In this paper, we focus on the interaction solutions of a (2 + 1)-dimensional Vakhnenko equation. By using Hirota’s transformation combined with the three-wave method and with symbolic computation, some interaction solutions which include interaction solutions between exponential and trigonometric functions and interaction solutions between exponential and trigonometric and hyperbolic functions are presented.

1. Introduction

It is well known that nonlinear partial differential equations (NPDEs) and their solutions play a significant role in interpreting many important phenomena in nonlinear sciences. A variety of powerful methods are developed for finding the exact solutions of NPDEs, such as Hirota’s method [1, 2], simplified Hirota’s method [3, 4], the Lie symmetry analysis method [5, 6], the simplest equation method [5, 6], the invariant subspace method [7], and the nonlinear steepest descent method [8]. Very recently, the lump and interaction solutions [9–11] have attracted the attention of many scholars because of lump’s applications in nonlinear optics, physics, oceanography, etc, and the interaction solutions are valuable in analyzing the nonlinear dynamics of waves in shallow water and can be used for forecasting the appearance of rogue waves [12, 13].

In order to describe high-frequent wave propagations in a relaxing medium, the Vakhnenko equation [14], the generalized Vakhnenko equation [15], and the modified generalized Vakhnenko equation [16] were presented. Many different kinds of valuable results have been obtained [17–23]. Vakhnenko and Parkes [17] obtained the two-loop soliton solution for the Vakhnenko equation using Hirota’s bilinear method. Vakhnenko et al. [18] derived a Bäcklund transformation both in the bilinear and in ordinary form for the generalized Vakhnenko equation and found the exact N-soliton solution via the inverse scattering method. Wazwaz [19] derived multiple soliton solutions and multiple singular soliton solutions for the Vakhnenko equation, the generalized Vakhnenko equation, and the modified generalized Vakhnenko equation by the simplified form of the bilinear method. Wang and Chen [20] investigated the integrability of the modified generalized Vakhnenko equation and presented the quasiperiodic solution by applying Hirota direct method and Riemann theta function. Brunelli and Sakovich [21] obtained a bi-Hamiltonian formulation for the Vakhnenko equation via the Miura-type transformations. The dynamical behaviours and exact traveling wave solutions of the modified generalized Vakhnenko equation were studied in [22]. Hashemi et al. [23] determined the Lie symmetry group, the corresponding symmetry reductions, and invariant solutions of the modified generalized Vakhnenko equation by the Li group analysis method, and so on.

In 2008, Victor et al. [24] initially derived a (2 + 1)-dimensional Vakhnenko equation

\[
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial t} + u \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \right) u + u = 0, \tag{1}
\]
which is to model high-frequency wave perturbations in relaxing high-rate active barothropic media and involves two spatial variables \(x, y\) and a temporal variable \(t\). With the aid of symbolic computation and Hirota’s method, Victor et al. [24] unearthed some typical solitary wave solutions to equation (1) and depicted single- and multivalued solutions depending on the dissipative parameter. Morrison and Parkes [15, 16] showed that 

\[
\text{\text{two spatial variables which is to model high-frequency wave perturbations in relaxing high-rate active barothropic media and involves two spatial variables} x, y \text{ and a temporal variable } t.}
\]

In this paper, we investigate interaction solutions of equation (1) via Hirota’s transformation [26] and three-wave methods [27–30].

2. Interaction Solutions of Equation (1)

Under the transformation \(W = 6(\ln f)_X\), equation (4) becomes the Hirota bilinear equation:

\[
(D^2_x f + D^2_X f + D^2_t f)(f \cdot f) = 0,
\]

where \(f = f(T_1, T_2, X)\) is a real function, and the Hirota bilinear differential operator \(D^m_x D^n_X\) was defined by [26]

\[
D^m_x D^n_X (f \cdot g) = \left(\frac{\partial}{\partial X} \frac{\partial}{\partial x} \right)^m \left(\frac{\partial}{\partial t} \frac{\partial}{\partial x} \right)^n (f(x, t)g(x', t')).
\]

In fact, equation (6) is equivalent to one special case of a generalized Bogoyavlensky–Konopelchenko equation [31] upon combining \(T_1\) and \(T_2\) as \(t\).

