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Research Article

Synchronization Analysis for Stochastic Inertial Memristor-Based Neural Networks with Linear Coupling

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This paper concerns the synchronization problem for a class of stochastic memristive neural networks with inertial term, linear coupling, and time-varying delay. Based on the interval parametric uncertainty theory, the stochastic inertial memristor-based neural networks (IMNNs for short) with linear coupling are transformed to a stochastic interval parametric uncertain system. Furthermore, by applying the Lyapunov stability theorem, the stochastic analysis approach, and the Halanay inequality, some sufficient conditions are obtained to realize synchronization in mean square. The established criteria show that stochastic perturbation is designed to ensure that the coupled IMNNs can be synchronized better by changing the state coefficients of stochastic perturbation. Finally, an illustrative example is presented to demonstrate the efficiency of the theoretical results.

1. Introduction

The memristor [1] is a kind of a nonlinear resistor with memory and nanoscale, which is widely applied in chaotic circuits, artificial neural networks, and so on. In [2], the relevant mechanisms of neural networks, such as long-term potentiation and spike time-dependent plasticity, are presented by applying basic electric circuits, and more complex mechanisms are constructed to mimic the synaptic connections in a (human) brain. During neuron transmission, synchronous resonance is a very important biological phenomenon. In recent years, a lot of systems have been investigated to realize synchronization such as time-varying switched systems, MNNs, and BAM neural networks [3–19]. In [8], a new switching pinning controller was designed to finite-time synchronization in nonlinear coupled neural networks by regulating a parameter. Therefore, it is necessary for synchronization to design a suitable controller, such as impulsive controller [11, 12, 20–22], nonchattering controller [19], and switching controller [6–8]. Based on parametric uncertainty and state dependency in the connection weight matrices of MNNs, the connection weight matrices jump in certain intervals. Duan and Huang [23] proposed periodicity and dissipativity for memristor-based neural networks with mixed delays involving both time-varying delays and distributed delays via using Mawhin-like coincidence theorem, inclusion theory, and M-matrix properties. The authors established two different types of exponential synchronization criteria for the coupled MNNs based on the master-slave (drive response) concept and discontinuous state feedback controller, and simultaneously, an estimation of the exponential synchronization rate was estimated (see [24]). It is worth pointing out that, the authors in [25, 26] added the linear coupling and interval term into MNNs to achieve two different synchronization via applying the Halanay inequality [27] and the Lyapunov method. However, there were essential differences between the synchronization results established by these two literature studies. In [25], the differential inclusion method was applied to transform the coupled connection weight matrices; moreover, a discontinuous controller was designed to ensure that multiple IMMNs can be synchronized. Li and Zheng [26] demonstrated that the coupled connection weight matrices can be decomposed by interval analysis [28],

which by weakening the matrices satisfies the conditions. Besides, the new synchronization criteria for IMMNs with linear coupling were established.

As we know, noise plays an important role in synchronization since it can stabilize an unstable system. In recent years, many scholars are very interested in synchronization of stochastic networks with time-varying delays [2, 29–38]. In 2013, the Jensen integral inequality was improved by the so-called Wirtinger-based integral inequality [33]. Furthermore, Gao et al. [29] showed that a state feedback controller and an adaptive updated law used to guarantee stochastic memristor-based neural networks with noise disturbance can be asymptotically synchronized. In [38], the authors investigated the synchronization of a stochastic multilayer dynamic network with time-varying delays and additive couplings by designing two pinning controllers. Therefore, taking stochastic perturbation into complex neural networks is very necessary and important.

Note that the stochastic systems were mainly fist-order neural networks in pervious works. In this paper, based on the model of [26], considering $x_i(t), x_i(t-\tau(t))$, $f_i(x_i(t)), f_i(x_i(t-\tau(t)))$ will produce errors, the new model of the stochastic coupled inertial memristor-based neural networks is constructed. Meanwhile, new results on synchronization in mean square are proposed. The main contributions of this paper are high-lighted as follows:

(i) Stochastic perturbation is taken into account in the second-order [39] coupled memristor-based neural networks with inertial term. Synchronization analysis becomes more challenging for the system with higher order and higher dimension.

(ii) The criterion for stochastic inertial memristor-based neural networks with linear coupling is proposed by applying the stochastic analysis techniques and the vector Lyapunov function method to realize synchronization in mean square.

(iii) An illustrative example is given to illustrate that system (1) can be synchronized under the coupled network with five nodes. Besides, system (1) has strong anti-interference.

2. Model Formulation and Preliminaries

In this paper, we consider the model of stochastic coupled inertial memristor-based neural networks (IMMNs for short) with N coupled identical nodes described by the following equation:

$$\begin{aligned} \mathrm{d}\left(\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\right) = \Bigg(&-D\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} - Cx_i((t)) + A(x_i(t))f(x_i(t)) + B(x_i(t))f(x_i(t-\tau(t))) \\ &+ c\sum_{j=1}^{N} G_{ij}\Gamma\left(\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} + x_j(t)\right)\Bigg)\mathrm{d}t + (\sigma_1 x_i((t)) + \sigma_2 x_i(t-\tau(t)) \\ &+ \sigma_3 f(x_i(t)) + \sigma_4 f(x_i(t-\tau(t))))\mathrm{d}w_t, \quad i \in N, \end{aligned} \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^{\top} \in R^n, i \in N$, $D = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n)$, $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$, and $\sigma_i = \mathrm{diag}(\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{in}), i = 1, 2, 3, 4$, are the constant positive definite matrices and $0 < \tau(t) < \tau$; the connection memristive weight matrix $A(x_i(t)) = [a_{kj}(x_{ij}(t))]_{n\times n}$ and the delayed connection of the current voltage characteristics $B(x_i(t)) = [b_{kj}(x_{ij}(t))]_{n\times n}$ satisfy the following conditions:

$$\begin{aligned} a_{kj}(x_{ij}(t)) &= \begin{cases} a_{kj}^{*}, & |x_{ij}| < T_j, \\ b_{kj}^{**}, & |x_{ij}| > T_j, \end{cases} \\ b_{kj}(x_{ij}(t)) &= \begin{cases} a_{kj}^{*}, & |x_{ij}| < T_j, \\ b_{kj}^{**}, & |x_{ij}| < T_j, \end{cases} \end{aligned} \tag{2}$$

where $T > 0$ is the switching jump and $a_{kj}^{*}, a_{kj}^{**}, , b_{kj}^{*}$, and b_{kj}^{**} are all constants, $k, j \in n$. The network coupling strength $c > 0$ is a constant, and $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) \in R^{n\times n}$ is the inner coupling matrix. $G = (G_{ij})_{N\times N}$ is the constant coupling configuration matrix representing the topological structure of the system. $G_{ij} > 0$ is defined as a link from node i to node j, otherwise, $G_{ij} = 0$. Besides, G satisfies

$$G_{ii} = -\sum_{j=1, j\neq i}^{N} G_{ij}, \quad i \in N. \tag{3}$$

The initial condition associated with system (1) is given as $x_i(s) = \varphi_i(s) \in c^1([-\tau, 0], R^n), i \in N$. And, $f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t)))^{\top}$ denotes the output of the neuron unit, which satisfies the following assumption:

(H_1): for any two different $u, v \in R$, there exists a positive scalar $l_i > 0 (i \in n)$ such that $|f_i(u) - f_i(v)| \leq l_i|u - v|$.

