Research Article

A Research on Nonendangered Population Protection Facing Biological Invasion

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1. Introduction

Biological invasion is the introduction and establishment of a species beyond its natural range where it may proliferate and spread dramatically [1]. As being predicted, the rate of biological invasion is expected to increase with the acceleration of worldwide movement of people and goods [2, 3]. Biological invasions constitute a major environmental change driver, affecting conservation, agriculture, and human health [4, 5]. Inevitably, biological invasions result in species interactions. When local community members are challenged by biological invasions, they may face novel antagonists such as predators or competitors, or they may benefit from new prey, new and underutilized host plants, or even new mutualists [6]. It is well known that some invasion cases such as the diffusion of alien invasive plants in Albania and the Mediterranean [7] and the rampant diffusion of Asian carp in America have caused great disaster in nature niche. Since biological invasions contribute a lot to rapid environmental change [8], biodiversity loss, degradation of ecosystem structure, impairment of ecosystem services [9], and significant impacts on both natural and agricultural ecosystems [10], ecological invasion and subsequent influence are receiving increasing attention from scientists and governors. To prevent some species from extinction and destruction of niche caused by the ecological invasion, ecologists carry out some strategies such as the introduction of natural enemies, substitution of invasion species, physical prevention, and chemical prevention. These artificial controlling methods are executed periodically instead of being performed continuously. The time point to carry out the artificial assistance is decided by the density of the species, but not the fixed period.

Recently, some mathematicians improved a lot in the state feedback impulsive dynamics. They not only set up the basic framework but also proposed and proved some useful theorems. Chen revised and improved some basic frame definitions and proposed a series of theorems which established the foundation of this area and had been widely applied in dealing with practical problems [11]. Zhang et al. improved the theory about order-1 periodic solution in the model of the Internet worm control [12]. Wei et al. dedicated a lot in the area of homoclinic cycle and heteroclinic orbit with respect to some kinds of dynamic model [13–15].
Inspired by the new results in state feedback impulsive model [16–26], and classical work in limit cycle [26] the authors propose a state feedback impulsive model, which is more suitable than the ODE model, to describe the procedure of ecological invasion and artificial control.

This paper is organized as follows. Section 2 proposes a state feedback impulsive model which describes the artificial auxiliary as an impulse relying on the state of variables and introduces some preliminaries of state feedback impulsive system. The existence of homoclinic cycle and order-1 periodic solution is proved in Section 3, and its stability is proved in Section 4. Some numerical simulations will be exhibited in Section 5.

2. State Feedback Impulsive Artificial Assistance Model and Its Preliminaries

We focus on the artificial control by releasing some artificial breeding ones and getting rid of some invasive ones to help the local species to survive from competition with invasive ones.

2.1. Free Developing Model and Analysis. We consider the model

\[
\begin{align*}
\frac{dx}{dt} &= x(a - cy), \\
\frac{dy}{dt} &= y(d - ex)
\end{align*}
\] (1)

to describe the competition between the two species, where \(x\) and \(y\) are densities of local and invasive species separately, and these two species compete for the same natural resource. Considering the practical significance, we only discuss model (1) in the first quadrant. It is easy to find that model (1) has two equilibriums \(O(0, 0)\) and \(E(d/e, a/c)\), where \(O(0, 0)\) is an unstable node while \(E(d/e, a/c)\) is a saddle. The dynamical features of model (1) are shown in Figure 1. Since there is only one saddle in the first quadrant, we claim that there exists no limit cycle in the area \((x, y) \mid x > 0, y > 0\).

In Figure 1, unstable and stable manifolds of saddle \(E\), shown in heavy lines, are denoted with \(L_1\) and \(L_2\). Then, the first quadrant is separated into four parts \(G_1, G_2, G_3,\) and \(G_4\). The trajectories initiated from \(G_2\) and \(G_4\) will tend to \((\infty, 0)\) which means the local species can survive the invasion without artificial assistance. On the contrary, the trend of trajectories starting from \(G_1\) and \(G_3\) implies that the local species will experience a critical shrink after a certain period, and the invasion will lead to the extinction of local species sooner or later. The endangered species should have already been protected in special ways, and in this paper, we focus on the common protection of nonendangered local species during the invasion, so we only study the trajectories in \(G_1\).

