

Research Article

Network Dynamics of a Fractional-Order Phase-Locked Loop with Infinite Coexisting Attractors

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We have investigated a fractional-order phase-locked loop characterised by a third-order differential equation. The integer-order mathematical model of the phase-locked loop (PLL) is first converted to fractional order using the Caputo-Fabrizio method. The stability of the equilibrium points is discussed in detail in both parameter and fractional-order domain. The proposed fractional-order phase-locked loop (FOPLL) model shows multiple coexisting attractors which was not discussed in the earlier literature of PLL. The significance of these infinite coexisting attractors is that they exist in the operation region of the PLL between $[-\pi, \pi]$ which increases the complexity of operation of the PLLs. Mainly when such FOPLLs are used in large-scale networks, the synchronisation of the FOPLLs becomes complicated and will result in unstable control conditions. Hence, studying the network dynamics of such FOPLLs is significant which motivates us to investigate the synchronisation phenomenon of the FOPLLs constructed in a square network. We could show that, because of the multiple coexisting attractors, the FOPLLs show various synchronisation phenomena, and more importantly in the chaotic region for lower fractional-order values, we could show that the FOPLLs are synchronised and this finding is very useful to completely analyse the FOPLL networks in high-frequency operations.

1. Introduction

The invention of the phase-locked loop (PLL) device is going to have its centenary in almost a decade with its formulation in 1932 by Bellescize [1]. Since then, the PLL devices have found their applications in many semiconductor integrated circuits and communication systems [2]. It was in [3, 4] that the first known mathematical model of the PLL is proposed, and since then, the interest is to investigate the dynamics of the PLLs. The simplest form of a PLL consists of a voltage-controlled oscillator (VCO) and a phase detector in the feedback which is to generate the phase error between the input and the oscillator signals. The PLL gets locked to the frequency when the loop error (phase) becomes constant [5]. Hence, the PLL devices are mostly used in communication systems to estimate a message signal from a noisy environment [6, 7]. In most of the cases, the PLL is used along with a low-pass filter to remove the low-frequency

oscillations and the model of PLL with such LPF is also studied [6, 8]. It is this model which was of interest to many researchers and hence there have been many published works discussing the bifurcation phenomenon [8], chaos and its dynamical properties [9, 10], and so on. This model of PLL is normally called the third-order model.

The complex dynamical behaviour of such third-order PLLs is widely discussed as seen in [6, 8, 11]. In most of the real-world applications of the PLLs, they are used in cascaded or in parallel connections [12] in which synchronisation between the PLLs is very important to determine their performance. The PLL is a highly nonlinear system which finds its applications in high-frequency areas; the linear system theory cannot sufficiently help us to understand the characteristics of the PLL. On the other hand, the high nonlinear phenomenon in PLL cannot be properly explored using nonlinear concepts such as stability, bifurcation, and Lyapunov exponents. Even though these tools help us to

understand the complex behaviour in the PLL system, they could not help us to understand the synchronised behaviour of the PLL. PLL is still regarded as the most common method for determination of the phase angle and frequency of grid voltages in smart grid networks. Due to the unbalance and noises created in the grid voltages combining with the nonlinear behaviour of the PLLs, their network behaviour is unpredictable [13].

When we consider the third-order model of the PLL, there have been several pieces of literature discussing the chaotic oscillations and the parameter dependence of such complex oscillations [6, 14, 15]. But all these discussions are limited to integer-order analysis. It was well established in the literature that fractional-order analysis can very well match real-time systems [16–19]. It was shown that models with memory can be effectively modelled using fractional calculus and we could explore some complex dynamical properties when using fractional-order analysis [20–24]. There have been studies on the study of network synchronisation in many excitable media models with memory [25–29] but all these are integer-order investigations. Also, there have been no many investigations on the synchronisation of real-time circuits like PLL when coupled together in networks.

As the model of the PLL falls under the jerk system category, we have also analysed the recent literature on such systems. Real phenomenon like time-reversible Hamiltonian chaos in Nosé-Hoover dynamic system can be effectively modelled with jerk equations [30]. In 1997, Sprott [31] formulated a series of chaotic jerk systems with quadratic nonlinearity. Since jerk systems with chaotic behaviour are simple in structure, it easily overcomes the challenges in implementation. Initially, formulation of such jerk systems is with introducing polynomial function, but later a significant number of jerk systems are demonstrated with piecewise-linear function [32, 33]. Jerk system with tanh-cubic nonlinearity is investigated for special properties such as multiscroll and multiple coexisting attractors [34]. In order to realize the hyperbolic nonlinearity, a pair of semiconductor diodes are connected in antiparallel [35]. The occurrence of an additional source of entropy results from multiple attractors with an appropriate range of system parameters corresponding to chaotic region can be significant in secure communication application and physical random number generators [30, 36]. Chaotic jerk system with different types of equilibria is investigated, and circuit realization is achieved [37]. A chaotic jerk system holding both self-excited and hidden attractors is investigated for the effect of time delay [38, 39] on secure communication, and the results revealed the existence of special properties such as multistability [40]. A class of unexcited jerk systems with megastability and its analog and microcontroller-based embedded system design is analysed for compatibility of such complicated systems in real-time implementation [41].