For stochastic systems, the It $\hat{o}$ formula plays an important role in the synchronization. Consider a general stochastic system $\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}w_t$ on $t > t_0$ with an initial value $x(t_0) = x_0 \in R^n$, where $f\colon \in R^n \times R^+ \longrightarrow R^{n\times m}$ and $g\colon \in R^n \times R^+ \longrightarrow R^{n\times m}$. Denote a general nonnegative function $V(x, t)$ on $R^n \times R^+$ to be continuously twice differentiable in x and once differentiable in t, an stochastic differential operator $t\mathrm{d}V(x, t) = \mathscr{L}V(x, t)\mathrm{d}t + V_x(x, t)g(x, t)\mathrm{d}w_t$,

where $\mathscr{L}V(x, t) = V_t(x, t) + V_x(x, t)f(x, t) + (1/2)\mathrm{trac}[g^{\top}(x, t)V_{xx}(x, t)g(x, t)]$, $V_t(x, t) = \partial V(x, t)/\partial t$, $V_x(x, t) = (\partial V(x, t)/\partial x_1, \partial V(x, t)/\partial x_1, \ldots, \partial V(x, t)/\partial x_n)$,

$V_{xx}(x,t) = (\partial^2 V(x,t)/\partial x_i \partial x_j)_{n\times m}$, and $E[\mathrm{d}V(x,t)] = E[\mathscr{L}V((x,t)\mathrm{d}t]$.

Considering $a_{kj}(x_{ij})$ and $b_{kj}(x_{ij})$ are bounded, therefore, $A(x_i(t)) \in [\underline{A}, \overline{A}], B(x_i(t)) \in [\underline{B}, \overline{B}]$, where $\underline{A} = (\underline{a}_{kj})_{n\times n}, \overline{A} = (\overline{a}_{kj})_{n\times n}, \underline{B} = (\underline{b}_{kj})_{n\times n}$, and $\overline{B} = (\overline{b}_{kj})_{n\times n}$ with $\underline{a}_{kj} = \min\{a_{kj}^*, a_{kj}^{**}\}$, $\overline{a}_{kj} = \max\{a_{kj}^*, a_{kj}^{**}\}$, $\underline{b}_{kj} = \min\{b_{kj}^*, b_{kj}^{**}\}$, and $\overline{b}_{kj} = \max\{b_{kj}^*, b_{kj}^{**}\}$.

By introducing the variable transformation,

$$r_i(t) = \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} + x_i(t). \tag{4}$$

System (1) can be transformed into

$$\begin{cases} \mathrm{d}x_i(t) = [-x_i(t) + r_i(t)]\mathrm{d}t, \\ \mathrm{d}r_i(t) = \left(-\Theta x_i(t) - \Lambda r_i(t) + [\underline{A}, \overline{A}] f(x_i(t)) + [\underline{B}, \overline{B}] f(x_i(t-\tau(t))) + c\sum_{j=1}^{N} G_{ij}\Gamma r_i(t)\right)dt \\ + (\sigma_1 x_i(t) + \sigma_2 x_i(t-\tau(t)) + \sigma_3 f(x_i(t)) + \sigma_4 f(x_i(t-\tau(t))))\mathrm{d}\omega_t, \quad i \in N, \end{cases} \tag{5}$$

where $\Theta = I + C - D, \Lambda = D - I$.

Based on interval uncertainty theory, the intervals $[\underline{A}, \overline{A}]$ and $[\underline{B}, \overline{B}]$ can be decomposed into $[\underline{A}, \overline{A}] = A_0 + [-1,1]H_A$ and $[\underline{B}, \overline{B}] = B_0 + [-1,1]H_B$, where $A_0 = (\underline{A} + \overline{A})/2, H_A = (1/2)(\overline{A} - \underline{A})$ and $B_0 = (\underline{B} + \overline{B})/2, H_B = (1/2)(\overline{B} - \underline{B})$.

Then, system (5) can be equivalently expressed as

$$\begin{cases} \mathrm{d}x_i(t) = [-x_i(t) + r_i(t)]\mathrm{d}t, \\ \mathrm{d}r_i(t) = \left(-\Theta x_i(t) - \Lambda r_i(t) + A_0 f(x_i(t)) + B_0 f(x_i(t-\tau(t))) + E\Delta(t) + c\sum_{j=1}^{N} G_{ij}\Gamma r_i(t)\right)\mathrm{d}t \\ + (\sigma_1 x_i(t) + \sigma_2 x_i(t-\tau(t)) + \sigma_3 f(x_i(t)) + \sigma_4 f(x_i(t-\tau(t))))\mathrm{d}\omega_t, \quad i \in N, \end{cases} \tag{6}$$

where

$$\begin{aligned} E\Delta((t)) &= \frac{1}{2}[-1,1](\overline{A} - \underline{A}) f(x_i(t)) + \frac{1}{2}[-1,1](\overline{B} - \underline{B}) f(x_i(t-\tau(t))) \\ &= [-1,1](H_A f(x_i(t)) + H_B f(x_i(t-\tau(t)))), \end{aligned} \tag{7}$$

and $E\Delta(t)$ is satisfied:

$$(E\Delta(t))^\top (E\Delta(t)) \le (H_A f(x_i(t)))^\top (H_A f(x_i(t))) + (H_B f(x_i(t-\tau(t))))^\top (H_B f(x_i(t-\tau(t)))). \tag{8}$$

Let $x(t) = (x_1(t)^\top, x_2(t)^\top, \ldots, x_N(t)^\top)^\top$, $r(t) = (r_1(t)^\top, r_2(t)^\top, \ldots, r_N(t)^\top)^\top$, $\mathbf{f}(x(t)) = (f(x_1(t))^\top, f(x_2(t))^\top, \ldots, f(x_N(t))^\top)^\top$, $\Theta = I_N \otimes \Theta, \Lambda = I_N \otimes \Lambda, \mathbf{G} = G \otimes \Gamma, \mathbf{A}_0 = I_N \otimes A_0, \mathbf{B}_0 = I_N \otimes B_0$, $\mathbf{E}\Delta(t) = I_N \otimes E\Delta(t), \mathbf{H}_A = I_N \otimes H_A, \mathbf{H}_B = I_N \otimes H_B$, and $\sigma_i = I_N \otimes \sigma_i, i = 1,2,3,4$. For simplicity, we use x, r, and x_τ instead of $x(t), r(t)$, and $x(t-\tau(t))$ in the following sections. Then, system (6) can be written as

$$\begin{cases} \mathrm{d}x = (-x + r)\mathrm{d}t, \\ \mathrm{d}r = (-\boldsymbol{\Theta}x - \boldsymbol{\Lambda}r + \mathbf{A}_0\mathbf{f}(x) + \mathbf{B}_0\mathbf{f}(x_\tau) + \mathbf{E}\boldsymbol{\Delta}(t) + c\mathbf{G}r)\mathrm{d}t \\ + (\boldsymbol{\sigma}_1 x + \boldsymbol{\sigma}_2 x_\tau + \boldsymbol{\sigma}_3\mathbf{f}(x) + \boldsymbol{\sigma}_4\mathbf{f}(x_\tau))\mathrm{d}\omega_t, \quad i \in N. \end{cases} \tag{9}$$

Definition 1. The stochastic coupled IMMNs (5) are said to be globally synchronized in the mean square sense if $E[\|x_i(t)-x_j(t)\|^2] \longrightarrow 0$ as $t \longrightarrow +\infty$ for any given initial conditions $\psi_i(0)$, where $i, j = 1, 2, \ldots, N$.

Lemma 1 (see [1]). *Let G be an $N \times N$ matrix in the set $T(R;k)$. Then, the $(N-1)\times(N-1)$ matrix H defined by $H = MGJ$ satisfies $MG = HM$, where G and J are given, respectively, by*

$$M = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}_{(N-1)\times N},$$
$$J = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{N\times(N-1)}, \tag{10}$$

and $T(\widehat{R}, K)$ is the set of matrices with entries in R such that the sum of the entries in each row is equal to R.

Lemma 2. *For any positive definite symmetric constant matrix $M \in R^{n\times n}$, if there exist the scalars $r_1 < r_2$ and vector function w: $[r_1, r_2] \longrightarrow R^n$ such that the concerned integrations are well defined, then the following inequality holds:*

$$\left(\int_{r_1}^{r_2} w(s)\mathrm{d}s\right)^{\top} M \int_{r_1}^{r_2} w(s)\mathrm{d}s \le r_{12} \int_{r_1}^{r_2} w^{\top}(s) M w(s)\mathrm{d}s, \tag{11}$$

where $r_{12} = r_2 - r_1$.

