Research Article

On the Group Controllability of Leader-Based Continuous-Time Multiagent Systems

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The group controllability of continuous-time multiagent systems (MASs) with multiple leaders is considered in this paper, where the entire group is compartmentalized into a few subgroups. The group controllability concept of continuous-time MASs with multiple leaders is put forward, and the group controllability criteria are obtained for switching and fixed topologies, respectively. Finally, the numerical simulations are given to prove the validity of the theoretical results.

1. Introduction

In recent decades, distributed coordination control of networked MASs has become a hot and challenging issue in lots of areas, such as applied mathematics, control theory, mechanics, engineering, and neurobiology [1–10]. Studies in this area include several basic problems, such as stability, consensus and synchronization [11], containment [12], controllability [13], and formation control and tracking control [14].

The controllability problem is a key essential problem in modern control theory and attracts increasing attention due to its wide applications in engineering. An MAS is controllable if each dynamic follower can attain its desirable configuration from any initial state during finite time by regulating some leaders. In [15], Tanner first put forward the controllability problem of networked systems in a leader-following framework, where a certain agent was acted as the leader (the external control input), and an algebraic feature based on eigenvalues and eigenvectors of such system's Laplacian matrices was derived by nearest neighbor rules. Based on this, Liu et al. [16] discussed the

controllability of discrete-time MASs with a single leader based on nearest neighbor rules and derived a simple controllable condition for such a system on switching topology. Afterwards, further studies on the controllability of MASs have mainly been concentrated from graph-theoretic and algebraic-theoretic points of view, respectively. At present, many works on the controllability of MASs from the perspective of graph theory have concentrated on the basis of partitions of graph topology, such as equitable partition/relaxed equitable partition/external equitable partition in [17], connected component partition in [18], and selection of leaders [19]. Further research studies on the controllability were presented for some different special topology graphs, such as path graphs [20], cycle graphs [21], multichain topologies [22], stars and trees [23], two-time-scale topologies [24], and regular graphs [25]. Lots of algebraic controllable conditions of MASs were characterized in [26, 27].

The aforementioned results on the controllability of MASs just contained a single group. However, in engineering practice, a single group can be compartmentalized into some subgroups with the improvement of MASs'

complexity [28]. It is a very challenging work to study the controllability problem of MASs with multiple subgroups and multiple leaders considering the control law, the information topology structure between different subgroups, and the effect of dynamical leaders acting on the follower agents, which will be highlighted in this paper. More recently, the group controllability of continuous-time/discrete-time MASs leaderless with different topologies and communication restrictions in [29, 30] was studied, respectively.

Motivated by the results of previous studies, this paper aims at the group controllability of continuous-time MASs consisting of some different subgroups by adjusting the leaders. The main contributions of this paper are summed up as follows:

- (1) Different from the group controllability problem of continuous-time MASs under the leaderless framework studied based on the fixed topology in [30], the current work has considered the group controllability of continuous-time MASs under the leader-follower framework with fixed topology and switching topology, respectively, which can be expressed by the system matrices. It is obvious that different models can lead to completely different features for MASs with leaders.
- (2) The concepts of the group controllability of continuous-time MASs with multiple leaders are proposed based on switching and fixed topologies, respectively.
- (3) Sufficient and/or necessary algebraic- and graph-theoretic group controllable characterizations of continuous-time MASs with multiple leaders under the group consensus protocol are established from the system's Laplacian matrices.
- (4) The effects of subgroups and leaders on the group controllability are discussed.

The rest of this work is arranged as follows. The problem formulation is stated in Section 2. Section 3 builds the group controllability of MASs with multiple leaders. Numerical example and simulations are given in Section 4. Finally, Section 5 summarises the conclusion.

2. Problem Formulation

Consider a continuous-time MAS consisting of N agents governed by

$$\dot{x}_i(t) = u_i(t), \quad i = 1, \dots, N, \quad (1)$$

where $x_i \in \mathbb{R}$ is the state and $u_i \in \mathbb{R}$ is the control input, respectively.

In engineering practice, the whole group can be compartmentalized into a few subgroups. Without loss of generality, in this paper, such MAS consisting of $m + n + l + k$ ($m, n, l, k > 1$) agents is compartmentalized into subgroup ($\mathcal{G}_1, x^1$) and subgroup ($\mathcal{G}_2, x^2$), as shown in Figure 1.

FIGURE 1: Topology $\mathcal{G}$.

Denote $\ell_1 \triangleq \{1, \dots, m\}$, $\ell_2 \triangleq \{m+1, \dots, m+n\}$, $\ell_{1l} \triangleq \{m+n+1, \dots, m+n+l\}$, $\ell_{2l} \triangleq \{m+n+l+1, \dots, m+n+l+k\}$, $\mathcal{V}_1 \triangleq \{v_1, \dots, v_m\}$, and $\mathcal{V}_2 \triangleq \{v_{m+1}, \dots, v_{m+n}\}$; then, $\ell = \ell_1 \cup \ell_2$, $\ell_l = \ell_{1l} \cup \ell_{2l}$, and $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$. $\mathcal{N}_i$ is the i th agent's neighbor set with $\mathcal{N}_{1i} = \{v_j \in \mathcal{V}_1 : (v_j, v_i) \in \mathcal{G}\}$ and $\mathcal{N}_{2i} = \{v_j \in \mathcal{V}_2 : (v_j, v_i) \in \mathcal{G}\}$, where $\mathcal{N}_i = \mathcal{N}_{1i} \cup \mathcal{N}_{2i}$ and $\mathcal{N}_{1i} \cap \mathcal{N}_{2i} = \emptyset$. $\mathcal{N}_{1p}$ and $\mathcal{N}_{2p}$, respectively, represent the leaders' neighbor sets of subgroups 1 and 2.