Motivated by these studies, our interest now is to propose the fractional-order model of the PLL and explore its dynamical properties using stability of equilibrium points, bifurcation plots, and so on. The main aim of this paper is to

study the network behaviours of the proposed fractional-order PLL in a network. For this, we have constructed a mathematical model of the network coupled together as a ring structure and investigate the various synchronisation patterns formed in the network.

2. Mathematical Model

The third-order differential equation representing the mathematical model of a phase-locked loop (PLL) is presented in [6, 14, 15] given by

$$\ddot{X} + (3 - K)\dot{X} + \dot{X} + KG \sin(X) = 0, \quad (1)$$

where the variable X defines the phase difference between the voltage-controlled oscillator (VCO) and the phase detector while the parameters K and G denote the LFG (low-frequency gain) of the filter and the range of PLL. It is to be noted that both parameters should be positive constants as defined originally in [14]. The dimensionless model of the PLL (1) is derived as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -KG \sin(x) - y - (3 - K)z. \end{aligned} \quad (2)$$

The aim of this paper is to discuss the fractional-order dynamics of the PLL (2) and to show some unexplored dynamics of the system including infinitely coexisting attractors. The fractional-order PLL (FOPLL) is defined as

$$\begin{aligned} \frac{d^{q_x} x}{dt} &= x, \\ \frac{d^{q_y} y}{dt} &= z, \\ \frac{d^{q_z} z}{dt} &= -KG \sin(x) - y - (3 - K)z, \end{aligned} \quad (3)$$

where $[q_x; q_y; q_z]$ are the fractional orders of the state variables. To numerically simulate system (5), we use the modified Adam-Bashforth method [42] for the Caputo-Fabrizio (CF) fractional operator [43]. By definition, the general form of a CF fractional operator can be written in the form

$$\begin{aligned} {}_0^{\text{CF}}D_t^q \varphi(t) &= F(t, X(t)), \\ \varphi(0) &= \varphi_0, \end{aligned} \quad (4)$$

whose solution is given by the definition

$$\begin{aligned} X(n+1) &= X(n) + \left(\frac{(1-q)}{M\Gamma(q)} + \frac{3q\Delta t}{2M(q)} \right) F(t_n, X_n) \\ &\quad - \left(\frac{(1-q)}{M\Gamma(q)} + \frac{q\Delta t}{2M(q)} \right) F(t_{n-1}, X_{n-1}), \end{aligned} \quad (5)$$

where $M(q)$ is the normalisation factor such that $M(0) = M(1) = 1$ [5] and Δt represents the integration step size. Using (5) and (3), the discrete FOPLL model is derived as

$$\begin{aligned}
x(n+1) &= x(n) + \left(\frac{(1-q_x)}{M\Gamma(q_x)} + \frac{3q_x\Delta t}{2M(q_x)} \right) x(n) - \left(\frac{(1-q_x)}{M\Gamma(q_x)} + \frac{q_x\Delta t}{2M(q_x)} \right) x(n-1), \\
y(n+1) &= y(n) + \left(\frac{(1-q_y)}{M\Gamma(q_y)} + \frac{3q_y\Delta t}{2M(q_y)} \right) y(n) - \left(\frac{(1-q_y)}{M\Gamma(q_y)} + \frac{q_y\Delta t}{2M(q_y)} \right) y(n-1), \\
z(n+1) &= z(n) + \left(\frac{(1-q_z)}{M\Gamma(q_z)} + \frac{3q_z\Delta t}{2M(q_z)} \right) (-KG \sin(x(n)) - y(n) - (3-k)z(n)) - \left(\frac{(1-q_z)}{M\Gamma(q_z)} + \frac{q_z\Delta t}{2M(q_z)} \right) \\
&\quad \cdot \begin{pmatrix} -KG \sin(x(n-1)) - y(n-1) \\ -(3-k)z(n-1) \end{pmatrix}.
\end{aligned} \tag{6}$$

For the numerical analysis of the FOPLL, we choose the parameters as $K = 2.8$, $G = 1$ and commensurate fractional order as $q_x = q_y = q_z = q = 0.98$. By fixing the initial conditions of states y and z , we could plot different coexisting attractors for various initial conditions of state variable x as shown in Figure 1.

3. Numerical Analysis and Discussions

As x can take the values $[-\pi, \pi]$, the equilibrium points of the FOPLL system are $[0, 0, 0]$ and $[\pi, 0, 0]$. Considering that the fractional orders of the FOPLL are commensurate, we can find the characteristic polynomial of the FOPLL as

$$\lambda^3 + A\lambda^2 + \lambda + B \cos(x_E) = 0, \tag{7}$$

where $A = (3 - K)$, $B = KG$, and $x_E = 0$ or π . The equilibrium points of the commensurate FOPLL system are stable if

$$q \frac{\pi}{2} < \arg(\text{eig}(\lambda)). \tag{8}$$

The equilibrium point $[\pi, 0, 0]$ is always unstable for any logical combination of the fractional order q and parameters K and G ; hence, we are interested in discussing the stability of the origin which is another equilibrium point. By Routh-Hurwitz criterion, we can conclude that, for $A \geq 0$, the values $1 < K \leq 3$ and $A - B > 0$ which give the condition $K < 3(1 + G)^{-1}$. Using (6) and the condition, we can derive the stability regions of the equilibrium points as shown in Figure 2. We could see that as we decrease the fractional order, the stable region increases while the unstable region shrinks as seen in Figures 2(c) and 2(d). Further by fixing $K = 2.5$ and $G = 1$, the Hopf bifurcation point is found at $q = 0.779$.