Consider equation (6) as well as the following novel test function:

\[
f = b_1 e^{\xi_1} + b_2 e^{-\xi_1} + b_3 \sin(\xi_2) + b_4 \cos(\xi_3) + b_5 \sinh(\xi_4),
\]

which is a combination of \(f = k_1 e^{\xi_1} + k_2 e^{-\xi_1} + k_3 \sin(\xi_2) + k_4 \cos(\xi_3) + k_5 \sinh(\xi_4)\) [28] and \(f = k_1 e^{\xi_1} + k_2 e^{-\xi_1} + k_3 \sin(\xi_2) + k_4 \sinh(\xi_3)\) [29], where \(\xi_i = \omega_i X + c_i T_1 + k_i T_2, \omega_i, c_i, k_i, b_j (1 \leq i \leq 4, 1 \leq j \leq 5)\) are unknown constants to be determined later.

Substituting (8) into (6), we can obtain an algebraic system of \(\omega_i, c_i, k_i, b_j (1 \leq i \leq 4, 1 \leq j \leq 5)\).

Case 1. Choosing \(b_5 = 0\) and with the aid of Maple, we present some solutions of the algebraic system as follows:

\[
\begin{align*}
b_1 &= 0, b_2 = b_2, b_3 = b_3, b_4 = b_4, b_5 = 0, \\
\omega_1 &= \sqrt{3} \omega_2, c_1 = \frac{\sqrt{3} - 4 \omega_1 k_1}{4 \omega_2}, \\
k_1 &= k_1, \omega_2 = \omega_2, c_2 = \frac{1 - 4 \omega_2 k_2}{4 \omega_2}, k_2 = k_2, \omega_3 = \omega_2, c_3 = \frac{1 - 4 \omega_2 k_3}{4 \omega_2}, k_3 = k_3,
\end{align*}
\]

where \(\omega_2 \neq 0\).
where $b_2 \neq 0$, $\omega_1 \neq 0$, and $\omega_3 \neq \pm \sqrt{3} \omega_1$.

where $\omega_2 \neq 0$. 

\[ b_1 = b_1, \]
\[ b_2 = 0, \]
\[ b_3 = b_3, \]
\[ b_4 = b_4, \]
\[ b_5 = 0, \]
\[ \omega_1 = \sqrt{3} \omega_2, \]
\[ c_1 = \frac{\sqrt{3} + 4 \omega_2 k_1}{4 \omega_2}, \]
\[ k_1 = k_1, \]
\[ k_2 = k_2, \]
\[ \omega_3 = -\omega_2, \]
\[ c_3 = \frac{1 + 4 \omega_2 k_3}{4 \omega_2}, \]
\[ k_3 = k_3, \]
The parameters in set (9) generate the following class of interaction solutions to equation (6) as

\[ f_1 = b_2 e^{-\left(\sqrt{3} \omega_1 x + (\sqrt{3} - 4\omega_k / 4\omega_2) T_1 + k T_2 \right)} + b_3 \sin \left( \omega_2 X + \frac{1 - 4\omega_k k_2}{4\omega_2} T_1 + k T_2 \right) + b_4 \cos \left( \omega_2 X + \frac{1 - 4\omega_k k_2}{4\omega_2} T_1 + k T_2 \right) \]

which further leads to furnish a class of interaction solutions to equation (1) as follows:

\[
\begin{cases}
    u(x, y, t) = \frac{6(G_1 f_1 - H_1^2)}{f_1}, \\
x = T_1 + \frac{6H_1}{f_1} + x_0, \\
y = T_2 + \frac{6H_1}{f_1} + y_0, \\
t = X,
\end{cases}
\]

where \( b_2, b_3, b_4, \omega_2 (\neq 0), k_1, k_2, k_3, x_0, y_0 \) are arbitrary constants and