Inspired by [30], the control input u_i is designed as follows:

$$u_i = \begin{cases} \sum_{j \in \mathcal{N}_{1i}} a_{ij}(x_j(t) - x_i(t)) + \sum_{j \in \mathcal{N}_{2i}} a_{ij}x_j(t) \\ \quad + \sum_{q \in \mathcal{N}_{1p}} b_{iq}(y_q(t) - x_i(t)), & i \in \ell_1, \\ \sum_{j \in \mathcal{N}_{2i}} a_{ij}(x_j(t) - x_i(t)) + \sum_{j \in \mathcal{N}_{1i}} a_{ij}x_j(t) \\ \quad + \sum_{q \in \mathcal{N}_{2p}} b_{iq}(y_q(t) - x_i(t)), & i \in \ell_2, \end{cases} \quad (2)$$

where $a_{ij} \in \mathbb{R}$, $b_{iq} \geq 0$, $\forall i, j, q \in \ell_1, \ell_2$.

Remark 1. It is noted that the (i, j) th entry of the system's adjacency matrix, denoted as a_{ij} , in this paper, can be allowed to be negative, which makes it more difficult and complex to discuss the group controllability problem since there are negative factors in the coupling links between different subgroups.

Supposed that $x^1 \triangleq (x_1, \dots, x_m)^T$ and $x^2 \triangleq (x_{m+1}, \dots, x_{m+n})^T$ are the state vectors of follower agents in $\mathcal{G}_1$ and $\mathcal{G}_2$ and $y^1 \triangleq (y_1, \dots, y_l)^T$ and $y^2 \triangleq (y_{l+1}, \dots, y_{l+p})^T$ are the state vectors of the leader agents in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then, the dynamics of the followers in system (1) becomes

$$\begin{aligned} \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} &= \begin{bmatrix} -L_1 - R_1 & C_1 \\ C_2 & -L_2 - R_2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \\ &\triangleq \begin{bmatrix} A_1 & C_1 \\ C_2 & A_2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}, \end{aligned} \quad (3)$$

where $L_1 = [l_{ij}] \in \mathbb{R}^{m \times m}$ and $L_2 = [l_{ij}] \in \mathbb{R}^{n \times n}$ are the Laplacian matrices of graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, $R_1 = \text{diag}(\sum_{q \in \mathcal{N}_{1q}} b_{1q}, \dots, \sum_{q \in \mathcal{N}_{1q}} b_{mq}) \in \mathbb{R}^{m \times m}$, $R_2 = \text{diag}(\sum_{q \in \mathcal{N}_{2q}} b_{m+1,q}, \dots, \sum_{q \in \mathcal{N}_{2q}} b_{m+n,q}) \in \mathbb{R}^{n \times n}$,

$$\begin{aligned} A_1 &\triangleq -L_1 - R_1 \in \mathbb{R}^{m \times m}, \quad B_1 \in \mathbb{R}^{m \times l}, \quad C_1 \in \mathbb{R}^{m \times n}, \\ A_2 &\triangleq -L_2 - R_2 \in \mathbb{R}^{n \times n}, \quad B_2 \in \mathbb{R}^{n \times k}, \quad C_2 \in \mathbb{R}^{n \times m}. \end{aligned} \quad (4)$$

Remark 2. Furthermore, because a_{ij} can be allowed to be negative, nonzero controller gains can be appropriately selected as long as L_1 and L_2 are Laplacian matrices. In essence, the group controllability of continuous-time leader-based MASs cannot be affected by the controller gains.

In order to discuss the group controllability problem of system (3), its equivalent augmented system can be described by

$$\begin{cases} \dot{x}^1 = A_1 x^1 + [C_1 \ B_1] \begin{bmatrix} x^2 \\ y^1 \end{bmatrix}, \\ \dot{x}^2 = A_2 x^2 + [C_2 \ B_2] \begin{bmatrix} x^1 \\ y^2 \end{bmatrix}, \end{cases} \quad (5)$$

where (x^2, y^1) and (x^1, y^2) are the inputs of subgroup $(\mathcal{G}_1, x^1)$ and subgroup $(\mathcal{G}_2, x^2)$, respectively.

3. Group Controllability Analysis

This section discusses the group controllability of continuous-time leader-based MASs and establishes the group controllability criteria by adjusting appropriate leaders with switching and fixed topologies, respectively.