The artificial control is designed as follows. When the density of invasive species reaches a certain level which is marked with \(h\), manual intervention will be implemented. Through some technical skills, such as spraying biocide, artificial killing, and releasing natural enemies, the invasive species will be partly removed. At the same time, a certain number of local species are put into circulation to replenish its density. Since the perishing of invasive species and the replenishing of local species are carried out according to the actual state of the species and all the artificial auxiliary can be finished in a relatively short period compared with the long-term struggle between the two species, the invasion and protection procedure should be described with a state feedback impulsive model. Without loss of generality, we assume the density of invasive species before mankind help is bigger than the ordinate of equilibrium \(E\), i.e., \(h > a/c\). If the mankind help is strong enough to maintain the state of the species in areas \(G_3\) and \(G_4\), the local system will persist. However, the power of mankind is always limited, which means the invasive species cannot be eliminated thoroughly and the local species can only be complemented with limited amount. So, we also assume that the state of two species is located in area \(G_1\) and its boundaries after impulse.

2.2. Construction of the State Feedback Impulsive Model. The state feedback impulse control works in the following procedure. Once the density of invasive species \(y\) rises to the threshold value \(h\), people will cull some of them at the rate of \(\beta\) and put in some artificial local ones at the same time. The number of artificial local ones put in the circumstance is infected by density of remainder local species. In circumstantial depiction, we put in more artificial ones to keep the local species permanence if the density of local species is lower, i.e., it is inversely proportional to the density of local species \(x\) before the impulse. To construct the model, we denote the intersection of unstable manifold \(L_1\) and the threshold hold line \(y = h\) as \(A(x_h, h)\), and point \(A^+(x_h, (1 - \beta)h)\) denotes the intersection of stable manifold \(L_2\); the line consisted of the points after impulse. We also denote the intersection of threshold hold line \(y = h\) and the horizontal isocline \(dy/dt = 0\) as \(P(x_p, y_p)\) (see Figure 2). Then, the model can be described as

\[
\begin{align*}
\frac{dx}{dt} &= x(a - cy), \\
\frac{dy}{dt} &= y(d - ex)
\end{align*}
\]
Without loss of generality, the initial point \( M(x, y) \) should be restricted not in impulse set \( M[x, y] \), i.e., \( P \in \Omega = R^2 \setminus M[x, y] \). Define \( N = \varphi(M) \), and we call it phase set. The state feedback impulsive system can be denoted with \((\Omega, f, \varphi, M)\), where \( f(x, y) = (P(x, y), Q(x, y)), (x, y) \notin M[x, y] \). And the mapping of the state feedback impulsive system is expressed with \( f \circ \varphi(\bullet, t) \).

It is obvious that line segment \( \overline{AP} \) is impulsive set of model (2), while a certain part of line \( y = (1 - \beta)h \) is the corresponding phase set. While the trajectory \( L \) reaches the impulsive set \( \overline{AP} \) at point \( M_G \), the impulsive function \( \varphi \) maps it to the point \( N_G \), i.e., \( f \circ \varphi(G, t) = N_G \).

Definition 2 (see [11]). Since \( f \circ \varphi(\bullet, t) \) is a mapping on itself, there exists a point \( G^* \) in phase set \( N \) and a corresponding moment \( t^* \) satisfying \( f(G^*, t^*) = M_G \); moreover, \( \varphi(G^*) = \varphi(f(G^*, t^*)) = G^* \in N \), i.e., \( f \circ \varphi(G^*, t^*) = G^* \). Then, \( f \circ \varphi(G^*, t^*) \) is an order-1 periodic solution of model (3). And the trajectory from \( G^* \) to \( M_G \) controlled by \( f \) and the mapping segment line from \( M_G \) to \( G^* \) controlled by \( \varphi \) make up an order-1 limited cycle. Furthermore, if there is a saddle on the order-1 limited cycle, then they form an order-1 homoclinic cycle.