To calculate the Lyapunov exponents (LEs) of the FOPLL system, we have used the Wolfs algorithm [44] modified to accommodate the Caputo-Fabrizio solvers designed using (6). For a fixed initial condition of $[1, 0, 0]$ with fractional order $q = 0.98$ and parameters $K = 2.6$ and $G = 1$, we could calculate the LEs as $[0.93; 0; -1.87]$. Having discussed the stability of equilibriums, our interest now is focused on the investigation of the impact of system parameters and fractional orders on the dynamical behaviour of the FOPLL system. To do this, we derive and investigate the bifurcation

plots of the FOPLL system considering fractional order as the control parameter and plotting the local maximum value of the state variable “ y ” as shown in Figure 3. The fractional order $q \in [0.82, 1]$ and we have used forward continuation by changing the initial conditions to end values of the state variables for every value control parameter “ q .” The FOPLL system takes a period-doubling route to chaos and shows signs of hysteresis when doubling from period 2 to period 4. We could see many cascades of period-doubling routes leading to chaotic regions.

In Figure 4, we have showed the bifurcation of the FOPLL system with parameter K by keeping the fractional order $q = 0.98$ and parameter $G = 1$. The parameter K is varied between $[2.4, 3]$ and we have plotted the local maxima of y using forward continuation as shown in Figure 4. We could see that the FOPLL shows the property of anti-monotonicity between $K \in [2.42, 2.55]$ with period-doubling and period-halving behaviours seen in this region. For further values of K , the FOPLL shows cascades of chaotic regions through period-doubling routes from each region.

4. Network Dynamics of FOPLL

In many communication systems and network applications, the synchronisation of the networks assures the efficiency of information processing [45]. Some studies have shown that the synchronous state frequencies and their stability conditions for fully connected PLL networks depend on the free-running frequencies and node phase detector gains [46, 48]. Hence, analysing the network behaviours of the FOPLL is important, and to do this, we have constructed a network of FOPLL which is coupled with the nearby $2P$. The FOPLL network structure is defined as

$$\begin{aligned}
D^q x_i &= y_i, \\
D^q y_i &= z_i, \\
D^q z_i &= -KG \sin(x_i) - y_i - (3 - K)z_i + \frac{D}{2P} \sum_{j=i-P}^{i+P} (z_j - z_i),
\end{aligned} \tag{9}$$

where $i = 1$ to N and N is the number of FOPLLs in the network. For simulations, we took $N = 100$ and $P = 25$ while

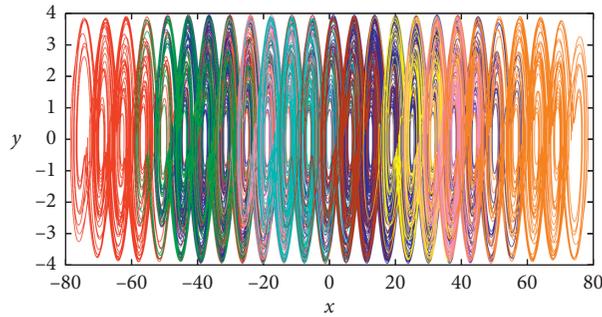


FIGURE 1: Different coexisting attractors of the FOPLL system for $K=2.8$ and $G=1$ with fractional order $q=0.98$. The initial conditions are $y=0$ and $z=0$ with x initial condition varying from $-\pi$ to $+\pi$ with a step size of $\pi/5$.