\[
G_1 = \omega_2^2 \left( 3b_2 e^{-\left(\sqrt{3} \omega_1 x + (\sqrt{3} - 4\omega_k / 4\omega_2) T_1 + k T_2 \right)} - b_3 \sin \left( \omega_2 X + \frac{1 - 4\omega_k k_2}{4\omega_2} T_1 + k T_2 \right) - b_4 \cos \left( \omega_2 X + \frac{1 - 4\omega_k k_2}{4\omega_2} T_1 + k T_2 \right) \right),
\]

\[
H_1 = \omega_2 \left( -b_2 \sqrt{3} e^{-\left(\sqrt{3} \omega_1 x + (\sqrt{3} - 4\omega_k / 4\omega_2) T_1 + k T_2 \right)} + b_3 \cos \left( \omega_2 X + \frac{1 - 4\omega_k k_2}{4\omega_2} T_1 + k T_2 \right) - b_4 \sin \left( \omega_2 X + \frac{1 - 4\omega_k k_2}{4\omega_2} T_1 + k T_2 \right) \right).
\]

The parameters in set (10) generate the following class of interaction solutions to equation (6) as

\[
f_2 = \frac{b_2^2 \omega_2^2 \left( a_1^3 - 3\omega_2^3 \right)}{4b_2^2 \omega_2^3 \left( a_1^3 - 3\omega_2^3 \right)} e^{-\left(\sqrt{3} \omega_1 x + (\sqrt{3} - 4\omega_k / 4\omega_2) T_1 + k T_2 \right)} + b_3 e^{-\left(\sqrt{3} \omega_1 x + (\sqrt{3} - 4\omega_k / 4\omega_2) T_1 + k T_2 \right)} + b_4 \cos \left( \omega_3 X + \frac{\omega_3 - k_3 \left( a_1^3 + \omega_2^3 \right)}{\omega_1^3 + \omega_3^3} T_1 + k T_2 \right),
\]

which further leads to furnish a class of interaction solutions to equation (1) as follows:

\[
\begin{cases}
    u(x, y, t) = \frac{6(G_2 f_2 - H_2^2)}{f_2^2}, \\
x = T_1 + \frac{6H_2}{f_2} + x_0, \\
y = T_2 + \frac{6H_2}{f_2} + y_0, \\
t = X,
\end{cases}
\]

where \( b_3 (\neq 0), b_4, \omega_1 (\neq 0), \omega_3 (\neq \pm \sqrt{3} \omega_1), k_1, k_3, x_0, y_0 \) are arbitrary constants and

\[
G_2 = \frac{b_2^2 \omega_2^2 \left( a_1^3 - 3\omega_2^3 \right)}{4b_2^2 \omega_2^3 \left( a_1^3 - 3\omega_2^3 \right)} e^{-\left(\sqrt{3} \omega_1 x + (\sqrt{3} - 4\omega_k / 4\omega_2) T_1 + k T_2 \right)} + b_3 \omega_1 e^{-\left(\sqrt{3} \omega_1 x + (\sqrt{3} - 4\omega_k / 4\omega_2) T_1 + k T_2 \right)} - b_4 \omega_3 \cos \left( \omega_3 X + \frac{\omega_3 - k_3 \left( a_1^3 + \omega_2^3 \right)}{\omega_1^3 + \omega_3^3} T_1 + k T_2 \right),
\]

\[
H_2 = \frac{b_2^2 \omega_2^2 \left( a_1^3 - 3\omega_2^3 \right)}{4b_2^2 \omega_2^3 \left( a_1^3 - 3\omega_2^3 \right)} e^{-\left(\sqrt{3} \omega_1 x + (\sqrt{3} - 4\omega_k / 4\omega_2) T_1 + k T_2 \right)} - b_3 \omega_1 e^{-\left(\sqrt{3} \omega_1 x + (\sqrt{3} - 4\omega_k / 4\omega_2) T_1 + k T_2 \right)} - b_4 \omega_3 \sin \left( \omega_3 X + \frac{\omega_3 - k_3 \left( a_1^3 + \omega_2^3 \right)}{\omega_1^3 + \omega_3^3} T_1 + k T_2 \right).
\]
Setting $b_2 = -(b_4 \omega_3/2 \omega_1) \sqrt{(\omega_1^2 - 3 \omega_2^2)/(3 \omega_1^2 - \omega_2^2)}$, solution (15) becomes

