theta) &:= \frac{\theta m}{d b_0 g(\theta^{1/(p-1)})}, \\ \mu_{2*}(\theta) &:= \frac{\theta \sigma_q}{d' \beta_1 \gamma(\theta^{1/(q-1)} K_2)}, \\ \mu_2^*(\theta) &:= \frac{\theta m}{d' \beta_0 \gamma(\theta^{1/(q-1)})}, \end{aligned} \quad (4)$$

where $d > 1$ and $(1/d) + (1/d') = 1$; also assume

$$\Lambda = \{\theta > \theta_0 : \lambda_{1*}(\theta) < \lambda_1^*(\theta), \mu_{1*}(\theta) < \mu_1^*(\theta), \lambda_{2*}(\theta) < \lambda_2^*(\theta), \mu_{2*}(\theta) < \mu_2^*(\theta)\}. \quad (5)$$

2. Existence Result

We give the following two definitions before we give our main result.

Definition 1. Let $(u, v) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$, (u, v) be said a weak solution of (1) if it satisfies

$$\begin{aligned} \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \xi dx - \int_{\Omega} |u|^{p-2} u \cdot \xi dx &= \lambda_1 \int_{\Omega} a(x) f(v) \xi dx + \mu_1 \int_{\Omega} \alpha(x) h(u) \xi dx \text{ in } \Omega, \\ \int_{\Omega} |\nabla v|^{q-2} \nabla v \cdot \nabla \zeta dx - \int_{\Omega} |v|^{q-2} v \cdot \zeta dx &= \lambda_2 \int_{\Omega} b(x) g(u) \zeta dx + \mu_2 \int_{\Omega} \beta(x) \gamma(v) \zeta dx \text{ in } \Omega, \end{aligned} \quad (6)$$

for all $(\xi, \zeta) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

Definition 2. A pair of nonnegative functions $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$ in $W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ is called a weak

subsolution and supersolution of (1) if they satisfy $(\underline{u}, \underline{v}), (\overline{u}, \overline{v}) = (0, 0)$ on $\partial\Omega$:

$$\begin{aligned} \int_{\Omega} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla \xi dx - \int_{\Omega} |\underline{u}|^{p-2} \underline{u} \cdot \xi dx &\leq \lambda_1 \int_{\Omega} a(x) f(\underline{v}) \xi dx + \mu_1 \int_{\Omega} \alpha(x) h(\underline{u}) \xi dx \text{ in } \Omega, \\ \int_{\Omega} |\nabla \underline{v}|^{q-2} \nabla \underline{v} \cdot \nabla \zeta dx - \int_{\Omega} |\underline{v}|^{q-2} \underline{v} \cdot \zeta dx &\leq \lambda_2 \int_{\Omega} b(x) g(\underline{u}) \zeta dx + \mu_2 \int_{\Omega} \beta(x) \gamma(\underline{v}) \zeta dx \text{ in } \Omega, \\ \int_{\Omega} |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \xi dx - \int_{\Omega} |\overline{u}|^{p-2} \overline{u} \cdot \xi dx &\geq \lambda_1 \int_{\Omega} a(x) f(\overline{v}) \xi dx + \mu_1 \int_{\Omega} \alpha(x) h(\overline{u}) \xi dx \text{ in } \Omega, \\ \int_{\Omega} |\nabla \overline{v}|^{q-2} \nabla \overline{v} \cdot \nabla \zeta dx - \int_{\Omega} |\overline{v}|^{q-2} \overline{v} \cdot \zeta dx &\geq \lambda_2 \int_{\Omega} b(x) g(\overline{u}) \zeta dx + \mu_2 \int_{\Omega} \beta(x) \gamma(\overline{v}) \zeta dx \text{ in } \Omega, \end{aligned} \quad (7)$$

for all $(\xi, \zeta) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

We shall establish the following result.

Theorem 1. Assume that the conditions (H1) – (H3) hold, $a(x)$, $b(x)$, $\alpha(x)$, and $\beta(x)$ are in $L^\infty(\Omega)$ and $\Lambda \neq \emptyset$. Let

$$I = \cup_{\theta \in \Lambda} [\lambda_{1*}(\theta), \lambda_1^*(\theta)] \times [\mu_{1*}(\theta), \mu_1^*(\theta)] \times [\lambda_{2*}(\theta), \lambda_2^*(\theta)] \times [\mu_{2*}(\theta), \mu_2^*(\theta)]. \quad (8)$$

Then, problem (1) has a positive weak solution for each $(\lambda_1, \mu_1, \lambda_2, \mu_2) \in I$.