Lemma 3. *Given any real matrics X and Y and $Q > 0$ with appropriate dimensions, then the following matrix inequality holds:*

$$X^{\top}Y + Y^{\top}X \le X^{\top}QX + Y^{\top}Q^{-1}Y. \tag{12}$$

Lemma 4 (see [40]). *The LMI $\begin{bmatrix} S_{11}(x) & S_{12}(x) \\ S_{12}^{\top}(x) & S_{22}(x) \end{bmatrix} > 0$, where $S_{11}(x) = S_{11}^{\top}(x), S_{22}(x) = F^{\top}(x)$, and $S_{12}(x)$ depend on x, is equivalent to each of the following conditions:*

(i) $S_{11}(x) > 0, S_{22}(x) - S_{12}^{\top}(x)S_{11}^{-1}(x)S_{12}(x) > 0$

(ii) $S_{22}(x) > 0, S_{11}(x) - S_{12}^{\top}(x)S_{22}^{-1}(x)S_{12}(x) > 0$

3. Main Results

Theorem 1. *Under the assumption (H_1), $0 < \tau(t) < \tau$, and $\dot{\tau}(t) \le \mu\,(\mu > 0)$, the stochastic coupled IMMNs (1) are globally synchronized in mean square sense if there exist positive definite symmetric matrices $P, Q, V_i \in R^{n\times n}$, $i = 1, 2, 3, 4, 5, 6$, and positive diagonal matrices $R, S, T, \widetilde{R}, S_1, S_2, S_3, S_4, S_5 \in R^{n\times n}$, such that the following matrix inequalities hold: $\Phi < 0$ and $\Pi - \Psi\Phi^{-1}\Psi > 0$, where*

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \star & \Phi_{22} & \Phi_{23} \\ \star & \star & \Phi_{33} \end{bmatrix},$$

$$\Psi = \begin{bmatrix} P - Q\Theta_1 - \tau^2 R - V_1 + \frac{1}{\tau}V_1S^{-1}V_4 & V_1 + \frac{1}{\tau}V_1S^{-1}V_5 & V_1 + \frac{1}{\tau}V_1S^{-1}V_6 \\ -V_2 + \frac{1}{\tau}V_2S^{-1}V_4 & V_2 + \frac{1}{\tau}V_2S^{-1}V_5 & V_2 + \frac{1}{\tau}V_2S^{-1}V_6 \\ -V_3 + \frac{1}{\tau}V_3S^{-1}V_4 & V_3 + \frac{1}{\tau}V_3S^{-1}V_5 & V_3 + \frac{1}{\tau}V_3S^{-1}V_6 \end{bmatrix}, \tag{13}$$

$$\Pi = \begin{bmatrix} \Pi_{44} + \gamma_{44} & V_4 + \frac{1}{\tau}V_4S^{-1}V_5 & V_4 + \frac{1}{\tau}V_4S^{-1}V_6 \\ \star & 2V_5 + \frac{1}{\tau}V_5S^{-1}V_5 & V_5 + \frac{1}{\tau}V_5S^{-1}V_6 \\ \star & \star & 2V_6 - \widetilde{R} + \frac{1}{\tau}V_6S^{-1}V_6 \end{bmatrix},$$

with

$$
\begin{aligned}
\Phi_{11} &= -2P + \left(\tau^2 - 1\right)R + H_1^\top\left(Q + \tau^2 S\right)H_1 + \frac{1}{\tau}V_1 S^{-1} V_1 + \tau^2 \boldsymbol{\Theta}_1^\top \tilde{R}\boldsymbol{\Theta}_1 + \gamma_{11},\\
\Phi_{12} &= H_1^\top\left(Q + \tau^2 S\right)H_2 + \frac{1}{\tau}V_1 S^{-1} V_2,\\
\Phi_{13} &= R + \frac{1}{\tau}V_1 S^{-1} V_3,\\
\Phi_{22} &= H_2^\top\left(Q + \tau^2 S\right)H_2 + \frac{1}{\tau}V_2 S^{-1} V_2 + \gamma_{22},\\
\Phi_{23} &= \frac{1}{\tau}V_2 S^{-1} V_3,\\
\Pi_{44} &= \tau^2\left(cH - \boldsymbol{\Lambda}_1\right)^\top \tilde{R}\left(cH - \boldsymbol{\Lambda}_1\right) + \tau^2 R + 2Q\left(cH - \boldsymbol{\Lambda}_1\right) - 2V_4 + \frac{1}{\tau}V_4 S^{-1} V_4,\\
\gamma_{44} &= \left(Q + \tau^2\left(cH^\top - \boldsymbol{\Lambda}_1^\top\right)\tilde{R}\right)\left(A_0^{N-1} + H_A^{N-1}\right)S_1\left(A_0^{N-1^\top} + H_A^{N-1^\top}\right)\left(Q + \tau^2\tilde{R}\left(cH - \boldsymbol{\Lambda}_1\right)\right)\\
&\quad + \left(Q + \tau^2\left(cH^\top - \boldsymbol{\Lambda}_1^\top\right)\tilde{R}\right)\left(B_0^{N-1^\top} + H_B^{N-1^\top}\right)S_2\left(B_0^{N-1} + H_B^{N-1}\right)\left(Q + \tau^2\tilde{R}\left(cH - \boldsymbol{\Lambda}_1\right)\right),\\
\gamma_{11} &= 2\left(H_1^\top\left(Q + \tau^2 S\right)H_3 + \tau^2\boldsymbol{\Theta}_1^\top\tilde{R}\left(A_0^{N-1} + H_A^{N-1}\right)\right)L + L^\top\left(S_1^{-1} + S_4^{-1} + T\right)L\\
&\quad + L^\top\left(H_3^\top\left(Q + \tau^2 S\right)H_3 + \tau^2\left(A_0^{N-1^\top} + H_A^{N-1^\top}\right)\tilde{R}\left(A_0^{N-1} + H_A^{N-1}\right)\right)L\\
&\quad + L^\top\left(H_3^\top\left(Q + \tau^2 S\right)H_4 + \tau^2\left(A_0^{N-1^\top} + H_A^{N-1^\top}\right)\tilde{R}\left(B_0^{N-1} + H_B^{N-1}\right)\right)\\
&\quad \times S_5\left(H_4^\top\left(Q + \tau^2 S\right)H_3 + \tau^2\left(B_0^{N-1^\top} + H_B^{N-1^\top}\right)\tilde{R}\left(A_0^{N-1} + H_A^{N-1}\right)\right)L\\
&\quad + \left(H_1^\top\left(Q + \tau^2 S\right)H_4 + \tau^2\boldsymbol{\Theta}_1^\top\tilde{R}\left(B_0^{N-1} + H_B^{N-1}\right)\right)S_3\\
&\quad \times\left(H_4^\top\left(Q + \tau^2 S\right)H_1 + \tau^2\left(B_0^{N-1^\top} + H_B^{N-1^\top}\right)\tilde{R}\boldsymbol{\Theta}_1\right),\\
\gamma_{22} &= L^\top\left(S_2^{-1} + S_3^{-1} + S_5^{-1} + H_4^\top\left(Q + \tau^2 S\right)H_4 - (1-\mu)T\right)L + 2H_2^\top\left(Q + \tau^2 S\right)H_4 L\\
&\quad + \tau^2 L^\top\left(B_0^{N-1^\top} + H_B^{N-1^\top}\right)\tilde{R}\left(B_0^{N-1} + H_B^{N-1}\right)L + H_2^\top\left(Q + \tau^2 S\right)H_3 S_4 H_3^\top\left(Q + \tau^2 S\right)H_2.
\end{aligned}
\tag{14}
$$