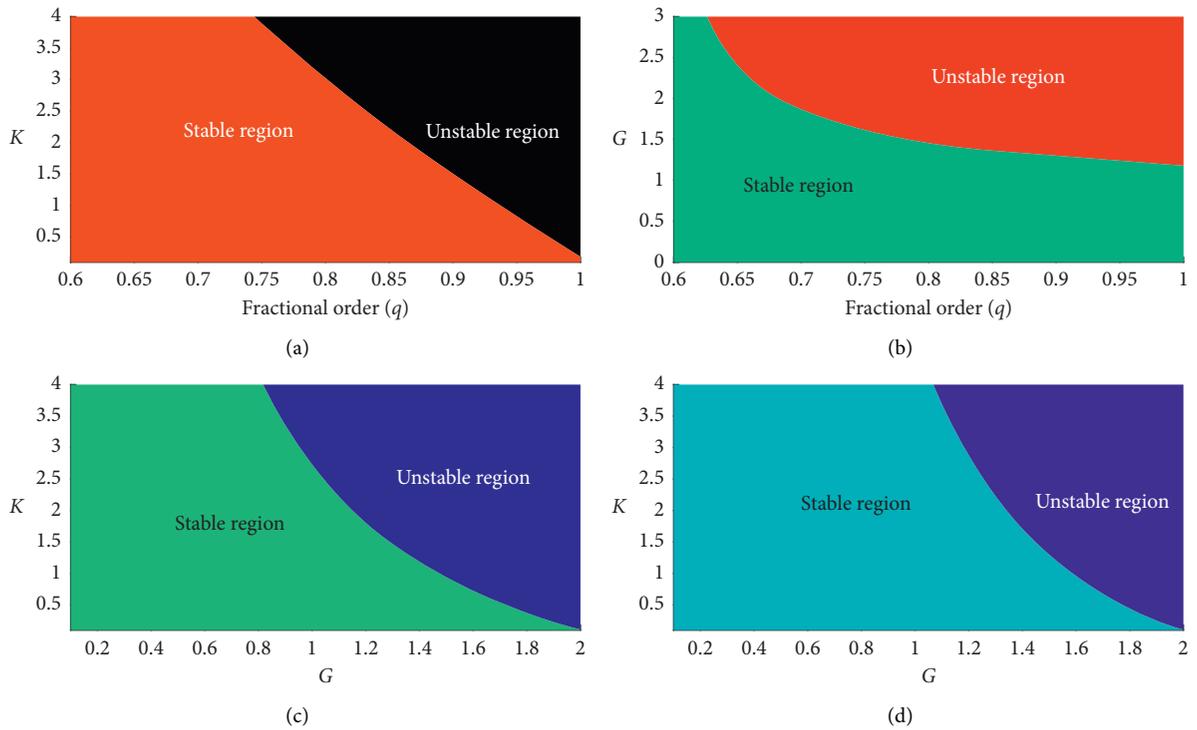


FIGURE 2: Stability regions of the equilibrium point $[0, 0, 0]$ of the FOPLL system for (a) various fractional orders and parameter K with parameter $G=1$; (b) various fractional orders and parameter G with $K=2.5$; (c) various values of parameters K and G with fractional orders $q=0.98$ and $q=0.9$ as shown in (c) and (d).

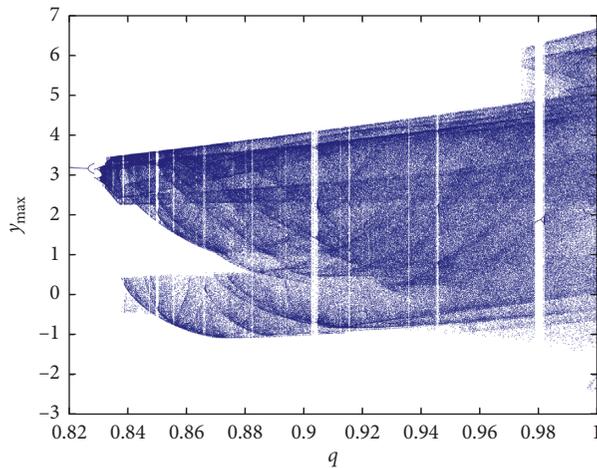


FIGURE 3: Bifurcation of the FOPLL system for $K=2.6$ and $G=1$ for various values of commensurate fractional order q .

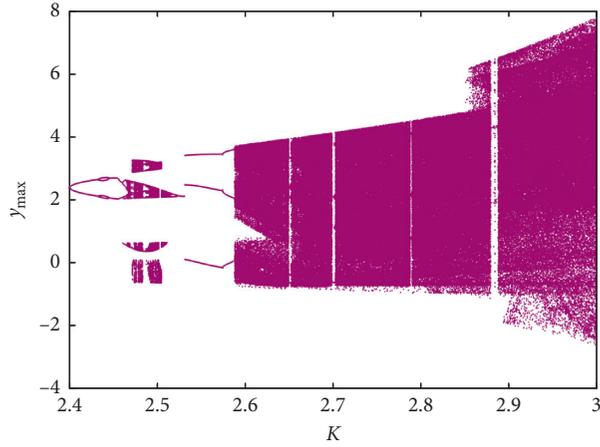


FIGURE 4: Bifurcation of the FOPLL system for $q = 0.98$ and $G = 1$ for various values of K .