Proof of Theorem 1. Let $(\lambda_1, \mu_1, \lambda_2, \mu_2) \in I$ and $\theta > \theta_0$ be such that $(\lambda_1, \mu_1, \lambda_2, \mu_2) \in [\lambda_{1*}(\theta), \lambda_1^*(\theta)] \times [\mu_{1*}(\theta), \mu_1^*(\theta)] \times [\lambda_{2*}(\theta), \lambda_2^*(\theta)] \times [\mu_{2*}(\theta), \mu_2^*(\theta)]$.

We shall verify that

$$\begin{aligned} \underline{u} &= \theta^{1/(p-1)} \left(\frac{p-1}{p} \right) \phi_p^{p/(p-1)}, \\ \underline{v} &= \theta^{1/(q-1)} \left(\frac{q-1}{q} \right) \phi_q^{q/(q-1)}, \end{aligned} \quad (9)$$

is a subsolution of (1). Let the test function $\xi(x) \in W_0^{1,p}(\Omega)$ with $\xi(x) \geq 0$. We have

$$\begin{aligned} \int_{\Omega} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla \xi dx - \int_{\Omega} |\underline{u}|^{p-2} \underline{u} \cdot \xi dx &\leq \int_{\Omega} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla \xi dx \\ &= \theta \int_{\Omega} \phi_p |\nabla \phi_p|^{p-2} \nabla \phi_p \cdot \nabla \xi dx \\ &= \theta \int_{\Omega} \left[|\nabla \phi_p|^{p-2} \nabla \phi_p \cdot \nabla (\phi_p \xi) - |\nabla \phi_p|^p \xi \right] dx \\ &= \theta \int_{\Omega} \left(\sigma_p \phi_p^p - |\nabla \phi_p|^p \right) \xi dx. \end{aligned} \quad (10)$$

Similarly,

$$\int_{\Omega} |\nabla \underline{v}|^{q-2} \nabla \underline{v} \cdot \nabla \zeta dx - \int_{\Omega} |\underline{v}|^{q-2} \underline{v} \cdot \zeta dx \leq \theta \int_{\Omega} \left(\sigma_q \phi_q^q - |\nabla \phi_q|^q \right) \zeta dx. \quad (11)$$

For all $\zeta(x) \in W_0^{1,q}(\Omega)$ with $\zeta(x) \geq 0$.

Now, on $\overline{\Omega_\delta}$ we have

$$\begin{aligned} \theta \int_{\overline{\Omega_\delta}} \left(\sigma_p \phi_p^p - |\nabla \phi_p|^p \right) \xi dx &\leq -\theta m \int_{\overline{\Omega_\delta}} \xi dx \\ &= -\theta m \left(\frac{1}{d} + \frac{1}{d'} \right) \int_{\overline{\Omega_\delta}} \xi dx \\ &\leq \left[-\lambda_1 a_0 f(\theta^{(1/q-1)}) - \mu_1 \alpha_0 h(\theta^{(1/p-1)}) \right] \int_{\overline{\Omega_\delta}} \xi dx \\ &\leq \int_{\overline{\Omega_\delta}} \left[-\lambda_1 a_0 f \left(\theta^{(1/q-1)} \left(\frac{q-1}{q} \right) \phi_q^{q/(q-1)} \right) - \mu_1 \alpha_0 h \left(\theta^{(1/p-1)} \left(\frac{p-1}{p} \right) \phi_p^{p/(p-1)} \right) \right] \xi dx \\ &\leq \int_{\overline{\Omega_\delta}} [\lambda_1 a(x) f(\underline{v}) + \mu_1 \alpha(x) h(\underline{u})] \xi dx \\ \theta \int_{\overline{\Omega_\delta}} \left(\sigma_q \phi_q^q - |\nabla \phi_q|^q \right) \zeta dx &\leq -\theta m \int_{\overline{\Omega_\delta}} \zeta dx \\ &= \left(\frac{-\theta m}{d} - \frac{\theta m}{d'} \right) \int_{\overline{\Omega_\delta}} \zeta dx \\ &\leq \left[-\lambda_2 b_0 g(\theta^{(1/p-1)}) - \mu_2 \beta_0 \gamma(\theta^{(1/q-1)}) \right] \int_{\overline{\Omega_\delta}} \zeta dx \\ &\leq \int_{\overline{\Omega_\delta}} \left[-\lambda_2 b_0 g \left(\theta^{(1/p-1)} \left(\frac{p-1}{p} \right) \phi_p^{p/(p-1)} \right) - \mu_2 \beta_0 \gamma \left(\theta^{(1/q-1)} \left(\frac{q-1}{q} \right) \phi_q^{q/(q-1)} \right) \right] \zeta dx \\ &\leq \int_{\overline{\Omega_\delta}} [\lambda_2 b(x) g(\underline{u}) + \mu_2 \beta(x) \gamma(\underline{v})] \zeta dx. \end{aligned} \quad (12)$$