Definition 3 (see [11]). Suppose impulse set \( M \) and phase set \( N \) are straight lines (see Figure 2). To any point \( G \in N \), define the absolutely value of its abscissas as its coordinate. The trajectory initiating from \( G \) intersects impulse set \( M \) at \( M_G \); then, the impulse function \( \varphi \) maps \( M_G \) to \( N_G \) in phase set \( N \), and \( N_G \) is the subsequent point of \( G \). Then, we define the successor function of \( G \) as \( F(G) = |x_{N_G}| - |x_G| \).

Remark 1. The sufficient and necessary condition of \( F(G) = 0 \) is that the solution from \( G \) point is an order-1 periodic solution of system (3).

Lemma 1 (see [11]). Successor function \( F(A) \) is continuous.

Lemma 2 (see [11]). In semicontinuous dynamic system \((\Omega, f, \varphi, M)\), there exist two points \( G_1 \) and \( G_2 \) in phase set \( N \); if \( F(G_1) \cdot (G_2) < 0 \), then there must exist a point \( G^* \) between \( G_1 \) and \( G_2 \) in phase set \( N \) satisfying \( F(G^*) = 0 \), i.e., \( f \circ \varphi(G^*, t) \) is the order-1 periodic solution.

Lemma 3. If the impulsive condition is expressed with \( \phi(x, y) = 0 \), then system (3) can be rewritten as

\[
\begin{aligned}
\frac{dx}{dt} &= P(x, y), \\
\frac{dy}{dt} &= Q(x, y), \\
\Delta x &= A(x, y), \\
\Delta y &= B(x, y),
\end{aligned}
\]

where \( \phi(x, y) \neq 0 \).
Assume it has an order-\(q\) periodic solution \(\Gamma(t) = (\xi(t), \eta(t))\), and the period is \(T\). Then, the order-1 periodic solution is orbitally asymptotically stable if the factor \(\mu_2\) satisfies \(|\mu_2| < 1\), where

\[
\begin{align*}
\mu_2 &= \prod_{k=1}^{q} \Delta_k \exp \left[ \int_0^T \left( \frac{\partial P}{\partial x} (\xi(t), \eta(t)) + \frac{\partial Q}{\partial y} (\xi(t), \eta(t)) \right) dt \right], \\
\Delta_k &= \frac{P_+ ((\partial B/\partial y)(\partial \phi/\partial x) - (\partial B/\partial x)(\partial \phi/\partial y) + (\partial \phi/\partial x)) + Q_+ ((\partial A/\partial x)(\partial \phi/\partial y) - (\partial A/\partial y)(\partial \phi/\partial x) + (\partial \phi/\partial y))}{P(\partial \phi/\partial x) + Q(\partial \phi/\partial y)}.
\end{align*}
\]

Here, \(P, Q, \partial A/\partial x, \partial A/\partial y, \partial B/\partial x, \partial B/\partial y, \partial \phi/\partial x, \) and \(\partial \phi/\partial y\) are the corresponding values at point \((\xi(t_k), \eta(t_k))\), and \(P_+\) and \(Q_+\) are calculated at \((\xi(t'_{k}), \eta(t'_{k}))\).

These definitions and lemmas about the state feedback impulsive system are of great significance in the following part of this paper.

### 3. Homoclinic Cycle and Homoclinic Bifurcation

In order to discuss the existence of homoclinic cycle and order-1 periodic solution of model (2), we choose \(\alpha\) as a key parameter. For a certain \(\alpha\), the homoclinic cycle exists and then it will disappear and bifurcate an order-1 periodic solution with the changing of \(\alpha\).

#### 3.1. Homoclinic Cycle of Model (2) about Parameter \(\alpha\)

The horizontal isocline \(I: dy/dt = 0\) of model (2) expressed as dot line in Figure 3 crosses impulsive set \(M: y = h\) and phase set \(N: y = (1 - \beta)h\) at \(P\) and \(Q\) (see Figure 3), and the vertical isocline \(I': dx/dt = 0\) is a line passing saddle \(E\) and parallelled with \(x\) axis. From the trajectory properties of model (2), we know that the intersection point \(A_i\) of unstable manifold \(L_2\) and the line \(y = h\) must locate on the left side of horizontal isocline \(I\) and on the upside of vertical isocline \(I'\), while the intersection point \(A^*\) of stable manifold \(L_1\) and line \(y = (1 - \beta)h\) must be on the upper right side.