Proof. For convenience, we set

$$
\begin{aligned}
g(t) &= -\boldsymbol{\Theta}x - \boldsymbol{\Lambda}r + \mathbf{A}_0\mathbf{f}(x) + \mathbf{B}_0\mathbf{f}(x_\tau) + \mathbf{E}\boldsymbol{\Delta}(t) + c\mathbf{G}r(t),\\
y(t) &= \boldsymbol{\sigma}_1 x + \boldsymbol{\sigma}_2 x_\tau + \boldsymbol{\sigma}_3\mathbf{f}(x) + \boldsymbol{\sigma}_4\mathbf{f}(x_\tau).
\end{aligned}
\tag{15}
$$

Consider the following Lyapunov–Krasovskii functional:

$$
V(t,x) = \sum_{i=1}^{6} V_i(t,x),
\tag{16}
$$

where

$$
\begin{aligned}
V_1(t,x) &= x(t)^\top M^\top P M x(t),\\
V_2(t,x) &= r(t)^\top M^\top Q M r(t),\\
V_3(t,x) &= \int_{t-\tau(t)}^{t} \mathbf{f}(x(s))^\top M^\top T M \mathbf{f}(x(s))\mathrm{d}s,\\
V_4(t,x) &= \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}(s)^\top M^\top R M \dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta,\\
V_5(t,x) &= \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} y(s)^\top M^\top S M y(s)\mathrm{d}s\,\mathrm{d}\theta,\\
V_6(t,x) &= \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} g(s)^\top M^\top \tilde{R} M g(s)\mathrm{d}s\,\mathrm{d}\theta,
\end{aligned}
\tag{17}
$$

for simplicity, we use $\dot{x}(t)$ instead of $\mathrm{d}x(t)/\mathrm{d}t$ in the paper.

By the It $\hat{o}$ formula, we can calculate $\mathscr{L}V(t,x)$ along system (9), and then we have

$$\mathscr{L}V(t,x) = \sum_{i=1}^{6} \mathscr{L}V_i(t,x), \tag{18}$$

and $\mathscr{L}V_i(t,x), i = 1,2,3,4,5,6$ are calculated along system (9) as follows:

$$\mathscr{L}V_1(t) = 2x^\top M^\top P(-x+r) \triangleq \xi(t)^\top \Omega_1 \xi(t), \tag{19}$$

where

$$\begin{aligned}
\xi(t)^\top &= \left(x(t)^\top M^\top, x(t-\tau(t))^\top M^\top, x(t-\tau)^\top M^\top, r(t)^\top M^\top, r(t-\tau)^\top M^\top, \left(\int_{t-\tau}^{t} Mg(s)\mathrm{d}s \right)^\top \right), \\
\Omega_1 &= \begin{bmatrix} -2P & 0 & 0 & P & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned} \tag{20}$$

$$\begin{aligned}
\mathscr{L}V_2(t) &= 2r^\top M^\top QMg(t) + y(t)^\top M^\top QMy(t) \\
&\le \xi(t)^\top \Omega_2 \xi(t) + 2x^\top M^\top H_1^\top QH_3 M\mathbf{f}(x) + 2x^\top M^\top H_1^\top QH_4 M\mathbf{f}(x_\tau) \\
&\quad + 2r^\top M^\top Q\left(\mathbf{A}_0^{N-1} + \mathbf{H}_{\mathbf{A}}^{N-1}\right) M\mathbf{f}(x) + 2r^\top M^\top Q\left(\mathbf{B}_0^{N-1} + \mathbf{H}_{\mathbf{B}}^{N-1}\right) M\mathbf{f}(x_\tau) \\
&\quad + 2x_\tau^\top M^\top H_2^\top QH_3 M\mathbf{f}(x) + 2x_\tau^\top M^\top H_2^\top QH_4 M\mathbf{f}(x_\tau) + \mathbf{f}(x)^\top M^\top H_3^\top QH_3 M\mathbf{f}(x) \\
&\quad + 2\mathbf{f}(x)^\top M^\top H_3^\top QH_4 M\mathbf{f}(x_\tau) + \mathbf{f}(x_\tau)^\top M^\top H_4^\top QH_4 M\mathbf{f}(x_\tau),
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
&M\boldsymbol{\sigma}_i = H_i M, (i = 1,2,3,4), M\boldsymbol{\Theta} = \boldsymbol{\Theta}_1 M, M\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_1 M, M\mathbf{A}_0 = \mathbf{A}_0^{N-1} M, \\
&M\mathbf{B}_0 = \mathbf{B}_0^{N-1} M, M\mathbf{H}_{\mathbf{A}} = \mathbf{H}_{\mathbf{A}}^{N-1} M, M\mathbf{H}_{\mathbf{B}} = \mathbf{H}_{\mathbf{B}}^{N-1} M, \\
&\Omega_2 = \begin{bmatrix} H_1^\top QH_1 & H_1^\top QH_2 & 0 & -Q\boldsymbol{\Theta}_1 & 0 & 0 \\ H_1^\top QH_2 & H_2^\top QH_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -Q\boldsymbol{\Theta}_1 & 0 & 0 & 2Q(cH - \boldsymbol{\Lambda}_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned} \tag{22}$$

$$\begin{aligned}
\mathscr{L}V_3(t) &= \mathbf{f}((x))^\top TM\mathbf{f}(x) - (1-\dot{\tau}(t))\mathbf{f}(x_\tau)^\top M^\top TM\mathbf{f}(x_\tau) \\
&\le \mathbf{f}(x)^\top M^\top TM\mathbf{f}(x) - (1-\mu)\mathbf{f}(x_\tau)^\top M^\top TM\mathbf{f}(x_\tau),
\end{aligned} \tag{23}$$

$$\mathscr{L}V_4(t) = \tau^2 \dot{x}(t)^\top M^\top RM\dot{x}(t) - \tau \int_{t-\tau}^{t} \dot{x}(s)^\top M^\top RM\dot{x}(s)\mathrm{d}s, \tag{24}$$

where $\dot{\tau}(t)$ is the derivative of $\tau(t)$.

By applying Lemma 2, one has

$$-\tau\int_{t-\tau}^{t}\dot{x}(s)^{\top}M^{\top}RM\dot{x}(s)\mathrm{d}s\le-\left(\int_{t-\tau}^{t}M\dot{x}(s)\mathrm{d}s\right)^{\top}R\left(\int_{t-\tau}^{t}M\dot{x}(s)\mathrm{d}s\right),\tag{25}$$

then $\mathscr{L}V_4(t)\le\xi(t)^{\top}\Omega_3\xi(t)$,where

$$\Omega_3=\begin{bmatrix}(\tau^2-1)R & 0 & R & -\tau^2R & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ R & 0 & -R & 0 & 0 & 0\\ -\tau^2R & 0 & 0 & \tau^2R & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\mathscr{L}V_5(t)=\tau^2y(t)^{\top}M^{\top}SMy(t)-\tau\int_{t-\tau}^{t}y(s)^{\top}M^{\top}SMy(s)\mathrm{d}s.\tag{26}$$

Recalling (9) and (15), it is easy to see that the following equalities hold:

$$0=2\xi(t)^{\top}V\left(r(t)-r(t-\tau)-\int_{t-\tau}^{t}g(s)\mathrm{d}s-\int_{t-\tau}^{t}y(s)\mathrm{d}w_s\right),\tag{27}$$

where
$V=\left(V_1(t)^{\top}\ \ V_2(t)^{\top}\ \ V_3(t)^{\top}\ \ V_4(t)^{\top}\ \ V_5(t)^{\top}\ \ V_6(t)^{\top}\right)^{\top}$.