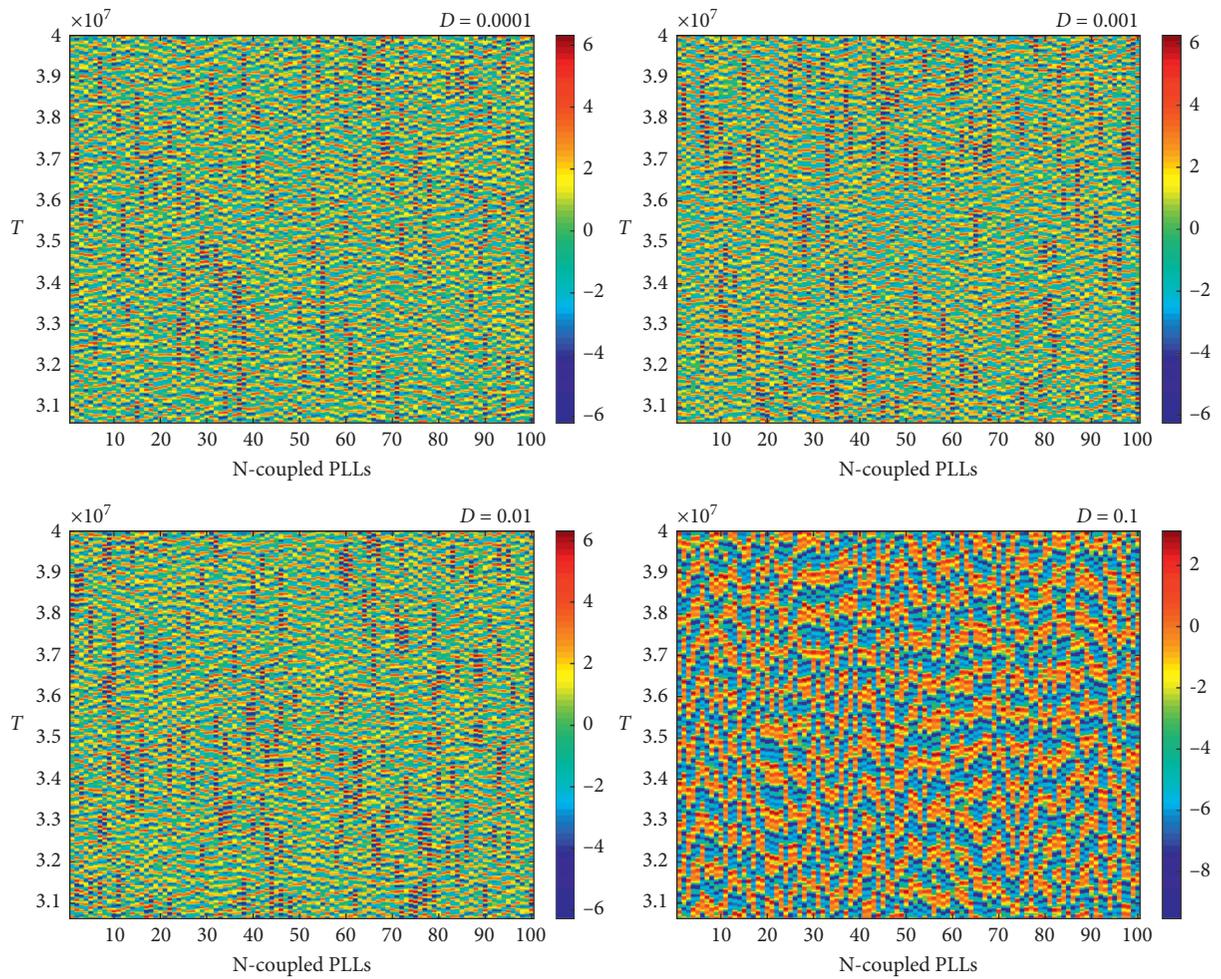


FIGURE 5: Spatiotemporal dynamics of the FOPLL network showing asynchronous behaviour of the FOPLL for the coupling strength $0.0001 \leq D \leq 0.1$. We have used random initial conditions between the range $[-\pi, \pi]$ for all the FOPLLs. The fractional order is fixed at $q = 0.98$ and the system parameters are fixed at $K = 2.6, G = 1$.

the other system parameters are fixed to their respective values. The coupling strength D and fractional order q are considered as the control parameters.

4.1. Effect of the Coupling Strength. Synchronised behaviour of the FOPLLs in the network is very important in order to ensure the efficient working of any large-scale communication