\[
\begin{align*}
\mathcal{F}_2 &= \frac{b_4 \omega_3}{\omega_1} \sqrt{\frac{\omega_1^2 - 3 \omega_2^2}{3 \omega_1^2 - \omega_2^2}} \sinh \left( \omega_1 X - \frac{\omega_3 - k_3 (\omega_1^2 + \omega_2^2)}{\omega_1^2 + \omega_3^2} T_1 + k_1 T \right) + b_1 \cos \left( \omega_3 X + \frac{\omega_3 - k_3 (\omega_1^2 + \omega_2^2)}{\omega_1^2 + \omega_3^2} T_1 + k_1 T \right),
\end{align*}
\]

which further leads to furnish a class of interaction solutions to equation (1) as follows:

\[
\begin{align*}
u(x, y, t) &= \frac{6(\mathcal{G}_2 \mathcal{F}_2 - \mathcal{H}_2)}{\mathcal{F}_2}, \\
x &= T_1 + \frac{6 \mathcal{H}_2}{\mathcal{F}_2} + x_0, \\
y &= T_2 + \frac{6 \mathcal{H}_2}{\mathcal{F}_2} + y_0, \\
t &= X,
\end{align*}
\]

where $b_1, \omega_1 (\neq 0), \omega_3 ((\omega_1^2 - 3 \omega_2^2) (3 \omega_1^2 - \omega_2^2) > 0), k_1, k_3, x_0, y_0$ are arbitrary constants and

\[
\begin{align*}
\mathcal{G}_2 &= b_4 \omega_3 \left( \omega_1 \sqrt{\frac{\omega_1^2 - 3 \omega_2^2}{3 \omega_1^2 - \omega_2^2}} \sinh \left( \omega_1 X - \frac{\omega_3 - k_3 (\omega_1^2 + \omega_2^2)}{\omega_1^2 + \omega_3^2} T_1 + k_1 T_2 \right) - \omega_3 \cos \left( \omega_3 X + \frac{\omega_3 - k_3 (\omega_1^2 + \omega_2^2)}{\omega_1^2 + \omega_3^2} T_1 + k_1 T_2 \right) \right), \\
\mathcal{H}_2 &= b_4 \omega_3 \left( \omega_1 \sqrt{\frac{\omega_1^2 - 3 \omega_2^2}{3 \omega_1^2 - \omega_2^2}} \cosh \left( \omega_1 X - \frac{\omega_3 - k_3 (\omega_1^2 + \omega_2^2)}{\omega_1^2 + \omega_3^2} T_1 + k_1 T_2 \right) - \sin \left( \omega_3 X + \frac{\omega_3 - k_3 (\omega_1^2 + \omega_2^2)}{\omega_1^2 + \omega_3^2} T_1 + k_1 T_2 \right) \right).
\end{align*}
\]

The parameters in set (11) generate the following class of interaction solutions to equation (6) as

\[
f_3 = b_4 e^{\sqrt{3} \omega_2 X + \sqrt{3} \omega_3 X + (\sqrt{3} + 4 \omega_2 k_1/4 \omega_2) T_1 + k_1 T_2} + b_3 \sin \left( \omega_2 X + \frac{1 - 4 \omega_2 k_2 T_1 + k_2 T_2}{4 \omega_2} \right) + b_3 \cos \left( \omega_2 X + \frac{1 + 4 \omega_2 k_2 T_1 - k_2 T_2}{4 \omega_2} \right) + b_4 \cos \left( \omega_2 X + \frac{1 - 4 \omega_2 k_3 T_1 + k_3 T_2}{4 \omega_2} \right) - b_4 \sin \left( \omega_2 X + \frac{1 + 4 \omega_2 k_3 T_1 - k_3 T_2}{4 \omega_2} \right).
\]

Case 2. Choosing $b_4 = 0$ and with the aid of Maple, we present some solutions of the algebraic system as follows:
where $\omega_2 \neq 0$, $(\omega_2^2 - 3\omega_4^2)(3\omega_2^2 - \omega_4^2)>0$.

