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Research Article

Upper and Lower Bounds for the Kirchhoff Index of the *n*-Dimensional Hypercube Network

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The hypercube Q_n is one of the most admirable and efficient interconnection network due to its excellent performance for some practical applications. The Kirchhoff index $\mathrm{Kf}(G)$ is equal to the sum of resistance distances between any pairs of vertices in networks. In this paper, we deduce some bounds with respect to Kirchhoff index of hypercube network Q_n.

1. Introduction

Network is usually modelled by a connected graph $G = (V_G, E_G)$ with order n, labeled as $V_G = \{v_1, v_2, \ldots, v_n\}$ and $E_G = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix $A(G)$ of G is a square matrix with n vertices, in which elements a_{ij} are 1 or 0, depending on whether there is an edge or not between vertices i and j. The degree diagonal matrix of G is denoted by $D(G) = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$, where $d_1, d_2, \ldots, d_n$ are the degree of vertices $v_1, v_2, \ldots, v_n$, respectively. Together with the adjacency and degree matrix, one arrives at the Laplacian matrix, whose expression can be written as $L(G) = D(G) - A(G)$. For other notations and graph theoretical terminologies that not state here, we follow [1].

Various parameters are always used to characterize and describe the complex networks of which the fundamental one is named as the distance d_{ij}, concerned as the shortest path between the vertices i and j in networks. Similarly considering the distance d_{ij}, Klein and Randić in 1993 presented a novel distance function, named as resistance distance [2]. Denote r_{ij} the resistance distance between two arbitrary vertices i and j in electrical networks by replacing every edge by a unit resistor [3–7]. The Kirchhoff index $\mathrm{Kf}(G)$ of networks is defined as

$$\mathrm{Kf}(G) = \sum_{i<j} r_{ij}(G). \tag{1}$$

The Kirchhoff index has attracted more and more attentions due to its practical applications in the fields of physical interpretations, electric circuit, and so on [8–11]. The Kirchhoff index of some product graphs, join graphs, and corona graphs were studied [5, 7]. The more results of the applications on the Kirchhoff index were explored in [12–14].

In what follows, the rest of the context is summarized. Section 2 proposes the main definition and preliminaries in our discussion. Some bounds on the Kirchhoff index of hypercubes Q_n are deduced in Section 3. We conclude the paper in Section 4.

2. Definition and Preliminaries

In this section, we recall some basic definition in graph theory. The hypercube network Q_n may be constructed from the family of subsets of a set with a binary string of length n, by making a vertex for each possible subset and joining two vertices by an edge whenever the corresponding subsets differ in a single binary string. The hypercube network Q_n admits several definitions of which one is stated as below [15].

The hypercube network Q_n is repeatedly constructed by making two copies of Q_{n-1}, written as Q_{n-1}^0 and Q_{n-1}^1, respectively. Meanwhile, adding repeatedly 2^{n-1} edges as below, let $V(Q_{n-1}^0) = \{0U = 0u_2u_3\ldots u_n\colon u_i = 0 \text{ or } 1\}$ and $V(Q_{n-1}^1) = \{1V = 1v_2v_3\ldots v_n\colon v_i = 0 \text{ or } 1\}$. A node $0U = 0u_2u_3\ldots u_n$ of

Q_{n-1}^0 is linked to another node $1V = 1v_2v_3 \ldots v_n$ of Q_{n-1}^1 if and only if $u_i = v_i$ for each $i, 2 \le i \le n$.

The hypercube network Q_n obtained more and more admirable concentrations due to its surprising properties, for instance, symmetry, regular structure, strong connectivity, small diameter, and so on [16, 17]. For more results on the hypercube network and its applications, see [18–21].

Next, we recall the formula for the Kirchhoff index in the hypercube Q_n with $n \ge 2$.

Theorem 1 (see [3]). *For the hypercube network Q_n with $n \ge 2$,*

$$\mathrm{Kf}(Q_n) = 2^n \sum_{i=1}^{n} \binom{n}{i} \frac{1}{2i}, \tag{2}$$

where $2i\,(i = 1, \ldots, n)$ is the eigenvalue of the Laplacian matrix of the hypercube network and the binomial coefficients $\binom{n}{i}$ are the multiplicities of the eigenvalues $2i$.

Theorem 2 (see [22]).

$$\lim_{n \longrightarrow \infty} \sum_{i=0}^{n-1} \frac{n}{2^i (n-i)} = 2. \tag{3}$$

The authors of [23] obtained a closed-form formula for the Kirchhoff index of the d-dimensional hypercube and found the asymptotic value $2^{2d}/d$ by using probabilistic tools. The result of Theorem 3 is obtained by directly calculating the eigenvalues of the Laplacian matrix of the hypercube network, which is different from the technique in [23].