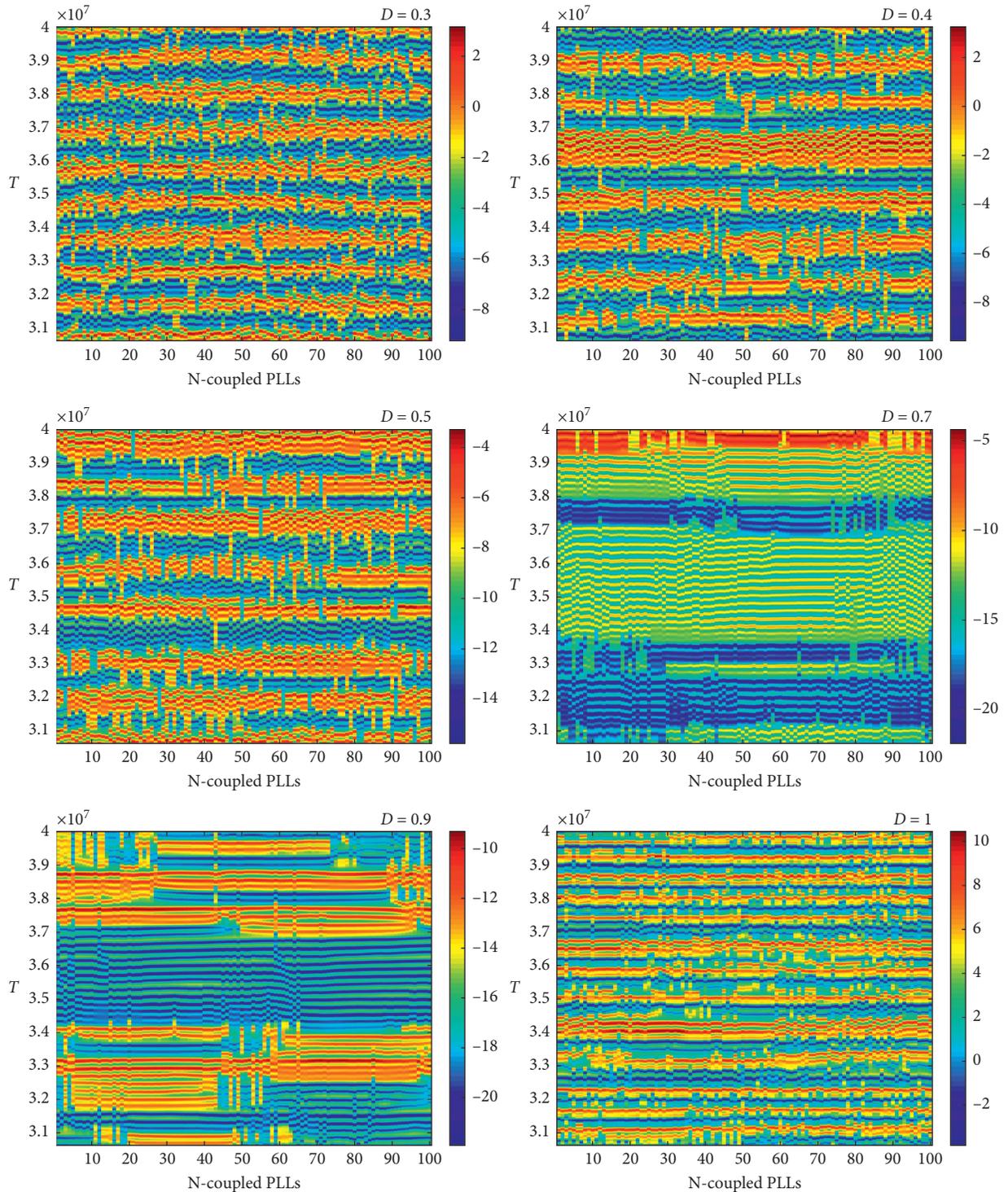


FIGURE 6: Spatiotemporal dynamics of the FOPLL network showing chimera-like behaviour for the coupling strength $0.3 \leq D \leq 1$. We have used random initial conditions between the range $[-\pi, \pi]$ for all the FOPLLs. The fractional order is fixed at $q = 0.98$ and the system parameters are fixed at $K = 2.6, G = 1$.

systems. This synchronisation depends on the coupling strength between the FOPLLs in the network. In order to understand the various synchronisation behaviours, we use the coupling strength D as the control parameter and capture the spatiotemporal dynamics of the network as shown in Figure 5. We could see that, for $D < 0.01$, the network is nearly in a

mutated state showing only few FOPLLs oscillating. This is because when the coupling strength is low, the oscillations in the ring network are damped by the nearby nonoscillating neurons. Also, the number of nearby neurons considered is 25, which indirectly increases the minimum required threshold coupling strength to remain synchronised. By increasing