3.1. Group Controllability on Switching Topology. Similar to literature [29], corresponding system (5) with switching topology can be described as

$$\begin{cases} \dot{x}^1 = A_{1\sigma(t)} x^1 + [C_1 \ B_1]_{\sigma(t)} \begin{bmatrix} x^2 \\ y^1 \end{bmatrix}, \\ \dot{x}^2 = A_{2\sigma(t)} x^2 + [C_2 \ B_2]_{\sigma(t)} \begin{bmatrix} x^1 \\ y^2 \end{bmatrix}, \end{cases} \quad (6)$$

where the switching path $\sigma(t): \mathbb{R}^+ \rightarrow \{1, \dots, K\}$ can be described by a piecewise constant scalar function, which presents the coupling links of such time-variant system, and K is the number of probable switching topologies. Moreover, we select $(A_{ip}, [C_{ip} B_{ip}])$ ($i = 1, 2, p = 1, 2, \dots, K$) to achieve the system realizations when $\sigma(t) = p$.

Some relevant important concepts will be introduced in the following, and more details can be seen in [29].

Definition 1 (see [29], switching sequence). A finite scalars' set $\pi \triangleq \{i_0, \dots, i_{P-1}\}$ is said to be a switching sequence, where $P \in (0, \infty)$ indicates the length of π , and a switching path $\sigma(p)$ is defined as $\sigma(p) = i_p$ if $p \in \underline{P}$ ($\underline{P} \triangleq 0, 1, \dots, P-1$) with $i_p \in \{1, 2, \dots, P\}$ be the index of the p th realization.

Definition 2 (group switching controllability). A nonzero state x of system (6) attains group switching controllability if

- (1) There are a time instant $P \in (0, \infty)$, a switching path $\sigma: \underline{P} \rightarrow 1, \dots, K$, and the input (x^2, y^1) for $t \in \underline{P}$ such that $x^1(0) = x^1$ and $x^1(P) = 0$
- (2) There are a time instant $P \in (0, \infty)$, a switching path $\sigma: \underline{P} \rightarrow 1, \dots, K$, and the input (x^1, y^2) for $t \in \underline{P}$ such that $x^2(0) = x^2$ and $x^2(P) = 0$

Definition 3 (see [29], column space). For a preset matrix $B_{p \times m} = [b_1, \dots, b_m]$, the column space $\mathcal{R}(B)$ is spanned by vectors $b_1, b_2, \dots, b_m$, denoted as $\mathcal{R}(B) \triangleq \text{span}\{b_1, \dots, b_m\}$.

Lemma 1. (see [29]). For matrices $A_i \in \mathbb{R}^{p \times m_i}$, $i \in \underline{r} \triangleq \{1, 2, \dots, r\}$ and $B = [A_1, A_2, \dots, A_r] \in \mathbb{R}^{p \times m}$, $m = \sum_{i=1}^r m_i$, then $\mathcal{R}(B) = \sum_{i=1}^r \mathcal{R}(A_i)$.

Definition 4 (see [29], cyclic invariant subspace). For $A \in \mathbb{R}^{N \times N}$ and a linear subspace $\mathcal{W} \subseteq \mathbb{R}^N$, $\langle A | \mathcal{W} \rangle$ is called as the $\mathcal{R}$ -cyclic invariant subspace indicated as $\langle A | \mathcal{W} \rangle \triangleq \sum_{i=1}^K A^{i-1} \mathcal{W}$.

For notational simplicity, let $\langle A_1 | (C_1 B_1) \rangle \triangleq \langle A_1 | \mathcal{R}(C_1 B_1) \rangle \triangleq \langle A_1 | \mathcal{R}(r_{ij}) \rangle$ for $i = 1$ and $j = 1, 2, \dots, l+n$, and $\langle A_2 | (C_2 B_2) \rangle \triangleq \langle A_2 | \mathcal{R}(r_{ij}) \rangle$ for $i = 2$ and $j = 1, 2, \dots, m+k$, where $(C_i B_i) \triangleq (r_{ij})$. For system (6), the subspace sequence is defined as

$$\begin{aligned} \mathcal{W}_{11} &= \sum_{i=1}^K \langle A_{1i} | r_{1i} \rangle, \mathcal{W}_{12} = \sum_{i=1}^K \langle A_{1i} | \mathcal{W}_{11} \rangle, \dots, \mathcal{W}_{1m} \\ &= \sum_{i=1}^K \langle A_{1i} | \mathcal{W}_{1(m-1)} \rangle, \\ \mathcal{W}_{21} &= \sum_{i=1}^K \langle A_{2i} | r_{2i} \rangle, \mathcal{W}_{22} = \sum_{i=1}^K \langle A_{2i} | \mathcal{W}_{21} \rangle, \dots, \mathcal{W}_{2n} \\ &= \sum_{i=1}^K \langle A_{2i} | \mathcal{W}_{2(n-1)} \rangle. \end{aligned} \quad (7)$$

Lemma 2. System (6) attains group switching controllability iff $\mathcal{W}_{1m} = \mathbb{R}^m$ and $\mathcal{W}_{2n} = \mathbb{R}^n$.