3. Main Results

In this section, one will estimate the Kirchhoff index of n-dimensional hypercube, i.e., our goal is to estimate the quantity:

$$2^n \sum_{i=1}^{n} \binom{n}{i} \frac{1}{2i}. \tag{4}$$

Theorem 3. *For the hypercube network Q_n with $n \ge 2$, then*

$$\frac{4^n}{n}\left(\frac{n}{n+1} - \frac{n(n+2)}{2^{n+1}(n+1)}\right) \le \mathrm{Kf}(Q_n). \tag{5}$$

Consider that

$$\begin{aligned}
\sum_{i=1}^{n} \binom{n}{i} \frac{1}{i} &\ge \sum_{i=1}^{n} \binom{n}{i} \frac{1}{i+1} \\
&= \binom{n}{1}\frac{1}{2} + \binom{n}{2}\frac{1}{3} + \ldots + \binom{n}{n}\frac{1}{n+1} \\
&= \frac{1}{2}\frac{n!}{1!(n-1)!} + \frac{1}{3}\frac{n!}{2!(n-2)!} + \ldots + \frac{1}{n+1} \\
&= \frac{1}{n+1}\frac{(n+1)!}{2!(n-1)!} + \frac{1}{n+1}\frac{(n+1)!}{3!(n-2)!} + \ldots + \frac{1}{n+1} \\
&= \sum_{i=1}^{n} \binom{n+1}{i+1} \frac{1}{n+1}.
\end{aligned} \tag{6}$$

By virtue of

$$\sum_{i=1}^{n} \binom{n+1}{i+1} = 2^{n+1} - n - 2. \tag{7}$$

By means of calculating the right of equation (6), one can establish the following identity:

$$\sum_{i=1}^{n} \binom{n+1}{i+1} \frac{1}{n+1} = \frac{2^{n+1} - n - 2}{n+1}. \tag{8}$$

Since

$$\begin{aligned}
\frac{2^{n+1} - n - 2}{n+1} &\le \sum_{i=1}^{n} \binom{n}{i} \frac{1}{i} = 2 \sum_{i=1}^{n} \binom{n}{i} \frac{1}{2i}, \\
\frac{2^{n+1} - n - 2}{n+1} &\le 2 \sum_{i=1}^{n} \binom{n}{i} \frac{1}{2i}.
\end{aligned} \tag{9}$$

Hence,

$$\frac{2^n 2^n}{n+1} - \frac{2^{n-1}(n+2)}{n+1} = 2^{n-1}\frac{2^{n+1} - n - 2}{n+1} \le \mathrm{Kf}(Q_n) = 2^n \sum_{i=1}^{n} \binom{n}{i} \frac{1}{2i}. \tag{10}$$

Simply, from the left of the above inequality, we obtain

$$\frac{4^n}{n}\left(\frac{n}{n+1} - \frac{n(n+2)}{2^{n+1}(n+1)}\right) \le \mathrm{Kf}(Q_n). \tag{11}$$

Apparently, the left of the above inequality converges to the asymptotic value $2^{2d}/d$ for large enough n. The proof of lower bound is completed.

For the upper bound, we have similar theorem to consider as follows.

Theorem 4. *For the hypercube networks Q_n with $n \ge 2$, then*

$$\mathrm{Kf}(Q_n) \le \frac{4^n}{n}\left(\frac{2n}{n+1} - \frac{n+2}{2^n(n+1)}\right), \tag{12}$$

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{2i} C_n^i &\le \sum_{i=1}^{n} \frac{1}{i+1} C_n^i \\
&= \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 + \ldots + \frac{1}{n+1}C_n^n \\
&= \frac{1}{2}\frac{n!}{1!(n-1)!} + \frac{1}{3}\frac{n!}{2!(n-2)!} + \ldots + \frac{1}{n+1} \\
&= \frac{1}{n+1}\frac{(n+1)!}{2!(n-1)!} + \frac{1}{n+1}\frac{(n+1)!}{3!(n-2)!} + \ldots + \frac{1}{n+1} \\
&= \sum_{i=1}^{n} \frac{1}{n+1} C_{n+1}^{i+1}.
\end{aligned} \tag{13}$$

Based on equation (8), we can obtain that

$$2^n \sum_{i=1}^{n} \frac{C_n^i}{2i} = \mathrm{Kf}(Q_n) \le 2^n \frac{2^{n+1} - n - 2}{n+1}. \tag{14}$$

Hence,

$$\mathrm{Kf}(Q_n) \le 2^n \frac{2^{n+1} - n - 2}{n+1} = \frac{4^n}{n}\left(\frac{2n}{n+1} - \frac{n+2}{2^n(n+1)}\right). \quad (15)$$

The above estimate looks a little complicated. The upper bound is roughly twice the asymptotic value. Hence, a new upper bound is explored as follows.