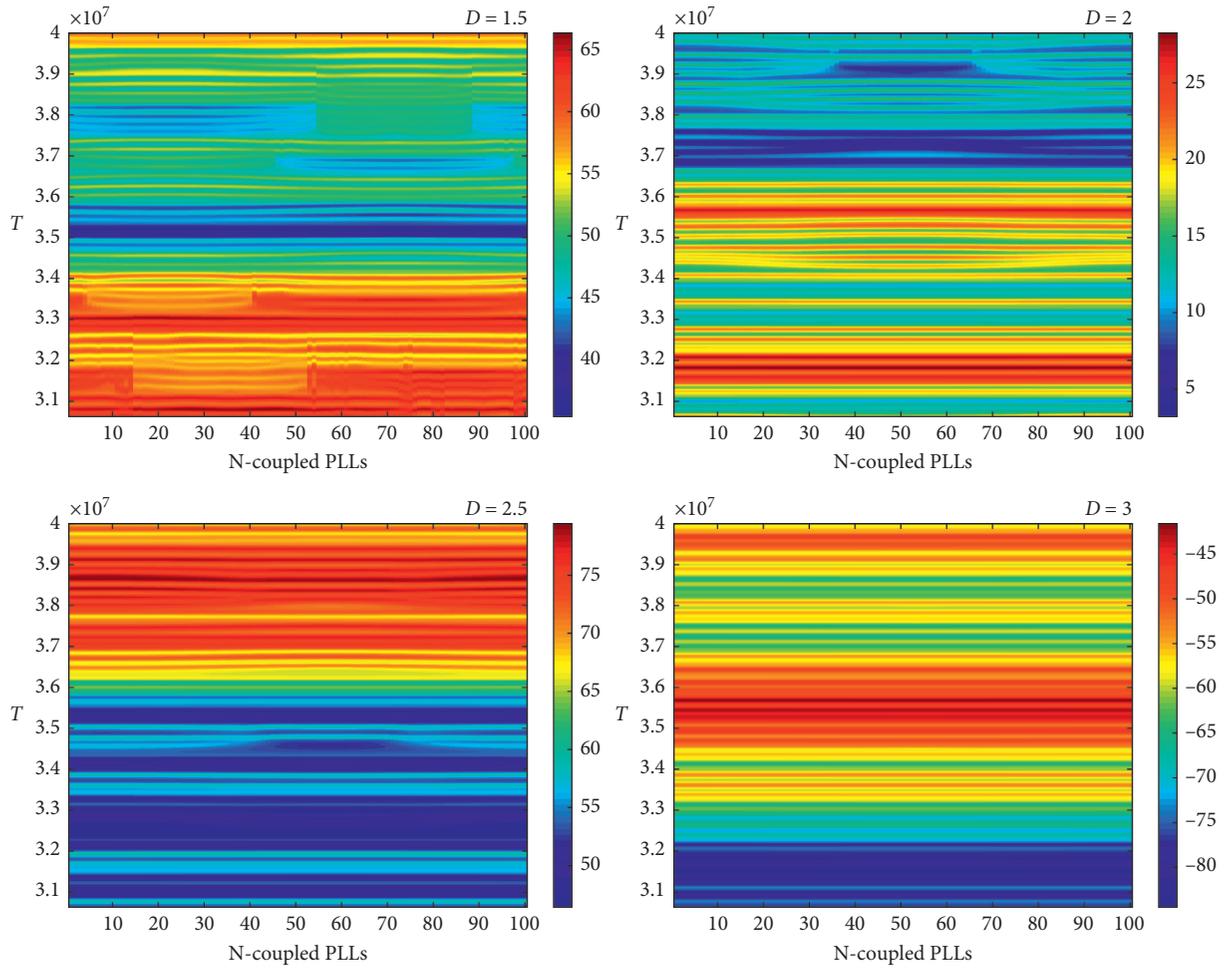


FIGURE 7: Spatiotemporal dynamics of the FOPLL network showing synchronisation of FOPLLs for the coupling strength $1.5 \leq D \leq 3$. We have used random initial conditions between the range $[-\pi, \pi]$ for all the FOPLLs. The fractional order is fixed at $q = 0.98$ and the system parameters are fixed at $K = 2.6, G = 1$.

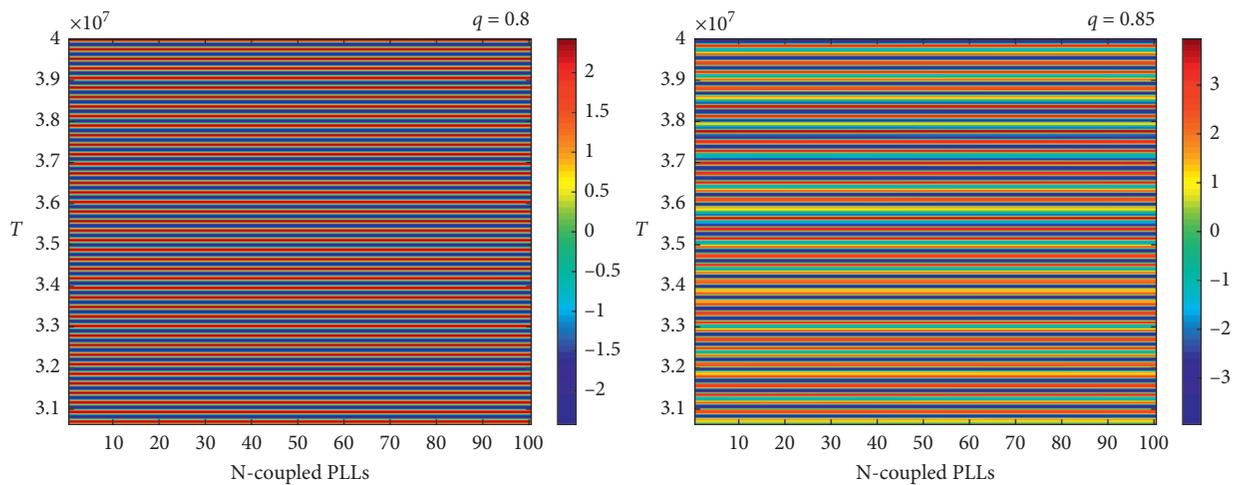


FIGURE 8: Spatiotemporal behaviour of the FOPLL network for fractional orders $q = 0.8$ and $q = 0.85$. We have used random initial conditions with coupling strength $D = 1$ and parameters $K = 2.6$ and $G = 1$.

$D = 0.1$, we could observe that all the FOPLLs oscillate and tend to show their tendency to synchronise (red and blue colours) among the nearby FOPLLs.

By increasing the coupling strength to $D = 0.3$, we could capture the emerging chimera-like patterns shown in Figure 6. We should note that the FOPLLs show infinite