According to the control of model (2), \(L_2\), unstable manifold of saddle, \(E\), comes out from \(E\) and reaches impulsive set \(M\) at point \(A\). Restricting \(\alpha = 1\), \(A\) will be mapped by the impulsive mapping \(\varphi\) of model (2), i.e.,

\[
\varphi(x_A, h) = (x_A^*, (1 - \beta)h), \quad \varphi: A \rightarrow A^*.
\]

After the impulsive mapping, the trajectory moves along \(L_1\), the stable manifold of \(E\). So, trajectory \(EA\), impulsive line \(AA^*\), and trajectory \(A^*E\) form a closed cycle, and saddle \(E\) locates on it. Then, the homoclinic cycle forms (see Figure 3).

**Theorem 1.** If \(\alpha = 1\), then there exists an order-1 homoclinic cycle in model (2).

#### 3.2. Homoclinic Bifurcation about Parameter \(\alpha\)

\(\alpha > 1\) indicates that putting abundant artificial local ones can maintain the permanence of local species. But in practice, the massive release cannot be satisfied every time for the deficiency of artificial breeding capability. Congruous with the above, we only consider the case that the phase point locates in \(G_1\) in this paper.

In this section, we assume \(\alpha < 1\), so the phase point of \(A\) which is denoted as \(N_A\) locates on the left side of \(A^*\). Take another point \(B^*(x_{B^*}, (1 - \beta)h)\) on the left side of \(A^*\) and assume \(x_{B'} = x_{A'} - e\), where \(e > 0\) is a small enough number, which means \(B^*\) is sufficiently close to \(A^*\). Following the fundamental theories of ordinary differential equation, there exists a unique trajectory \(L\) passing through \(B^*\) and intersecting with the impulse set \(M\) at point \(B(x_{B}, h)\). According to the continuous property of solution with respect to the initial value, point \(B\) locates on the right side of \(A\) and is close enough to \(A\). Furthermore, based on the continuous characteristic of impulse function \(\varphi\) with respect to the independent variable, the phase point \(N_B = \varphi(B)\) is also close enough to \(N_A\), i.e., \(N_B\) is on the left side of \(B^*\), \(F(B^*) < 0\) (see Figure 4).

Then, we consider the trajectory passing \(Q\). Following the properties of trajectories, there exists only one trajectory passing through \(Q\) denoted as \(L^*\); it arrives at the image set \(M: y = h\) at \(D\), then the impulsive function \(\varphi\) maps \(D\) to \(N_B\).

The following discussion is based section, we will discuss the orbital stabi on the location of \(\eta\) (see Figure 5).

**Case 1.** \(x_{N_D} = x_Q = d/e\). This condition ensures the phase points \(N_D\) and \(Q\) coincide and the successor function of \(Q\) is \(F(Q) = 0\), i.e., the trajectory \(QD\) and the impulsive mapping \(DN_D\) make up an order-1 limit cycle (see Figure 5). We denote the value of \(e\) under this situation as \(\alpha^*\).

**Case 2.** \(x_{N_D} > x_Q = d/e\), i.e., \(\alpha^* < \alpha < 1\). With this condition, we have that the successor function of \(Q\) is \(F(Q) = x_{N_D} - x_Q > 0\). According to Lemma 2, there exists a point \(C \in N\) (between \(Q\) and \(B^*\)) satisfying \(F(C) = 0\). That means there exists an order-1 periodic solution of system (2) (see Figure 6).

**Case 3.** \(x_{N_D} < x_Q = d/e\), i.e., \(\alpha < \alpha^*\). Here, we can get a contrary successor function of \(Q\), i.e., \(F(Q) = x_{N_D} - x_Q < 0\). Denote the intersection of \(L\) and \(N\) (\(L\) moves upside) as \(B^*\); then, \(F(B^*) > 0\). Also, from Lemma 2, there exists a point \(C^* \in N\) (between \(B^*\) and \(Q\)) satisfying \(F(C^*) = 0\). That means there exists an order-1 periodic solution of system (2) (see Figure 7).