Proof. Similar proof can be referred from that of Lemma 2 in [29]; here, it is omitted. $\square$

Theorem 1. System (6) attains group switching controllability if

$$\begin{aligned} \sum_{i=1}^K \mathcal{R}(r_{1i}) &= \mathbb{R}^m, \\ \sum_{i=1}^K \mathcal{R}(r_{2i}) &= \mathbb{R}^n. \end{aligned} \quad (8)$$

Proof. Obviously, for $i = 1, 2, \dots, K$,

$$\begin{aligned}\mathcal{R}(r_{li}) &\subseteq \mathcal{R}(r_{li}) + \mathcal{R}(A_{li}r_{li}) + \cdots + \mathcal{R}(A_{li}^{m-1}r_{li}) \\ &= \langle A_{li} | r_{li} \rangle,\end{aligned}\quad (9)$$

we have

$$\begin{aligned}\mathbb{R}^m &= \mathcal{R}(r_{11}) + \mathcal{R}(r_{12}) + \cdots + \mathcal{R}(r_{1K}) \\ &\subseteq \langle A_{11} | r_{11} \rangle + \langle A_{12} | r_{12} \rangle + \cdots + \langle A_{1K} | r_{1K} \rangle \\ &= \mathcal{W}_{11} \subseteq \mathcal{W}_{12} \subseteq \cdots \subseteq \mathcal{W}_{1m}.\end{aligned}\quad (10)$$

On the contrary, it is easy to know $\mathcal{W}_{1m} \subseteq \mathbb{R}^m$. Therefore, we can have $\mathcal{W}_{1m} = \mathbb{R}^m$. For the subspace $\mathcal{W}_{2n}$, we can also have the similar result $\mathcal{W}_{2n} = \mathbb{R}^n$. From Lemmas 1 and 2, the assertion holds. $\square$

Remark 3. Theorem 1 provides an important and simple method to check the controllability of continuous-time MASs with leaders by designing a switching path. At the same time, it is noted that the group controllability of continuous-time MASs with leaders can depend on leader-to-follower information communications (i.e., matrices B_i) and the subgroup-to-subgroup information communications (i.e., matrices C_i) regardless of the internal information communications between subgroups (i.e., matrices A_i) whether the topology of the internal network is fixed or switching, that is, the controllability of the subgroups is not required, which provides an important convenience for designing a switching path to ensure the group controllability for continuous-time MASs with switching topology.

In the following, some special important cases are discussed.

3.2. Group Controllability on Fixed Topology. When $\sigma(t) = 1$, system (6) (equivalently, (5)) presents an MAS based on fixed topology.

Definition 5 (see [30] (group controllability)). A nonzero state x of system (5) attains group controllability if

- (1) There are a finite time $T \in J$ and the input (x^2, y^1) such that $x^1(0) = x^1$ and $x^1(T) = 0$
- (2) There are a finite time $T \in J$ and the input (x^1, y^2) such that $x^2(0) = x^2$ and $x^2(T) = 0$

Lemma 3. System (5) attains group controllability iff $\text{rank}(Q_1) = m$ and $\text{rank}(Q_2) = n$, where

$$\begin{aligned}Q_1 &= [(C_1 B_1) : A_1(C_1 B_1) : A_1^2(C_1 B_1) : \cdots : A_1^{m-1}(C_1 B_1)], \\ Q_2 &= [(C_2 B_2) : A_2(C_2 B_2) : A_2^2(C_2 B_2) : \cdots : A_2^{n-1}(C_2 B_2)].\end{aligned}\quad (11)$$

Here, Q_1 is the controllability matrix of $(\mathcal{G}_1, x^1)$, and Q_2 is the controllability matrix of $(\mathcal{G}_2, x^2)$.

Proof. The result is obvious from Definition 5. $\square$

Remark 4. From Lemma 3, it is too complex to compute the controllability matrices of system (5). On this basis, the group controllability of such MAS with leaders is shown by the technique of PBH rank test.

Theorem 2 (PBH rank test). System (5) attains group controllability iff system (5) satisfies

- (1) $\text{rank}(sI - A_1, C_1, B_1) = m$ and $\text{rank}(tI - A_2, C_2, B_2) = n$, $\forall s, t \in \mathbb{C}$, where $\mathbb{C}$ is a complex number set
- (2) $\text{rank}(\lambda_i I - A_1, C_1, B_1) = m$ and $\text{rank}(\mu_i I - A_2, C_2, B_2) = n$, where $\lambda_i (\forall i = 1, \dots, m)$ and $\mu_i (\forall i = 1, \dots, n)$ are, respectively, the eigenvalues of A_1 and A_2

Proof. Obviously, if condition (1) holds, condition (2) absolutely holds. Therefore, it is only necessary to prove that condition (1) is true.

Necessity: by contradiction, supposed that $\exists s \in \mathbb{C}$; then,

$$\text{rank}(sI - A_1, C_1, B_1) < m, \quad (12)$$

and then the rows of $[sI - A_1, C_1, B_1]$ are linearly dependent. Thus, $\exists \alpha' (\neq 0)$ such that $\alpha' [sI - A_1, C_1, B_1] = 0$. Therefore,

$$\begin{aligned}\alpha' A_1 &= \alpha' s I, \\ \alpha' C_1 &= 0, \\ \alpha' B_1 &= 0.\end{aligned}\quad (13)$$