By Lemma 3, we have

$$2\xi(t)^{\top}V\int_{t-\tau}^{t}y(s)\mathrm{d}w_s\le\frac{1}{\tau}\xi(t)^{\top}VS^{-1}V^{\top}\xi(t)+\tau\left(\int_{t-\tau}^{t}y(s)\mathrm{d}w_s\right)^{\top}S\left(\int_{t-\tau}^{t}y(s)\mathrm{d}w_s\right).\tag{28}$$

Then,

$$\begin{aligned}\mathscr{L}V_5(t)\le{}&\xi(t)^{\top}\Omega_4\xi(t)+\tau\left(\int_{t-\tau}^{t}My(s)\mathrm{d}w_s\right)^{\top}\left(\int_{t-\tau}^{t}My(s)\mathrm{d}w_s\right)\\&-\tau\int_{t-\tau}^{t}y(s)^{\top}M^{\top}SMy(s)\mathrm{d}s+2\tau^2x^{\top}M^{\top}H_1^{\top}SH_3M\mathbf{f}(x)\\&+2\tau^2x^{\top}M^{\top}H_1^{\top}SH_4M\mathbf{f}(x_\tau)+2\tau^2x_\tau^{\top}M^{\top}H_2^{\top}SH_3M\mathbf{f}(x)\\&+2\tau^2x_\tau^{\top}M^{\top}H_2^{\top}SH_4M\mathbf{f}(x_\tau)+\tau^2\mathbf{f}(x)^{\top}M^{\top}H_3^{\top}SH_3M\mathbf{f}(x)\\&+2\tau^2\mathbf{f}(x)^{\top}M^{\top}H_3^{\top}SH_4M\mathbf{f}(x_\tau)+\tau^2\mathbf{f}(x_\tau)^{\top}M^{\top}H_4^{\top}SH_4M\mathbf{f}(x_\tau),\end{aligned}\tag{29}$$

where

$$\Omega_4=\begin{bmatrix}\tau^2H_1^{\top}SH_1 & \tau^2H_1^{\top}SH_2 & 0 & -V_1 & V_1 & V_1\\ \star & \tau^2H_2^{\top}SH_2 & 0 & -V_2 & V_2 & V_2\\ \star & \star & 0 & -V_3 & V_3 & V_3\\ \star & \star & \star & -2V_4 & V_4 & V_4\\ \star & \star & \star & \star & 2V_5 & V_5\\ \star & \star & \star & \star & \star & 2V_6\end{bmatrix}+\frac{1}{\tau}VS^{-1}V^{\top}.\tag{30}$$

Using the property of the Itô isometry, we have

$$E\left(\left(\int_{t-\tau}^{t}My(s)\mathrm{d}w_s\right)^{\top}S\left(\int_{t-\tau}^{t}My(s)\mathrm{d}w_s\right)\right)=E\left(\int_{t-\tau}^{t}y(s)^{\top}M^{\top}SMy(s)\mathrm{d}s\right).\tag{31}$$

Based on Lemma 2 and (15), we have

$$
\begin{aligned}
\mathscr{L}V_6(t) &= \tau^2 g(t)^\top M^\top \widetilde{R} M g(t) - \tau \int_{t-\tau}^{t} g(s)^\top M^\top \widetilde{R} M g(s)\mathrm{d}s \\
&\le \tau^2 g(t)^\top M^\top \widetilde{R} M g(t) - \left(\int_{t-\tau}^{t} M g(s)\mathrm{d}s\right)^\top \widetilde{R}\left(\int_{t-\tau}^{t} M g(s)\mathrm{d}s\right) \\
&\le \xi(t)^\top \Omega_5 \xi(t) + 2\tau^2 x^\top M^\top \boldsymbol{\Theta}_1^\top \widetilde{R}\left(\mathbf{A}_0^{N-1} + \mathbf{H}_A^{N-1}\right) M\mathbf{f}(x) \\
&\quad + 2\tau^2 x^\top M^\top \boldsymbol{\Theta}_1^\top \widetilde{R}\left(\mathbf{B}_0^{N-1} + \mathbf{H}_B^{N-1}\right) M\mathbf{f}(x_\tau) \\
&\quad + 2\tau^2 r^\top M^\top \left(cH^\top - \boldsymbol{\Lambda}_1^\top\right)\widetilde{R}\left(\mathbf{A}_0^{N-1} + \mathbf{H}_A^{N-1}\right) M\mathbf{f}(x) \\
&\quad + 2\tau^2 r^\top M^\top \left(cH^\top - \boldsymbol{\Lambda}_1^\top\right)\widetilde{R}\left(\mathbf{B}_0^{N-1} + \mathbf{H}_B^{N-1}\right) M\mathbf{f}(x_\tau) \\
&\quad + \tau^2 \mathbf{f}(x)^\top M^\top \left(\mathbf{A}_0^{N-1^\top} + \mathbf{H}_A^{N-1^\top}\right)\widetilde{R}\left(\mathbf{A}_0^{N-1} + \mathbf{H}_A^{N-1}\right) M\mathbf{f}(x) \\
&\quad + 2\tau^2 \mathbf{f}(x)^\top M^\top \left(\mathbf{A}_0^{N-1^\top} + \mathbf{H}_B^{N-1^\top}\right)\widetilde{R}\left(\mathbf{B}_0^{N-1} + \mathbf{H}_B^{N-1}\right) M\mathbf{f}(x_\tau) \\
&\quad + \tau^2 \mathbf{f}(x_\tau)^\top M^\top \left(\mathbf{B}_0^{N-1^\top} + \mathbf{H}_B^{N-1^\top}\right)\widetilde{R}\left(\mathbf{B}_0^{N-1} + \mathbf{H}_B^{N-1}\right) M\mathbf{f}(x_\tau),
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
&M\mathbf{G} = HM, \\
&\Omega_5 = \begin{bmatrix}
\tau^2 \boldsymbol{\Theta}_1^\top \widetilde{R} \boldsymbol{\Theta}_1 & 0 & 0 & -\tau^2 \boldsymbol{\Theta}_1^\top \widetilde{R}\left(cH - \boldsymbol{\Lambda}_1\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\tau^2 \boldsymbol{\Theta}_1^\top \widetilde{R}\left(cH - \boldsymbol{\Lambda}_1\right) & 0 & 0 & \tau^2\left(cH^\top - \boldsymbol{\Lambda}_1^\top\right)\widetilde{R}\left(cH - \boldsymbol{\Lambda}_1\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\widetilde{R}
\end{bmatrix}.
\end{aligned}
\tag{33}
$$

Under assumption (H_1) and Lemma 3, we obtain

$$
\begin{aligned}
&2r^\top M^\top \left(Q + \tau^2\left(cH^\top - \boldsymbol{\Lambda}_1^\top\right)\widetilde{R}\right)\left(\mathbf{A}_0^{N-1} + \mathbf{H}_A^{N-1}\right) M\mathbf{f}(x) \\
&= 2\sum_{i=1}^{N-1} \left(r_i - r_{i+1}\right)^\top \left(Q + \tau^2\left(cH^\top - \boldsymbol{\Lambda}_1^\top\right)\widetilde{R}\right)\left(\mathbf{A}_0^{N-1} + \mathbf{H}_A^{N-1^\top}\right)\left(f(x_i) - f(x_{i+1})\right) \\
&\le 2\sum_{i=1}^{N-1} \left(r_i - r_{i-1}\right)^\top \left(Q + \tau^2\left(cH^\top - \boldsymbol{\Lambda}_1^\top\right)\widetilde{R}\right)\left(\mathbf{A}_0^{N-1} + \mathbf{H}_A^{N-1}\right) L\left(x_i - x_{i+1}\right) \\
&\le \sum_{i=1}^{N-1} \left(r_i - r_{i+1}\right)^\top \left(Q + \tau^2\left(cH^\top - \boldsymbol{\Lambda}_1^\top\right)\widetilde{R}\right)\left(\mathbf{A}_0^{N-1} + \mathbf{H}_A^{N-1}\right) S_1\left(\mathbf{A}_0^{N-1^\top} + \mathbf{H}_A^{N-1^\top}\right) \\
&\quad \times\left(Q + \tau^2\widetilde{R}\left(cH - \boldsymbol{\Lambda}_1\right)\right)\left(r_i - r_{i+1}\right) + \sum_{i=1}^{N-1} \left(x_i - x_{i+1}\right)^\top L^\top S_1^{-1} L\left(x_i - x_{i+1}\right) \\
&= r^\top M^\top \left(Q + \tau^2\left(cH^\top - \boldsymbol{\Lambda}_1^\top\right)\widetilde{R}\right)\left(\mathbf{A}_0^{N-1} + \mathbf{H}_A^{N-1}\right) S_1\left(\mathbf{A}_0^{N-1^\top} + \mathbf{H}_A^{N-1^\top}\right) \\
&\quad \times\left(Q + \tau^2\widetilde{R}\left(cH - \boldsymbol{\Lambda}_1\right)\right) Mr + x^\top M^\top L^\top S_1^{-1} LMx.
\end{aligned}
\tag{34}
$$