The parameters in set (24) generate the following class of interaction solutions to equation (6) as

$$f_4 = b_5 \omega_4 \omega_2 \left( \frac{\omega_2^2 - 3\omega_4^2}{3\omega_2^2 - \omega_4^2} \sin \left( \omega_2 X + c_2 T_1 + \frac{\omega_2 - c_2 (\omega_2^2 + \omega_4^2)}{\omega_2^2 + \omega_4^2} T_2 \right) + b_3 \sinh \left( \omega_4 X - \frac{\omega_4 + k_4 (\omega_2^2 + \omega_4^2)}{\omega_2^2 + \omega_4^2} T_1 + k_4 T_2 \right) \right),$$

which further leads to furnish a class of interaction solutions to equation (1) as follows:

$$u(x, y, t) = \frac{6(G_4 f_4 - H_4^2)}{f_4^2},$$

$$x = T_1 + \frac{6H_4}{f_4} + x_0,$$

$$y = T_2 + \frac{6H_4}{f_4} + y_0,$$

$$t = X,$$

where $b_5, \omega_2 (\neq 0), \omega_4 ((\omega_2^2 - 3\omega_4^2)(3\omega_2^2 - \omega_4^2)>0), c_2, k_4, x_0, y_0$ are arbitrary constants and

$$G_4 = b_5 \omega_4 \left( -\omega_4 \frac{\omega_2^2 - 3\omega_4^2}{3\omega_2^2 - \omega_4^2} \sin \left( \omega_2 X + c_2 T_1 + \frac{\omega_2 - c_2 (\omega_2^2 + \omega_4^2)}{\omega_2^2 + \omega_4^2} T_2 \right) + \omega_4 \sinh \left( \omega_4 X - \frac{\omega_4 + k_4 (\omega_2^2 + \omega_4^2)}{\omega_2^2 + \omega_4^2} T_1 + k_4 T_2 \right) \right),$$

$$H_4 = b_5 \omega_4 \left( \frac{\omega_2^2 - 3\omega_4^2}{3\omega_2^2 - \omega_4^2} \cos \left( \omega_2 X + c_2 T_1 + \frac{\omega_2 - c_2 (\omega_2^2 + \omega_4^2)}{\omega_2^2 + \omega_4^2} T_2 \right) + \cosh \left( \omega_4 X - \frac{\omega_4 + k_4 (\omega_2^2 + \omega_4^2)}{\omega_2^2 + \omega_4^2} T_1 + k_4 T_2 \right) \right).$$

The parameters in set (25) generate the following class of interaction solutions to equation (6) as

$$f_5 = b_1 e^{\omega_4 X - (1+4\omega_4 k_4/\omega_4) T_1 + k_4 T_2} + b_2 e^{-\omega_4 X - (1+4\omega_4 k_4/\omega_4) T_1 + k_4 T_2} - b_5 \sinh \left( \omega_4 X - \frac{1 - 4\omega_4 k_4}{4\omega_4} T_1 - k_4 T_2 \right),$$

where $b_1, b_2, b_3 = 0, b_4 = 0, b_5 = b_5,$

$$\omega_1 = \omega_1, c_1 = c_1, k_1 = k_1, \omega_2 = \omega_2, k_2 = \frac{\omega_2 - c_2 (\omega_2^2 + \omega_4^2)}{\omega_2^2 + \omega_4^2},$$

$$c_2 = c_2, \omega_4 = \omega_4, c_4 = \frac{\omega_4 + k_4 (\omega_2^2 + \omega_4^2)}{\omega_2^2 + \omega_4^2}, k_4 = k_4,$$

$$b_1 = b_1, b_2 = b_2, b_3 = 0, b_4 = 0, b_5 = b_5, \omega_1 = \omega_1, c_1 = -\frac{1 + 4\omega_4 k_4}{4\omega_1},$$