Moreover, we can have

$$\begin{aligned}\alpha' Q_1 &= \alpha' [C_1 A_1 C_1 A_1^2 C_1 \cdots A_1^{m-1} C_1 : B_1 A_1 B_1 A_1^2 B_1 \cdots A_1^{m-1} B_1] \\ &= [\alpha' C_1 \alpha' A_1 C_1 \alpha' A_1^2 C_1 \cdots \alpha' A_1^{m-1} C_1 : \alpha' B_1 \alpha' A_1 B_1 \alpha' A_1^2 B_1 \cdots \alpha' A_1^{m-1} B_1] \\ &= [\alpha' B_1 \alpha' A_1 B_1 \cdots \alpha' A_1^{m-1} B_1 : \alpha' C_1 \alpha' A_1 C_1 \cdots \alpha' A_1^{m-1} C_1] = 0.\end{aligned}\quad (14)$$

Since $\alpha \neq 0$, then there must be $\text{rank}(Q_1) < m$, which implies that system (5) is uncontrollable, contradicting to

the assertion that system (5) attains the group controllability. The necessity of (1) is proved.

Sufficiency: by contradiction, assumed that system (5) is uncontrollable, then $\exists \lambda \in \mathbb{C}$ of A_1 , which corresponds to the eigenvector $\beta (\neq 0)$ satisfying

$$\begin{aligned}\beta' A_1 &= \lambda \beta', \\ \beta' C_1 &= 0, \\ \beta' B_1 &= 0,\end{aligned}\tag{15}$$

and then $\beta' [\lambda I - A_1, C_1, B_1] = 0$ so that $\text{rank}(\lambda I - A_1, C_1, B_1) < m$. This contradicts to $\text{rank}(sI - A_1, C_1, B_1) = m$ for $\forall s \in \mathbb{C}$. The sufficiency of (1) is proved. $\square$

Theorem 3. If $L_i^T = L_i$ ($i = 1, 2$), system (5) attains group controllability iff

- (1) The eigenvalues of A_i are different
- (2) The eigenvectors of A_i are unorthogonal to at least one column of B_i or C_i

Proof. Since $L_1^T = L_1$, then $A_1^T = A_1$, which can be displayed as $A_1 = U_1 \Lambda_1 U_1^T$, where the columns of U_1 and the diagonal matrix Λ_1 are made up of orthogonal eigenvectors and eigenvalues of A_1 , respectively. Moreover, $U_1 U_1^T = I$; then, the controllability matrix of such a system can be expressed as

$$\begin{aligned}Q_1 &= [B_1 \quad A_1 B_1 \quad A_1^2 B_1 \quad \cdots \quad A_1^{m-1} B_1 : C_1 \quad A_1 C_1 \quad A_1^2 C_1 \quad \cdots \quad A_1^{m-1} C_1] \\ &= [U_1 U_1^T B_1 \quad U_1 \Lambda_1 U_1^T B_1 \quad U_1 \Lambda_1^2 U_1^T B_1 \quad \cdots \quad U_1 \Lambda_1^{m-1} U_1^T B_1 : U_1 U_1^T C_1 \quad U_1 \Lambda_1 U_1^T C_1 \quad \cdots \quad U_1 \Lambda_1^{m-1} U_1^T C_1] \\ &= U_1 [U_1^T B_1 \quad \Lambda_1 U_1^T B_1 \quad \Lambda_1^2 U_1^T B_1 \quad \cdots \quad \Lambda_1^{m-1} U_1^T B_1 : U_1^T C_1 \quad \Lambda_1 U_1^T C_1 \quad \Lambda_1^2 U_1^T C_1 \quad \cdots \quad \Lambda_1^{m-1} U_1^T C_1] \\ &\triangleq U_1 \widetilde{Q}_1,\end{aligned}\tag{16}$$

where $\widetilde{Q}_1 = [U_1^T B_1 \Lambda_1 U_1^T B_1 \Lambda_1^2 U_1^T B_1 \cdots \Lambda_1^{m-1} U_1^T B_1 : U_1^T C_1 \Lambda_1 U_1^T C_1 \Lambda_1^2 U_1^T C_1 \cdots \Lambda_1^{m-1} U_1^T C_1]$.

Since U_1 consists of the orthogonal eigenvectors of A_1 , then U_1 is nonsingular, which implies that $\text{rank}(Q_1) = \text{rank}(\widetilde{Q}_1)$. Let λ_i and η_i be the eigenvalues and their corresponding eigenvectors of A_1 , respectively, for $i = 1, 2, \dots, m$.

For the convenience of discussion, let $U_1^T B_1 \triangleq [r_1, \dots, r_l]$, $U_1^T C_1 \triangleq [p_1, \dots, p_n]$, $r_i \triangleq [r_{1i}, \dots, r_{mi}]^T$ with $r_{ki} = (\eta_k, b_i)$, ($k = 1, 2, \dots, m; i = 1, 2, \dots, l$), and $p_i = [p_{1i}, p_{2i}, \dots, p_{mi}]^T$ ($i = 1, 2, \dots, n$); then,