On the contrary, on $(\Omega/\overline{\Omega}_\delta)$ we have

$$\begin{aligned}
 \theta \int_{(\Omega/\overline{\Omega}_\delta)} (\sigma_p \phi_p^p - |\nabla \phi_p|^p) \xi dx &\leq \theta \sigma_p \int_{(\Omega/\overline{\Omega}_\delta)} \xi dx, \\
 &= \theta \sigma_p \left(\frac{1}{d} + \frac{1}{d'} \right) \int_{(\Omega/\overline{\Omega}_\delta)} \xi dx \\
 &\leq \left[\lambda_1 a_1 f(\theta^{(1/q-1)} K_2) + \mu_1 \alpha_0 h(\theta^{(1/p-1)} K_1) \right] \int_{(\Omega/\overline{\Omega}_\delta)} \xi dx \\
 &\leq \int_{(\Omega/\overline{\Omega}_\delta)} [\lambda_1 a(x) f(\underline{v}) + \mu_1 \alpha(x) h(\underline{u})] \xi dx,
 \end{aligned} \tag{13}$$

and similarly,

$$\theta \int_{(\Omega/\overline{\Omega}_\delta)} (\sigma_q \phi_q^q - |\nabla \phi_q|^q) \zeta dx \leq \int_{(\Omega/\overline{\Omega}_\delta)} [\lambda_2 b(x) g(\underline{u}) + \mu_2 \beta(x) \gamma(\underline{v})] \zeta dx. \tag{14}$$

Therefore, $(\underline{u}, \underline{v})$ is subsolution of problem (1).

Next, we construct a supersolution of (1). Let ω_r be a unique positive solution of

$$\begin{cases} -\Delta_r \omega_r = 1 \text{ in } \Omega, \\ \omega_r = 0 \text{ on } \partial\Omega, \end{cases} \tag{15}$$

for $r = p, q$. We denote

$$\begin{aligned}
 \overline{u} &= \frac{C}{\nu_p} \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \omega_p, \\
 \overline{v} &= \left[\left(\frac{\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty}{1 - \nu_q^{q-1}} \right) g \left(C \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \right)^{1/(q-1)} \right] \omega_q,
 \end{aligned} \tag{16}$$

where $\nu_r = \|\omega_r\|_\infty$, $r = p, q$, and $C > 0$ is a large number to be chosen later. We shall verify that $(\overline{u}, \overline{v})$ is a supersolution of (1) such that $(\overline{u}, \overline{v}) \geq (\underline{u}, \underline{v})$. By (H2) – (H3), we can choose C large enough so that

$$\begin{aligned}
 \left(\frac{C}{\nu_p} \right)^{p-1} &\geq f \left(\left[\left(\frac{\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty}{1 - \nu_q^{q-1}} \right) g \left(C \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \right)^{1/(q-1)} \right] \omega_q \right) \\
 &\quad + \mu_1 h \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \omega_p.
 \end{aligned} \tag{17}$$

Hence,

$$\begin{aligned}
 &\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \xi dx - \int_\Omega |\overline{u}|^{p-2} \overline{u} \cdot \xi dx \\
 &= \left(\frac{C}{\nu_p} \right)^{p-1} (\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty) \int_\Omega \xi dx.
 \end{aligned} \tag{18}$$