$$k_1 = k_1, \omega_2 = \omega_2, c_2 = c_2, k_2 = k_2, \omega_4 = -\omega_1, c_4 = \frac{1 - 4\omega_4 k_4}{4\omega_1}, k_4 = k_4,$$
which further leads to furnish a class of interaction solutions to equation (1) as follows:

\[
\begin{align*}
    u(x, y, t) &= \frac{6(G_5 f_5 - H_5^2)}{f_5^2}, \\
    x &= T_1 + \frac{6H_5}{f_5} + x_0, \\
    y &= T_2 + \frac{6H_5}{f_5} + y_0, \\
    t &= X,
\end{align*}
\]

where \( b_1, b_2, b_5, \omega_1 \) \((\neq 0)\), \( k_1, k_4, x_0, y_0 \) are arbitrary constants and

\[
    G_5 = \omega_1^2 \left( b_1 e^{(\omega_1 X - (1+4\omega_1 k_4)T_1 + k_4 T_2)} \\
    + b_2 e^{-(\omega_1 X - (1+4\omega_1 k_4)T_1 + k_4 T_2)} \\
    - b_3 \sinh \left( \omega_1 X - \frac{1 - 4\omega_1 k_4 T_2}{4\omega_1} \right) \right),
\]

\[
    H_5 = \omega_1 \left( b_1 e^{(\omega_1 X - (1+4\omega_1 k_4)T_1 + k_4 T_2)} \\
    - b_2 e^{-(\omega_1 X - (1+4\omega_1 k_4)T_1 + k_4 T_2)} \\
    - b_3 \cosh \left( \omega_1 X - \frac{1 - 4\omega_1 k_4 T_2}{4\omega_1} \right) \right).
\]

Setting \( b_2 = -b_1 \), solution (29) becomes

\[
    \bar{f}_5 = 2b_1 \sinh \left( \omega_1 X - \frac{1 + 4\omega_1 k_4 T_1}{4\omega_1} \right) \\
    - b_3 \sinh \left( \omega_1 X - \frac{1 - 4\omega_1 k_4 T_2}{4\omega_1} \right),
\]

which further leads to furnish a class of interaction solutions to equation (1) as follows:

\[
\begin{align*}
    u(x, y, t) &= \frac{6(G_5 \bar{f}_5 - \bar{H}_5^2)}{\bar{f}_5^2}, \\
    x &= T_1 + \frac{6\bar{H}_5}{\bar{f}_5} + x_0, \\
    y &= T_2 + \frac{6\bar{H}_5}{\bar{f}_5} + y_0, \\
    t &= X,
\end{align*}
\]

where \( b_1, b_5, \omega_1 \) \((\neq 0)\), \( k_1, k_4, x_0, y_0 \) are arbitrary constants and
\[ G_5 = \omega_1 \left( 2b_1 \cosh \left( \omega_1 X - \frac{1 + 4\omega_1 k_1}{4\omega_1} T_1 + k_1 T_2 \right) \right) \\
- b_5 \sinh \left( \omega_1 X - \frac{1 - 4\omega_1 k_1}{4\omega_1} T_1 - k_1 T_2 \right). \]

\[ H_5 = \omega_1 \left( 2b_1 \sinh \left( \omega_1 X - \frac{1 + 4\omega_1 k_1}{4\omega_1} T_1 + k_1 T_2 \right) \right) \\
- b_5 \cosh \left( \omega_1 X - \frac{1 - 4\omega_1 k_1}{4\omega_1} T_1 - k_1 T_2 \right). \]

### 3. Conclusions

In this paper, we explore a (2 + 1)-dimensional Vakhnenko equation by Hirota’s transformation combined with the three-wave method. With symbolic computation, five types of interaction solutions are obtained. It should be pointed out that some similar types of interaction solutions as that of the solution presented in this paper are not shown here for brevity. Future investigation may be applied to the method in this paper to search for lump solutions and interaction solutions between lump and other functions for the equation.

### Data Availability

The data used to support the findings of this study are included within the article. For more details, they are available from the corresponding author upon request.

### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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