$$\begin{aligned}
\widetilde{Q}_1 &= \left\{ (r_1, r_2, \dots, r_l), \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix} (r_1, r_2, \dots, r_l), \dots, \begin{bmatrix} \lambda_1^{m-1} & & & \\ & \lambda_2^{m-1} & & \\ & & \ddots & \\ & & & \lambda_m^{m-1} \end{bmatrix} (r_1, r_2, \dots, r_l) : \right. \\
&\quad \left. (p_1, p_2, \dots, p_n), \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix} (p_1, p_2, \dots, p_n), \dots, \begin{bmatrix} \lambda_1^{m-1} & & & \\ & \lambda_2^{m-1} & & \\ & & \ddots & \\ & & & \lambda_m^{m-1} \end{bmatrix} (p_1, p_2, \dots, p_n) \right\} \\
&\longrightarrow \{ (r_1, \Lambda_1 r_1, \dots, \Lambda_1^{m-1} r_1), \dots, (r_l, \Lambda_1 r_l, \dots, \Lambda_1^{m-1} r_l) : (p_1, \Lambda_1 p_1, \dots, \Lambda_1^{m-1} p_1), \dots, (p_n, \Lambda_1 p_n, \dots, \Lambda_1^{m-1} p_n) \} \\
&\longrightarrow \left\{ \begin{bmatrix} r_{11} & \lambda_1 r_{11} & \lambda_1^2 r_{11} & \dots & \lambda_1^{m-1} r_{11} \\ r_{21} & \lambda_2 r_{21} & \lambda_2^2 r_{21} & \dots & \lambda_2^{m-1} r_{21} \\ \vdots & \vdots & \vdots & & \vdots \\ r_{m1} & \lambda_m r_{m1} & \lambda_m^2 r_{m1} & \dots & \lambda_m^{m-1} r_{m1} \end{bmatrix}, \dots, \begin{bmatrix} r_{1l} & \lambda_1 r_{1l} & \lambda_1^2 r_{1l} & \dots & \lambda_1^{m-1} r_{1l} \\ r_{2l} & \lambda_2 r_{2l} & \lambda_2^2 r_{2l} & \dots & \lambda_2^{m-1} r_{2l} \\ \vdots & \vdots & \vdots & & \vdots \\ r_{ml} & \lambda_m r_{ml} & \lambda_m^2 r_{ml} & \dots & \lambda_m^{m-1} r_{ml} \end{bmatrix} \right\} : \\
&\quad \left\{ \begin{bmatrix} p_{11} & \lambda_1 p_{11} & \lambda_1^2 p_{11} & \dots & \lambda_1^{m-1} p_{11} \\ p_{21} & \lambda_2 p_{21} & \lambda_2^2 p_{21} & \dots & \lambda_2^{m-1} p_{21} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{m1} & \lambda_m p_{m1} & \lambda_m^2 p_{m1} & \dots & \lambda_m^{m-1} p_{m1} \end{bmatrix}, \dots, \begin{bmatrix} p_{1n} & \lambda_1 p_{1n} & \lambda_1^2 p_{1n} & \dots & \lambda_1^{m-1} p_{1n} \\ p_{2n} & \lambda_2 p_{2n} & \lambda_2^2 p_{2n} & \dots & \lambda_2^{m-1} p_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{mn} & \lambda_m p_{mn} & \lambda_m^2 p_{mn} & \dots & \lambda_m^{m-1} p_{mn} \end{bmatrix} \right\} \\
&\longrightarrow \left\{ \begin{bmatrix} r_{11} & & & \\ & r_{21} & & \\ & & \ddots & \\ & & & r_{m1} \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_m & \dots & \lambda_m^{m-1} \end{bmatrix}, \dots, \begin{bmatrix} r_{1l} & & & \\ & r_{2l} & & \\ & & \ddots & \\ & & & r_{ml} \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_m & \dots & \lambda_m^{m-1} \end{bmatrix} \right\} : \\
&\quad \left\{ \begin{bmatrix} p_{11} & & & \\ & p_{21} & & \\ & & \ddots & \\ & & & p_{m1} \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_m & \dots & \lambda_m^{m-1} \end{bmatrix}, \dots, \begin{bmatrix} p_{1n} & & & \\ & p_{2n} & & \\ & & \ddots & \\ & & & p_{mn} \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_m & \dots & \lambda_m^{m-1} \end{bmatrix} \right\} \\
&\longrightarrow \left\{ \begin{bmatrix} (\eta_1, b_1) & & & \\ & (\eta_2, b_1) & & \\ & & \ddots & \\ & & & (\eta_m, b_1) \end{bmatrix} M, \dots, \begin{bmatrix} (\eta_1, b_l) & & & \\ & (\eta_2, b_l) & & \\ & & \ddots & \\ & & & (\eta_m, b_l) \end{bmatrix} M, \dots, \right. \\
&\quad \left. \begin{bmatrix} (\eta_1, c_{11}) & & & \\ & (\eta_2, c_{11}) & & \\ & & \ddots & \\ & & & (\eta_m, c_{11}) \end{bmatrix} M, \dots, \begin{bmatrix} (\eta_1, c_{1n}) & & & \\ & (\eta_2, c_{1n}) & & \\ & & \ddots & \\ & & & (\eta_m, c_{1n}) \end{bmatrix} M \right\},
\end{aligned} \tag{17}$$

where $(\eta_1, b_l), (\eta_m, c_{1n})$ is the vector inner product and matrix

$$M = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_m & \lambda_m^2 & \dots & \lambda_m^{m-1} \end{bmatrix}, \tag{18}$$