Using (17),

$$\begin{aligned}
 &\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \xi dx - \int_\Omega |\overline{u}|^{p-2} \overline{u} \cdot \xi dx \\
 &\geq \lambda_1 \|a\|_\infty f \left(\left[\left(\frac{\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty}{1 - \nu_q^{q-1}} \right) g \left(C \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \right)^{1/(q-1)} \right] \omega_q \right) \int_\Omega \xi dx \\
 &\quad + \mu_1 \|a\|_\infty \int_\Omega h \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \xi dx \\
 &\geq \int_\Omega [\lambda_1 a(x) f(\overline{v}) + \mu_1 \alpha(x) h(\overline{u})] \xi dx.
 \end{aligned} \tag{19}$$

Next

$$\begin{aligned}
 & \int_{\Omega} |\nabla \bar{v}|^{q-2} \nabla \bar{v} \cdot \nabla \zeta \, dx - \int_{\Omega} |\bar{v}|^{q-2} \bar{v} \cdot \zeta \, dx \\
 &= \left\{ (\lambda_2 \|b\|_{\infty} + \mu_2 \|\beta\|_{\infty}) g \left(C \left(\frac{\lambda_1 \|a\|_{\infty} + \mu_1 \|\alpha\|_{\infty}}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \right) \right\} \omega_q \int_{\Omega} \xi \, dx \\
 &\geq \left[\begin{aligned} & \lambda_2 \|b\|_{\infty} g \left(C \left(\frac{\lambda_1 \|a\|_{\infty} + \mu_1 \|\alpha\|_{\infty}}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \right) \\ & + \mu_2 \|\beta\|_{\infty} g \left(C \left(\frac{\lambda_1 \|a\|_{\infty} + \mu_1 \|\alpha\|_{\infty}}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \right) \end{aligned} \right] \int_{\Omega} \xi \, dx.
 \end{aligned} \tag{20}$$

By (H3), choose C large so that

$$\begin{aligned}
 & g \left(C \left(\frac{\lambda_1 \|a\|_{\infty} + \mu_1 \|\alpha\|_{\infty}}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \right) \\
 &\geq \gamma \left(\left[\left(\frac{\lambda_2 \|b\|_{\infty} + \mu_2 \|\beta\|_{\infty}}{1 - \nu_q^{q-1}} \right) g \left(C \left(\frac{\lambda_1 \|a\|_{\infty} + \mu_1 \|\alpha\|_{\infty}}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \right)^{1/(q-1)} \right] \|\omega_q\|_{\infty} \right).
 \end{aligned} \tag{21}$$

Then, from (19), we have

$$\begin{aligned}
 & \int_{\Omega} |\nabla \bar{v}|^{q-2} \nabla \bar{v} \cdot \nabla \zeta \, dx - \int_{\Omega} |\bar{v}|^{q-2} \bar{v} \cdot \zeta \, dx \\
 &\geq \lambda_2 \|b\|_{\infty} g \left(C \left(\frac{\lambda_1 \|a\|_{\infty} + \mu_1 \|\alpha\|_{\infty}}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \right) \\
 &\quad + \mu_2 \|\beta\|_{\infty} \gamma \left(\left[\left(\frac{\lambda_2 \|b\|_{\infty} + \mu_2 \|\beta\|_{\infty}}{1 - \nu_q^{q-1}} \right) g \left(C \left(\frac{\lambda_1 \|a\|_{\infty} + \mu_1 \|\alpha\|_{\infty}}{1 - \nu_p^{p-1}} \right)^{1/(p-1)} \right)^{1/(q-1)} \right] \|\omega_q\|_{\infty} \right) \\
 &\geq \int_{\Omega} [b(x)g(\bar{u}) + \mu_2 \beta(x)\gamma(\bar{v})] \zeta \, dx.
 \end{aligned} \tag{22}$$

According to (19) and (20), we can conclude that $(\bar{u}, \bar{v})$ is a supersolution of (1). Furthermore, $\bar{u} \geq \underline{u}$ and $\bar{v} \geq \underline{v}$ for C large. Thus, there exists a solution $(u, v) \in W^{1,p}(\Omega) \cap C(\bar{\Omega}) \times W^{1,q}(\Omega) \cap C(\bar{\Omega})$ of (1) with $\underline{u} \leq u \leq \bar{u}$ and $\underline{v} \leq v \leq \bar{v}$. This completes the Proof of Theorem 1. $\square$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

Authors' Contributions

All authors contributed equally to this work. They have all read and approved the final manuscript.

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