From the discussion of those three cases, we have the following theorem.
Theorem 2. If $\alpha < \alpha < 1$ ($\alpha$ is the threshold to make sure the phase point is located in $G_1$) holds, then there exists an order-1 periodic solution to system (2).

Remark 2. From Theorems 1 and 2, we know that system (2) has an order-1 homoclinic cycle when $\alpha = 1$. For any $\alpha < \alpha < 1$, the order-1 homoclinic cycle breaks and bifurcates an order-1 periodic solution. Then, $\alpha = 1$ is a bifurcation point for system (2).

4. Stability of the Order-1 Periodic Solution

In this section, we will discuss the orbital stability of the order-1 periodic solution.

Theorem 3. Suppose $(\xi(t), \eta(t))$ is an order-1 periodic solution of model (2) which initiates from $(\xi_0, \eta_0) \in N$ and it arrives the impulsive set at $(\xi_1, \eta_1) \in M$. The order-1 periodic solution $(\xi(t), \eta(t))$ is orbitally asymptotically stable if

$$(H), \xi_0 < \frac{d}{e} \tag{7}$$

is satisfied.

Proof 1. Assume $\Gamma(t) = (\xi(t), \eta(t))$ is an order-1 periodic solution of system (2) and the period is $T$. Denote
Following the expression of system (2), we have

\[ P(x, y) = x(a - cy), \]
\[ Q(x, y) = y(d - ex), \]
\[ A(x, y) = \alpha(x_A - x), \]
\[ B(x, y) = -\beta y, \]
\[ \phi(x, y) = y - h. \]  

Then, their values can be calculated as

\[
\frac{\partial P}{\partial x} = a - cy, \\
\frac{\partial P}{\partial y} = -ex, \\
\frac{\partial Q}{\partial x} = -ey, \\
\frac{\partial Q}{\partial y} = d - ex, \\
\frac{\partial A}{\partial x} = -\alpha, \\
\frac{\partial A}{\partial y} = 0, \\
\frac{\partial B}{\partial x} = 0, \\
\frac{\partial B}{\partial y} = -\beta, \\
\frac{\partial \phi}{\partial x} = 0, \\
\frac{\partial \phi}{\partial y} = 1.
\]  

We also have

\[
\Delta_1 = \frac{(1 - \alpha)Q_1}{Q} = \frac{(1 - \alpha)(d - e\xi_0)}{(d - e\xi_1)}
\]
\[ = (1 - \alpha)(1 - \beta) \frac{d - e\xi_0}{d - e\xi_1}, \]
\[
\int_0^T \left( \frac{\partial P}{\partial x} (\xi(t), \eta(t)) + \frac{\partial Q}{\partial y} (\xi(t), \eta(t)) \right) dt \\
= \int_0^T \left( \frac{\dot{x}}{x} + \frac{\dot{y}}{y} \right) dt \\
= \ln \frac{\xi_1}{\xi_0}, \frac{\eta_1}{\eta_0},
\]

Then,

\[
\mu_2 = \Delta_1 \cdot \exp \int_0^T \left( \frac{\partial P}{\partial x} (\xi(t), \eta(t)) + \frac{\partial Q}{\partial y} (\xi(t), \eta(t)) \right) dt \\
= (1 - \alpha) \frac{d - e\xi_0}{d - e\xi_1} \frac{\xi_1}{\xi_0}.
\]
In area $G_1$, variable $x$ shows a continuous decrease when it is controlled by the first two equations of model (2). $(x_0, \eta_0)$ and $(\xi_1, \eta_1)$ are coordinates of the intersections of the order-1 periodic solution with phase set and impulsive set. It is obvious that the trajectories move from the intersection on phase set to the one on impulsive set, so we can draw the conclusion that $\xi_1 < \xi_0$. With these conditions, if

$$(H), \xi_0 < \frac{d}{c},$$

also holds, i.e., the intersection of the order-1 periodic solution and the phase set locates on the left side of the horizontal isocline, we have $0 < (d - \varepsilon \xi_0^2/d - \varepsilon \xi_1^2) < 1$. Then, $|\mu_2| < 1$ is satisfied. Following Lemma 3, we can draw the conclusion that the order-1 periodic solution is orbitally asymptotically stable. This completes the proof.