is a Vandermonde matrix. If $\tilde{Q}_1$ has full row rank needing that at least one block of $\tilde{Q}_1$ has full row rank, without loss of generality, the first block will be selected to discuss. It is known that Vandermonde matrix M is nonsingular if eigenvalues of A_1 are distinct so that the row rank of matrix $\tilde{Q}_1$ is decided by matrix $\text{diag}\{(\eta_1, b_1), (\eta_2, b_1), \dots, (\eta_m, b_1)\}$ or matrix $\text{diag}\{(\eta_1, c_{11}), (\eta_2, c_{11}), \dots, (\eta_m, c_{11})\}$. Since eigenvectors A_i are unorthogonal to at least one column of B_i or C_i ($i = 1, 2$), therefore, matrix $\text{diag}\{(\eta_1, b_1), (\eta_2, b_1), \dots, (\eta_m, b_1)\}$ or matrix $\text{diag}\{(\eta_1, c_{11}), (\eta_2, c_{11}), \dots, (\eta_m, c_{11})\}$ has full row rank, which means that Q_1 has full row rank. Similarly, $\tilde{Q}_2$ also has full row rank. This completes the proof.

Note that condition $L_i^T = L_i$ implies that the information weight from agent i to agent j is the same as that from agent j to agent i in the same subgroup, that is, the topological structure is symmetric for the subgroups. $\square$

Corollary 1. System (5) is uncontrollable if subgroups $(\mathcal{G}, x^1)$ and $(\mathcal{G}, x^2)$ are both complete graphs (see Figure 2) and $a_{ij} = b_{iq} (\forall i, j \in \ell_1, \ell_2, \forall q \in \ell_i)$, regardless of how to connect $(\mathcal{G}, x^1)$ and $(\mathcal{G}, x^2)$.

Proof. Because subgraphs $(\mathcal{G}, x^1)$ and $(\mathcal{G}, x^2)$ are both complete and $a_{ij} = b_{iq} (\forall i, j \in \ell_1, \ell_2, \forall q \in \ell_i)$, without loss of generality, let $a_{ij} = b_{iq} = 1 (\forall i, j \in \ell_1, \ell_2, \forall q \in \ell_i)$; then,

$$A_1 = \begin{bmatrix} -(m+l-1) & 1 & \cdots & 1 \\ 1 & -(m+l-1) & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -(m+l-1) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -(n+k-1) & 1 & \cdots & 1 \\ 1 & -(n+k-1) & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -(n+k-1) \end{bmatrix}. \quad (19)$$

By simple calculation, we can know that A_1 's eigenvalues are $\lambda_i = 0, -(m+l), \dots, -(m+l)$, and A_2 's eigenvalues are $\mu_i = 0, -(n+k), \dots, -(n+k)$. Then, A_1 has common eigenvalue $-(m+l)$, and A_2 has common eigenvalue $-(n+k)$, which are contrary to the conditions of Theorem 3. Thus, no matter how to connect subgroups $(\mathcal{G}_1, x^1)$ and $(\mathcal{G}_2, x^2)$, system (5) is uncontrollable. $\square$

Corollary 2. If $(\mathcal{G}, x^1)$ and $(\mathcal{G}, x^2)$ are both star graphs (see Figure 3) as well as $a_{ij} = b_{iq} (\forall i, j \in \ell_1, \ell_2, \forall q \in \ell_i)$ regardless of how to connect $(\mathcal{G}, x^1)$ and $(\mathcal{G}, x^2)$, then system (5) is uncontrollable.

Proof. Because subgraphs $(\mathcal{G}, x^1)$ and $(\mathcal{G}, x^2)$ are both star groups and $a_{ij} = b_{iq} (\forall i, j \in \ell_1, \ell_2, \forall q \in \ell_i)$, without loss of generality, let $a_{ij} = b_{iq} = 1 (\forall i, j \in \ell_1, \ell_2, \forall q \in \ell_i)$; therefore,

FIGURE 2: Complete graph.

FIGURE 3: Star graph.

$$A_1 = \begin{bmatrix} -m-l-1 & 1 & \cdots & 1 \\ 1 & -m-l-1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -m-l-1 \end{bmatrix}, \quad (20)$$

$$A_2 = \begin{bmatrix} -n-k-1 & 1 & \cdots & 1 \\ 1 & -n-k-1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -n-k-1 \end{bmatrix}.$$

By computing, we can also know that the eigenvalues of A_1 are $\lambda_i = 0, -(m+l+2), \dots, -(m+l+2)$, and the eigenvalues of A_2 are $\mu_i = 0, -(n+k+2), \dots, -(n+k+2)$. Then, A_1 has common eigenvalues $-(m+l+2)$, and A_2 has common eigenvalues $-(n+k+2)$, which contradict to condition (1) of Theorem 3. Thus, no matter how to connect subgroup $(\mathcal{G}_1, x^1)$ and subgroup $(\mathcal{G}_2, x^2)$, system (5) must be uncontrollable. $\square$

Remark 5. It is noted that there must exist a few leaders making the system to reach the desired state from the random initial state if system (5) is controllable. However, how to configure the leaders such that the desired formation can be achieved? That is, how to select the leaders (or design the inputs) with given initial state and desired state?

Here presents an algorithm for designing the leaders.