Similarly, we have

$$\begin{aligned}
&2r^\top M^\top\left(Q+\tau^2\left(cH^\top-\Lambda_1^\top\right)\widetilde{R}\right)\left(\mathbf{B}_0^{N-1}+\mathbf{H}_B^{N-1}\right)M\mathbf{f}\left(x_\tau\right)\\
&\quad\le\sum_{i=1}^{N-1}\left(r_i-r_{i+1}\right)^\top\left(Q+\tau^2\left(cH^\top-\Lambda_1^\top\right)\widetilde{R}\right)\left(\mathbf{B}_0^{N-1}+\mathbf{H}_B^{N-1}\right)S_2\\
&\qquad\times\left(\mathbf{B}_0^{N-1^\top}+\mathbf{H}_B^{N-1^\top}\right)\left(Q+\tau^2\widetilde{R}\left(cH-\Lambda_1\right)\right)\left(r_i-r_{i+1}\right)\\
&\qquad+\sum_{i=1}^{N-1}\left(x_i\left(t-\tau\left(t\right)\right)-x_{i+1}\left(t-\tau\left(t\right)\right)\right)^\top L^\top S_2^{-1}L\left(x_i\left(t-\tau\left(t\right)\right)-x_{i+1}\left(t-\tau\left(t\right)\right)\right)\\
&\quad=r^\top M^\top\left(Q+\tau^2\left(cH^\top-\Lambda_1^\top\right)\widetilde{R}\right)\left(\mathbf{B}_0^{N-1}+\mathbf{H}_B^{N-1}\right)S_2\left(\mathbf{B}_0^{N-1^\top}+\mathbf{H}_0^{N-1^\top}\right)\\
&\qquad\times\left(Q+\tau^2\widetilde{R}\left(cH-\Lambda_1\right)\right)Mr+x_\tau^\top M^\top L^\top S_2^{-1}LMx_\tau.
\end{aligned}\tag{35}$$

Next, it is easy to verify that

$$\begin{aligned}
&2x^\top M^\top\left(H_1\left(Q+\tau^2S\right)H_3+\tau^2\mathbf{\Theta}_1^\top\widetilde{R}\left(\mathbf{A}_0^{N-1}+\mathbf{H}_A^{N-1}\right)\right)M\mathbf{f}\left(x\right)\\
&\quad\le x^\top M^\top\left(2H_1^\top\left(Q+\tau^2S\right)H_3+2\tau^2\mathbf{\Theta}_1^\top\widetilde{R}\left(\mathbf{A}_0^{N-1}+\mathbf{H}_A^{N-1}\right)\right)LMx,
\end{aligned}\tag{36}$$

$$\begin{aligned}
&2x^\top M^\top\left(H_1\left(Q+\tau^2S\right)H_4+\tau^2\mathbf{\Theta}_1^\top\widetilde{R}\left(\mathbf{B}_0^{N-1}+\mathbf{H}_B^{N-1}\right)\right)M\mathbf{f}\left(x_\tau\right)\\
&\quad\le x^\top M^\top\left(H_1^\top\left(Q+\tau^2S\right)H_4+\tau^2\mathbf{\Theta}_1^\top\widetilde{R}\left(\mathbf{B}_0^{N-1}+\mathbf{H}_B^{N-1}\right)\right)S_3\\
&\qquad\times\left(H_4^\top\left(Q+\tau^2S\right)H_1+\tau^2\left(\mathbf{B}_0^{N-1^\top}+\mathbf{H}_B^{N-1^\top}\right)\widetilde{R}\mathbf{\Theta}_1\right)Mx+x_\tau^\top M^\top L^\top S_3^{-1}LMx_\tau,
\end{aligned}\tag{37}$$

$$\begin{aligned}
&2x_\tau^\top M^\top H_2^\top\left(Q+\tau^2S\right)H_3M\mathbf{f}\left(x\right)\\
&\quad\le x^\top M^\top L^\top S_4^{-1}LMx+x_\tau^\top M^\top H_2^\top\left(Q+\tau^2S\right)H_3S_4H_3^\top\left(Q+\tau^2S\right)H_2Mx_\tau,
\end{aligned}\tag{38}$$

$$2x_\tau^\top M^\top H_2^\top\left(Q+\tau^2S\right)H_4Mf\left(x_\tau\right)\le 2x_\tau^\top M^\top H_2^\top\left(Q+\tau^2S\right)H_4LMx_\tau,\tag{39}$$

$$\begin{aligned}
&\mathbf{f}\left(\left(x\right)^\top M^\top\left(\tau^2\left(\mathbf{A}_0^{N-1^\top}+\mathbf{H}_A^{N-1^\top}\right)\widetilde{R}\left(\mathbf{A}_0^{N-1^\top}+\mathbf{H}_A^{N-1^\top}\right)+H_3^\top\left(Q+\tau^2S\right)H_3+T\right)M\mathbf{f}\left(x\right)\right.\\
&\quad\le x^\top M^\top L^\top\left(T+H_3^\top\left(Q+\tau^2S\right)H_3+\tau^2\left(=\left(\mathbf{A}_0^{N-1^\top}+\mathbf{H}_A^{N-1^\top}\right)\widetilde{R}\left(\mathbf{A}_0^{N-1^\top}+\mathbf{H}_A^{N-1^\top}\right)\right)LMx,
\end{aligned}\tag{40}$$

$$\begin{aligned}
&2\mathbf{f}\left(x\right)^\top M^\top\left(\tau^2\left(\mathbf{A}_0^{N-1^\top}+\mathbf{H}_A^{N-1^\top}\right)\widetilde{R}\left(\mathbf{B}_0^{N-1}+\mathbf{H}_B^{N-1}\right)+H_3^\top\left(Q+\tau^2S\right)H_4\right)M\mathbf{f}\left(x_\tau\right)\\
&\quad\le x^\top M^\top L^\top\left(H_3^\top\left(Q+\tau^2S\right)H_4+\tau^2\left(\mathbf{A}_0^{N-1^\top}+\mathbf{H}_A^{N-1^\top}\right)\widetilde{R}\left(\mathbf{B}_0^{N-1}+\mathbf{H}_B^{N-1}\right)\right)\\
&\quad\times S_5\left(H_4^\top\left(Q+\tau^2S\right)H_3+\tau^2\left(\mathbf{B}_0^{N-1^\top}+\mathbf{H}_B^{N-1^\top}\right)\widetilde{R}\left(\mathbf{A}_0^{N-1}+\mathbf{H}_A^{N-1}\right)\right)LMx\\
&\quad+x_\tau^\top M^\top L^\top S_5^{-1}LMx_\tau,
\end{aligned}\tag{41}$$

$$\begin{aligned}
&\mathbf{f}\left(x_\tau\right)^\top M^\top\left(\tau^2\left(\mathbf{B}_0^{N-1^\top}+\mathbf{H}_B^{N-1^\top}\right)\widetilde{R}\left(\mathbf{B}_0^{N-1}+\mathbf{H}_B^{N-1}\right)\right)M\mathbf{f}\left(x_\tau\right)\\
&\quad+\mathbf{f}\left(x_\tau\right)^\top M^\top\left(H_4^\top\left(Q+\tau^2S\right)H_4-\left(1-\mu\right)T\right)M\mathbf{f}\left(x_\tau\right)\\
&\quad\le x_\tau^\top M^\top L^\top\left(H_4^\top\left(Q+\tau^2S\right)H_4-\left(1-\mu\right)T\right)LMx_\tau\\
&\quad+x_\tau^\top M^\top L^\top\left(\tau^2\left(\mathbf{B}_0^{N-1^\top}+\mathbf{H}_B^{N-1^\top}\right)\widetilde{R}\left(\mathbf{B}_0^{N-1}+\mathbf{H}_B^{N-1}\right)\right)LMx_\tau.
\end{aligned}\tag{42}$$