5. Numerical Simulation and Discussion

System (1) is a state feedback impulsive dynamic system whose corresponding system without impulse is model (2). In Section 3 and Section 4, we have proved that system (2) has either a homoclinic cycle or an order-1 periodic solution according to different parameter values. To verify the result, we show some results which display the conclusion more intuitively.

In the following numerical simulation of model (2), we assume $a = 5, c = 4, d = 6, e = 3$, and $h = 2.2$ while the parameters $\alpha$ and $\beta$ which control the intensity of impulse take different values.

Figure 8 shows that homoclinic cycles exist when $\alpha = 1$ while the parameter $\beta$ takes different values. It means the existence of homoclinic cycle is not influenced by the value of parameter $\beta$ as long as the impulsive set is located above the saddle $E$.

Figure 9 exhibits the situation when $\alpha > 1$. When $\alpha > 1$, the phase point will drop in area $G_1$ which means the trajectory definitely runs to $(\infty, 0)$. In reality, if there are plenty of artificial local ones can be invested in the competition together with getting rid of a certain amount of invasive ones, then the local species will success and sustain. If artificial breeding local species is easy, impulsive replenish of enough amount of artificial local ones can help the local species hold advantage in the competition with invasive ones. So, strengthening the local species with supplying plentiful amount in single impulse is the best strategy when dealing with ecological invasion.
Figure 10: Existence of order-1 periodic solution when $\alpha < 1$ with parameter values: $a = 5, c = 4, d = 6, e = 3, h = 2.2$, and $\beta = 0.2$ and initial values $(3.5, 2.4)$. The cases of $\alpha = 0.3$, $\alpha = 0.6$, and $\alpha = 0.9$ are shown with hazel, blue, and pink lines coordinately.

Figure 11: Continued.
When $\bar{\alpha} < \alpha < 1$, there exists order-1 periodic solution in model (2), and Figure 10 shows the specific situation. It means that removing some invasive species and supplying a certain amount of local species can keep the local species and prevent it from extinction. In reality, when the local species is not quite easy to breed, proper amount investment of local species and removal of invasive ones will keep the two species into periodic circumstances. Both local species and invasive species will be kept, and their amounts appear to have a complex relation with each other. The single situation and the time series of local species $x$ and invasive species $y$ are shown in Figure 11. It is easy to find that both time period and amplitude increase with the growing of $\alpha$ based on the fixed $\bar{\beta}$. This means that more supplement of artificial breeding of local species can make the local species steady when facing disturbance.

To different $\beta$, model (2) has an order-1 periodic solution for $\bar{\alpha} < \alpha < 1$. Figure 12 shows the order-1 periodic solutions of model (2) when $\bar{\beta} = 0.2$ and initial values $(3.5, 2.4)$. (a, b) $\alpha = 0.3$. (c, d) $\alpha = 0.6$. (e, f) $\alpha = 0.9$. (a), (c), and (e) show the order-1 periodic solution, while (b), (d), and (f) exhibit the time series of variables $x$ and $y$.

**Figure 11:** The order-1 periodic solution of model (2) and time series of $x$ and $y$ with parameters $a = 5, c = 4, d = 6, e = 3, h = 2.2$, and $\bar{\beta} = 0.2$ and initial values $(3.5, 2.4)$. (a, b) $\alpha = 0.3$. (c, d) $\alpha = 0.6$. (e, f) $\alpha = 0.9$. (a), (c), and (e) show the order-1 periodic solution, while (b), (d), and (f) exhibit the time series of variables $x$ and $y$. 

Based on the theoretical and numerical results, we can conclude that killing invasive species and investing artificial breeding local species into circumstances can help the local ones resist the ecological invasion. Investing plenty of local species in one time can even support the local species win the
If the supplement of local species is not enough, we should also replenish it as abundant as we can, since the more we invest, the stronger the local species will be.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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