Algorithm 1 (algorithm for designing leaders). For the given initial and desired states $x(0)$ and $x(t_1)$, MASs can reach the desired state during $[0, t_1]$, where $t_1 > 0$ is the final time. Suppose that MAS (5) is controllable; then, its Gram matrix is

$$W_c[0, t_1] \triangleq \int_0^{t_1} e^{-At} [CB][CB]^T e^{-A^T t} dt, \quad (21)$$

where $t \in [0, t_1]$. Since $W_c[0, t_1]$ is invertible, we can design a set of inputs (leaders) as

$$u(t) = -[CB]^T e^{-A^T t} W_c^{-1}[0, t_1] x(0). \quad (22)$$

Then, a solution of system (5) is

$$x(t_1) = e^{At_1} x(0) + \int_0^{t_1} e^{A(t_1-t)} [CB] u(t) dt, \quad (23)$$

which can make the system state from $x(0)$ to $x(t_1)$ during $[0, t_1]$. Notice that t_1 is the longest time to get a set of inputs.

4. Example and Simulations

A nine-agent system with followers 4 and a leader as subgroup 1 and followers 3 and a leader as subgroup 2 is described by Figure 4 with $a_{12} = a_{21} = 1, a_{23} = a_{32} = 2, a_{34} = a_{43} = 1, a_{45} = a_{54} = 1, a_{56} = a_{65} = 1, a_{67} = a_{76} = 1$; otherwise, $a_{ij} = 0$.

From Figure 4, the system matrices are as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \\ A_2 &= \begin{bmatrix} -2 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (24)$$

FIGURE 4: Topology with 9 agents.

By computing, we can get the eigenvalues of A_1 and A_2 are $\{-5.7711, -2.2430, -0.7904, -0.1955\}$ and $\{-4.6562, -0.5770, 2.2332\}$, respectively, and the corresponding eigenvectors are

$$\begin{aligned} \eta_1 &= \begin{bmatrix} 0.1262 \\ -0.6023 \\ 0.7714 \\ -0.1617 \end{bmatrix}, \\ \eta_2 &= \begin{bmatrix} 0.4941 \\ -0.6141 \\ -0.4795 \\ 0.3858 \end{bmatrix}, \\ \eta_3 &= \begin{bmatrix} -0.6023 \\ -0.1262 \\ 0.1617 \\ 0.7714 \end{bmatrix}, \\ \eta_4 &= \begin{bmatrix} -0.6141 \\ -0.4941 \\ -0.3858 \\ -0.4795 \end{bmatrix}; \\ \mu_1 &= \begin{bmatrix} 0.5972 \\ -0.7932 \\ 0.1192 \end{bmatrix}, \\ \mu_2 &= \begin{bmatrix} 0.7949 \\ 0.5656 \\ -0.2195 \end{bmatrix}, \\ \mu_3 &= \begin{bmatrix} -0.1067 \\ -0.2258 \\ -0.9683 \end{bmatrix}. \end{aligned} \quad (25)$$

FIGURE 5: A straight line.

FIGURE 6: A trapezoid.

We define the first column of B_1 as $b_{11} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, the first column of C_1 as $c_{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, the first column of B_2 as

$b_{21} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and the last column of C_2 as $c_{24} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, as well as calculate the inner products of every eigenvalue and the corresponding vectors as

$$\begin{aligned}
 (\eta_1, b_{11}) &= 0.7714, \\
 (\eta_2, b_{11}) &= -0.4795, \\
 (\eta_3, b_{11}) &= 0.1617, \\
 (\eta_4, b_{11}) &= -0.3858; \\
 (\eta_1, c_{11}) &= -0.1617, \\
 (\eta_2, c_{11}) &= 0.3858, \\
 (\eta_1, c_{11}) &= 0.7714, \\
 (\eta_1, c_{11}) &= -0.4795,
 \end{aligned} \tag{26}$$

which mean all the eigenvectors of A_1 are unorthogonal to the one column of B_1 or C_1 . At the same time,

$$\begin{aligned}(\mu_1, b_{21}) &= 0.1192, \\(\mu_2, b_{21}) &= -0.2195, \\(\mu_3, b_{21}) &= -0.9683, \\(\mu_1, c_{24}) &= 0.5972, \\(\mu_2, c_{24}) &= 0.7949, \\(\mu_3, c_{24}) &= -0.1067,\end{aligned}\tag{27}$$

which mean all the eigenvectors of A_2 are unorthogonal to any one column of B_2 or C_2 . Those imply that system (5) described by Figure 4 can attain the group controllability.

Figures 5 and 6 depict the initial states, final states, and moving trajectories of the followers of subgroup 1 and subgroup 2 described by the black star dots and the black circular dots, respectively. Beginning from random initial states, the followers of subgroup 1 and subgroup 2 can be finally governed to a straight-line alignment and a trapezoid alignment, respectively.

5. Conclusion

This paper has discussed the group controllability of continuous-time MASs with multiple leaders on switching and fixed topologies, respectively. Some useful and effective results of group controllability are obtained by the rank test and the PBH test. Specially, the group controllability of continuous-time MASs for some special topology graphs has also been studied.

Data Availability

In this paper, no data are needed; only mathematical derivation is needed.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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