Substituting (19)–(42) into (18), we arrive at

$$E\left[\mathscr{L}V\left(t\right)\right]\le E\left[\xi\left(t\right)^\top\Xi\xi\left(t\right)\right]\le\lambda_{\max}\left(\Xi\right)E\left[\left\|\xi\left(t\right)\right\|^2\right],\tag{43}$$

where

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ \star & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & \Xi_{26} \\ \star & \star & \Xi_{33} & \Xi_{34} & \Xi_{35} & \Xi_{36} \\ \star & \star & \star & \Xi_{44} & \Xi_{45} & \Xi_{46} \\ \star & \star & \star & \star & \Xi_{55} & \Xi_{56} \\ \star & \star & \star & \star & \star & \Xi_{66} \end{bmatrix}, \tag{44}$$

with

$$\Xi_{11} = -2P + \left(\tau^2 - 1\right)R + H_1^\top\left(Q + \tau^2 S\right)H_1 + \frac{1}{\tau}V_1 S^{-1} V_1 + \tau^2 \boldsymbol{\Theta}_1^\top \tilde{R} \boldsymbol{\Theta}_1 + \gamma_{11},$$

$$\Xi_{12} = H_1^\top\left(Q + \tau^2 S\right)H_2 + \frac{1}{\tau}V_1 S^{-1} V_2,$$

$$\Xi_{13} = R + \frac{1}{\tau}V_1 S^{-1} V_3,$$

$$\Xi_{14} = P - Q\boldsymbol{\Theta}_1 - \tau^2 R + \frac{1}{\tau}V_1 S^{-1} V_4 - V_1,$$

$$\Xi_{15} = V_1 + \frac{1}{\tau}V_1 S^{-1} V_5,$$

$$\Xi_{16} = V_1 + \frac{1}{\tau}V_1 S^{-1} V_6,$$

$$\Xi_{22} = H_2^\top\left(Q + \tau^2 S\right)H_2 + \frac{1}{\tau}V_2 S^{-1} V_2 + \gamma_{22},$$

$$\Xi_{23} = \frac{1}{\tau}V_2 S^{-1} V_3,$$

$$\Xi_{24} = -V_2 + \frac{1}{\tau}V_2 S^{-1} V_4,$$

$$\Xi_{25} = V_2 + \frac{1}{\tau}V_2 S^{-1} V_5,$$

$$\Xi_{26} = V_2 + \frac{1}{\tau}V_2 S^{-1} V_6,$$

$$\Xi_{33} = -R + \frac{1}{\tau}V_3 S^{-1} V_3,$$

$$\Xi_{34} = -V_3 + \frac{1}{\tau}V_3 S^{-1} V_4,$$

$$\Xi_{35} = V_3 + \frac{1}{\tau}V_3 S^{-1} V_5,$$

$$
\begin{aligned}
\Xi_{36} &= V_3 + \frac{1}{\tau} V_3 S^{-1} V_6, \\
\Xi_{44} &= \tau^2 (cH - \Lambda_1)^\top \tilde{R} (cH - \Lambda_1) + \tau^2 R + 2Q(cH - \Lambda_1) - 2V_4 + \frac{1}{\tau} V_4 S^{-1} V_4 + \gamma_{44}, \\
\Xi_{45} &= V_4 + \frac{1}{\tau} V_4 S^{-1} V_5, \\
\Xi_{46} &= V_4 + \frac{1}{\tau} V_4 S^{-1} V_6, \\
\Xi_{55} &= 2V_5 + \frac{1}{\tau} V_5 S^{-1} V_5, \\
\Xi_{56} &= V_5 - \tilde{R} + \frac{1}{\tau} V_5 S^{-1} V_6, \\
\Xi_{66} &= 2V_6 - \tilde{R} + \frac{1}{\tau} V_6 S^{-1} V_6, \\
\gamma_{11} &= 2\left(H_1^\top (Q + \tau^2 S) H_3 + \tau^2 \Theta_1^\top \tilde{R} \left(A_0^{N-1} + H_A^{N-1}\right)\right) L + L^\top \left(S_1^{-1} + S_4^{-1} + T\right) L \\
&\quad + L^\top \left(H_3^\top (Q + \tau^2 S) H_3 + \tau^2 \left(A_0^{N-1^\top} + H_A^{N-1^\top}\right) \tilde{R} \left(A_0^{N-1} + H_A^{N-1}\right)\right) L \\
&\quad + L^\top \left(H_3^\top (Q + \tau^2 S) H_4 + \tau^2 \left(A_0^{N-1^\top} + H_A^{N-1^\top}\right) \tilde{R} \left(B_0^{N-1} + H_B^{N-1}\right)\right) \\
&\quad \times S_5 \left(H_4^\top (Q + \tau^2 S) H_3 + \tau^2 \left(B_0^{N-1^\top} + H_A^{N-1^\top}\right) \tilde{R} \left(A_0^{N-1} + H_A^{N-1}\right)\right) L \\
&\quad + \left(H_1^\top (Q + \tau^2 S) H_4 + \tau^2 \Theta_1^\top \tilde{R} \left(B_0^{N-1} + H_B^{N-1}\right)\right) S_3 \\
&\quad \times \left(H_4^\top (Q + \tau^2 S) H_1 + \tau^2 \left(B_0^{N-1^\top} + H_B^{N-1^\top}\right) \tilde{R} \Theta_1\right), \\
\gamma_{22} &= L^\top \left(S_2^{-1} + S_3^{-1} + S_5^{-1} + H_4^\top (Q + \tau^2 S) H_4 - (1 - \mu) T\right) L + 2H_2^\top (Q + \tau^2 S) H_4 L \\
&\quad + \tau^2 L^\top \left(B_0^{N-1^\top} + H_B^{N-1^\top}\right) \tilde{R} \left(B_0^{N-1} + H_B^{N-1}\right) L + H_2^\top (Q + \tau^2 S) H_3 S_4 H_3^\top (Q + \tau^2 S) H_2, \\
\gamma_{44} &= \left(Q + \tau^2 (cH^\top - \Lambda_1^\top) \tilde{R}\right) \left(A_0^{N-1} + H_A^{N-1}\right) S_1 \left(A_0^{N-1^\top} + H_A^{N-1^\top}\right) \left(Q + \tau^2 \tilde{R} (cH - \Lambda_1)\right) \\
&\quad + \left(Q + \tau^2 (cH^\top - \Lambda_1^\top) \tilde{R}\right) \left(B_0^{N-1^\top} + H_B^{N-1^\top}\right) S_2 \left(B_0^{N-1} + H_B^{N-1}\right) \left(Q + \tau^2 \tilde{R} (cH - \Lambda_1)\right).
\end{aligned}
\tag{45}
$$

Based on Lemma 4 and the conditions of Theorem 1, we have the matrix $\Xi < 0$; then, $\lambda_{\max}(\Xi) < 0$.

Obviously, we obtain $E[\mathscr{L}V(t)] < 0$. Hence, it follows the stochastic stability theory that the stochastic coupled IMMNs (1) are globally synchronized in mean square sense. □

4. Conclusion

Based on interval uncertainty theory, the stochastic analysis techniques and the vector Lyapunov function method are applied to realize the global synchronization in mean square sense. Nevertheless, the criterion given in Theorem 1 is different from the results in the existing literature. Moreover, the time delay is dependent, and the upper bound of the delayed derivative is 0 or less than 1. Hence, Theorem 1 would be feasible and less conservative. It is worth noting that the upper bound of the random factor can be calculated when the state parameter of the system is selected based on Theorem 1. Similarly, the stochastic coupled IMMNs (1) have strong anti-interference. In addition, the coupling of various neural nodes can be described. The example in Section 5 fully illustrates these two points.