Theorem 5. *For the hypercube network Q_n with $n \ge 2$,*

$$\mathrm{Kf}(Q_n) \le \frac{4^n}{n}. \quad (16)$$

Following the identity which is obtained in [24],

$$\sum_{i=1}^{n} \frac{x^i}{i} = \left(\frac{1}{n} + \frac{1}{n-1} + \cdots + 1\right) + \sum_{i=1}^{n} \binom{n}{i} \frac{(x-1)^i}{i}. \quad (17)$$

Fixing $x = 2$, one arrives at

$$\sum_{i=1}^{n} \binom{n}{i} \frac{1}{i} = \sum_{i=1}^{n} \frac{2^i}{i} - \left(\frac{1}{n} + \frac{1}{n-1} + \cdots + 1\right). \quad (18)$$

Namely,

$$\sum_{i=1}^{n} \binom{n}{i} \frac{1}{i} = \sum_{i=1}^{n} \frac{2^i - 1}{i}. \quad (19)$$

According to equation (19) and Theorem 2, one obtains

$$\mathrm{Kf}(Q_n) = 2^{n-1} \cdot \sum_{i=1}^{n} \frac{2^i - 1}{i}. \quad (20)$$

Using equation (20), one has

$$\mathrm{Kf}(Q_n) \le 2^{n-1} \cdot \sum_{i=1}^{n} \frac{2^i}{i}. \quad (21)$$

On the contrary,

$$2^{n-1} \cdot \sum_{i=1}^{n} \frac{2^i}{i} = \frac{1}{n} \cdot 2^{2n-1} \cdot \sum_{i=0}^{n-1} \frac{n}{2^i(n-i)}. \quad (22)$$

Using Theorem 2 and substituting equations (22) to (21), one obtains the desired result:

$$\mathrm{Kf}(Q_n) \le \frac{4^n}{n}. \quad (23)$$

This has completed the proof.

4. Further Discussion

We, at this place, try another way to estimate the Kirchhoff index of n-dimensional hypercubes.

Theorem 6. *For the hypercube networks Q_n with $n \ge 2$, then*

$$\mathrm{Kf}(Q_n) = 2^{n-1} \sum_{i=1}^{n} \frac{2^i - 1}{i}. \quad (24)$$

Let $S_n = \sum_{i=1}^{n} C_n^i / i$, then

$$\begin{aligned} S_n - S_{n-1} &= \sum_{i=1}^{n} \frac{C_n^i}{i} - \sum_{i=1}^{n-1} \frac{C_{n-1}^i}{i} \\ &= \left[\frac{1}{n} + \sum_{i=1}^{n-1} \frac{C_n^i}{i}\right] - \sum_{i=1}^{n-1} \frac{C_{n-1}^i}{i} \\ &= \frac{1}{n} + \sum_{i=1}^{n-1} \frac{1}{i}\left[C_n^i - C_{n-1}^i\right] \\ &= \frac{1}{n} + \sum_{i=1}^{n-1} \frac{1}{i} C_{n-1}^{i-1}. \end{aligned} \quad (25)$$

Consequently,

$$n \cdot (S_n - S_{n-1}) = 1 + \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} = 1 + \sum_{i=1}^{n-1} C_n^i = 2^n - 1. \quad (26)$$

One can easily check that $S_1 = 1$. Hence, $S_n - S_{n-1} = (2^n/n) - (1/n)$.

By virtue of the above equality, we obtain

$$S_n = \sum_{i=1}^{n} \frac{2^i}{i} - \sum_{i=1}^{n} \frac{1}{i}. \quad (27)$$

Therefore,

$$\mathrm{Kf}(Q_n) = 2^{n-1} \sum_{i=1}^{n} \frac{2^i - 1}{i}. \quad (28)$$

The proof of Theorem 6 is completed.

Data Availability

The data used to support the findings of this study are available within paper.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

Data curation was carried out by J-B.L.; J-B.L. and J.Cao helped with the methodology; J.Z., Z-Y.S., and F.E. Alsaadi wrote the original draft. All authors read and approved the final manuscript.

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