5. Numerical Simulations

Now, we perform some numerical simulations to illustrate our analysis.

Example 1. Consider the following stochastic inertial memristor-based neural networks with five coupled identical nodes:

$$\begin{aligned} \mathrm{d}\left(\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\right) = \Bigg(&-D\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} - Cx_i(t) + A(x_i(t))f(x_i(t)) + B(x_i(t))f(x_i(t-\tau(t))) \\ &+ c\sum_{j=1}^{N} G_{ij}\Gamma\left(\left(\frac{dx_i(t)}{dt} + x_j(t)\right)\right)\mathrm{d}t + (\sigma_1 x_i(t) + \sigma_2 x_i(t-\tau(t)) \\ &+ \sigma_3 f(x_i(t)) + \sigma_4 f(x_i(t-\tau(t))))\mathrm{d}w_t, \quad i = 1,2,3,4,5, \end{aligned} \tag{46}$$

where the activation function $f(x_i(t)) = 0.6\tanh(x_i)$, the time delay $\tau(t) = e^t/(e^t+1)$, and the coupling strength $c = 0.5$. The system parameters are taken as

$$\begin{aligned} D &= \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}, \\ C &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \\ \Gamma &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ G &= \begin{bmatrix} -2.2 & 1 & 0 & 0 & 0.2 \\ 1 & -3.2 & 1.2 & 0 & 0 \\ 0 & 2 & -3.5 & 0.5 & 0 \\ 0 & 0 & 2 & -4.4 & 1.4 \\ 0.2 & 0 & 0 & 3 & -4.2 \end{bmatrix}, \\ \sigma_1 &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \\ \sigma_2 &= \begin{bmatrix} 1.8 & 0 \\ 0 & 1.8 \end{bmatrix}, \\ \sigma_3 &= \begin{bmatrix} 1.6 & 0 \\ 0 & 1.6 \end{bmatrix}, \\ \sigma_4 &= \begin{bmatrix} 1.4 & 0 \\ 0 & 1.4 \end{bmatrix}, \\ A(x_i(t)) &= \begin{bmatrix} a_{11}(x_{i1}(t)) & a_{12}(x_{i2}(t)) \\ a_{21}(x_{i1}(t)) & a_{22}(x_{i2}(t)) \end{bmatrix}, \\ B(x_i(t)) &= \begin{bmatrix} b_{11}(x_{i1}(t)) & b_{12}(x_{i2}(t)) \\ b_{21}(x_{i1}(t)) & b_{22}(x_{i2}(t)) \end{bmatrix}, \end{aligned} \tag{47}$$

with the memristor connection weights:

$$\begin{aligned} a_{11}(x) &= \begin{cases} 0.2, & |x| \le 0.1, \\ -0.2, & |x| > 0.1, \end{cases} \\ a_{12}(x) &= \begin{cases} 0.6, & |x| \le 0.1, \\ -0.6, & |x| > 0.1, \end{cases} \\ a_{21}(x) &= \begin{cases} 0.4, & |x| \le 0.1, \\ -0.4, & |x| > 0.1, \end{cases} \\ a_{22}(x) &= \begin{cases} 0.4, & |x| \le 0.1, \\ -0.4, & |x| > 0.1, \end{cases} \\ b_{11}(x) &= \begin{cases} 0.2, & |x| \le 0.1, \\ -0.2, & |x| > 0.1, \end{cases} \\ a_{12}(x) &= \begin{cases} 0.4, & |x| \le 0.1, \\ -0.4, & |x| > 0.1, \end{cases} \\ b_{21}(x) &= \begin{cases} 0.3, & |x| \le 0.1, \\ -0.3, & |x| > 0.1, \end{cases} \\ b_{22}(x) &= \begin{cases} 0.3, & |x| \le 0.1, \\ -0.3, & |x| > 0.1. \end{cases} \end{aligned} \tag{48}$$

Obviously, by calculation, we can get the Lipschitz constants $L = 0.6 * I_2$, and the upper bound of the delay $\tau = 1$.

In order to show the effectiveness of Theorem 1, we display the synchronization of each node $x_{ij}(t)(i = 1,2,3,4,5;\ j = 1,2)$ in Figure 1. Moreover, Figures 2 and 3 depict the synchronization error trajectories of $x_{i1}(t) - x_{11}(t)$ and $x_{i2}(t) - x_{12}(t), i = 1,2,3,4$.

Data Availability

No data were used to support this study.

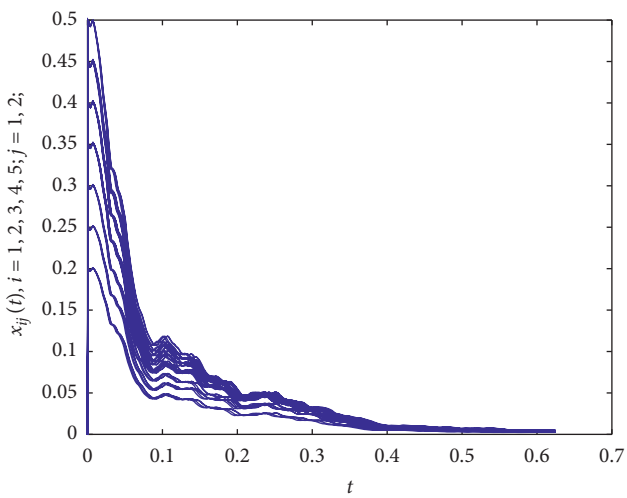

Figure 1: The synchronization of each node $x_{ij}(t)$ in Example 1.

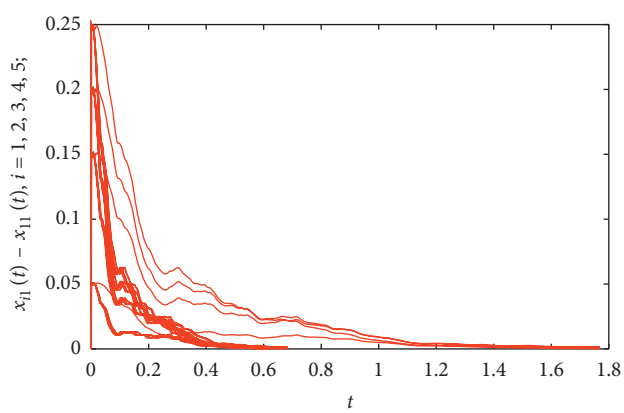

Figure 2: The synchronization error trajectories of $x_{i1}(t) - x_{11}(t)$ in Example 1.

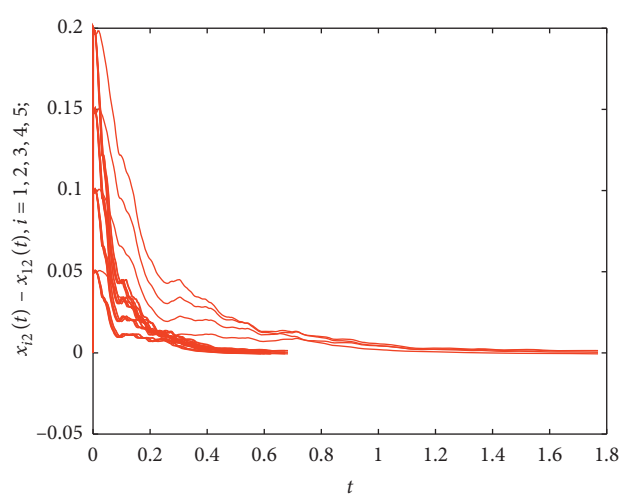

Figure 3: The synchronization error trajectories of $x_{i2}(t) - x_{12}(t)$ in Example 1.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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