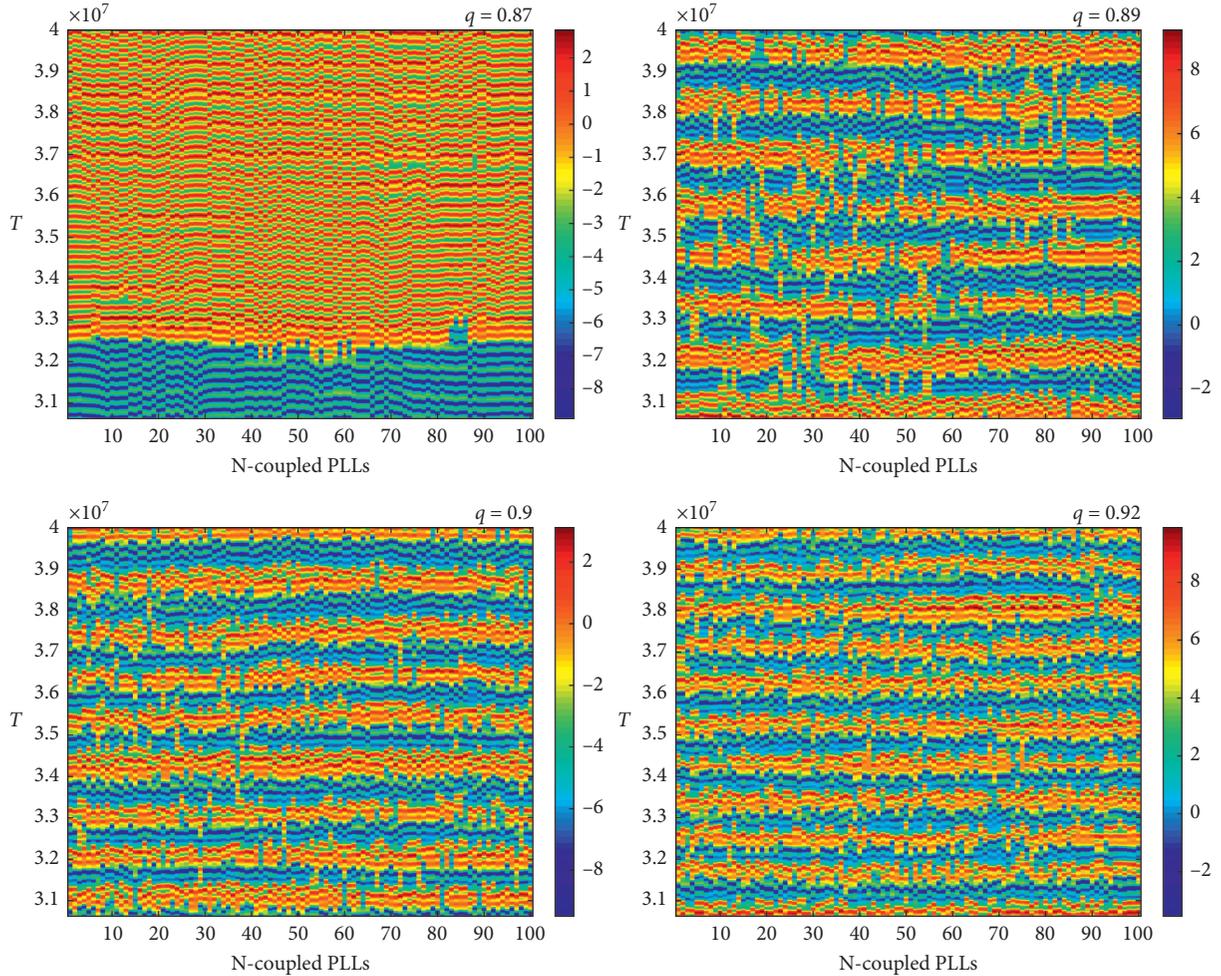


FIGURE 9: Spatiotemporal behaviour of the FOPLL network for different fractional orders. We have used random initial conditions with coupling strength $D=1$ and parameters $K=2.6$ and $G=1$.

coexisting attractors and detecting such chimera states when using random initial conditions in an extremely multistable oscillator is a very rare phenomenon. For the range $0.3 \leq D \leq 0.5$, the network shows chimera-like behaviour organised into two different sets of synchronous and asynchronous FOPLLs. When $D=0.7$, we could see a cluster synchronisation between FOPLLs which does not last long as when $D=0.9$, the chimera-like states reemerge and try to enter synchronisation when $D=1$.

For further increase in coupling strength to $D=1.5$, the cluster synchronisation converts to complete synchronisation with 70% of the FOPLLs in the network synchronised to the same phase while the remaining 30% experiences a phase shift. This is because of the negative and positive random initial conditions used. This type of phase-shifted synchronisation occurs for $1.5 \leq D \leq 2.5$ while the percentage of the antiphase FOPLLs reduces as D increases. When $D=3$, the FOPLLs in the network are completely synchronised as in Figure 7.

4.2. Impact of Fractional Order on Network Dynamics. In the previous section, we have shown the effect of coupling strength on the network dynamics, and in this section, we

wish to analyse the effect of fractional order on the network performance. Many pieces of literature have shown the importance of fractional-order analysis in real physical systems and hence this investigation is of great significance. In Figure 8, we have shown the synchronised FOPLL in the network for $D=1$ with fractional orders $q=0.8$ and $q=0.85$. Comparing with Figure 3, we could see that, for $q=0.85$, the FOPLL is already in chaotic state but still shows synchronised behaviour for the same value of q as seen in Figure 8. By this, we could understand that the synchronisation between FOPLLs in the network can be modelled using fractional orders in order to ensure better understanding of their network performance. It is also to be noted that such synchronisation is achieved in a FOPLL model showing infinite coexisting attractors and, in this simulation, we have used random initial conditions to ensure that each FOPLL is showing random attractors.

Increasing the value of fractional order for $q=0.87$, we could see that the network aligns to form a chimera-like states with coherent and incoherent FOPLLs and the behaviour further divides the network into small spatial groups forming coherent and incoherent states for $0.89 \leq q \leq 0.92$ as seen in Figure 9. As the FOPLL shows many coexisting

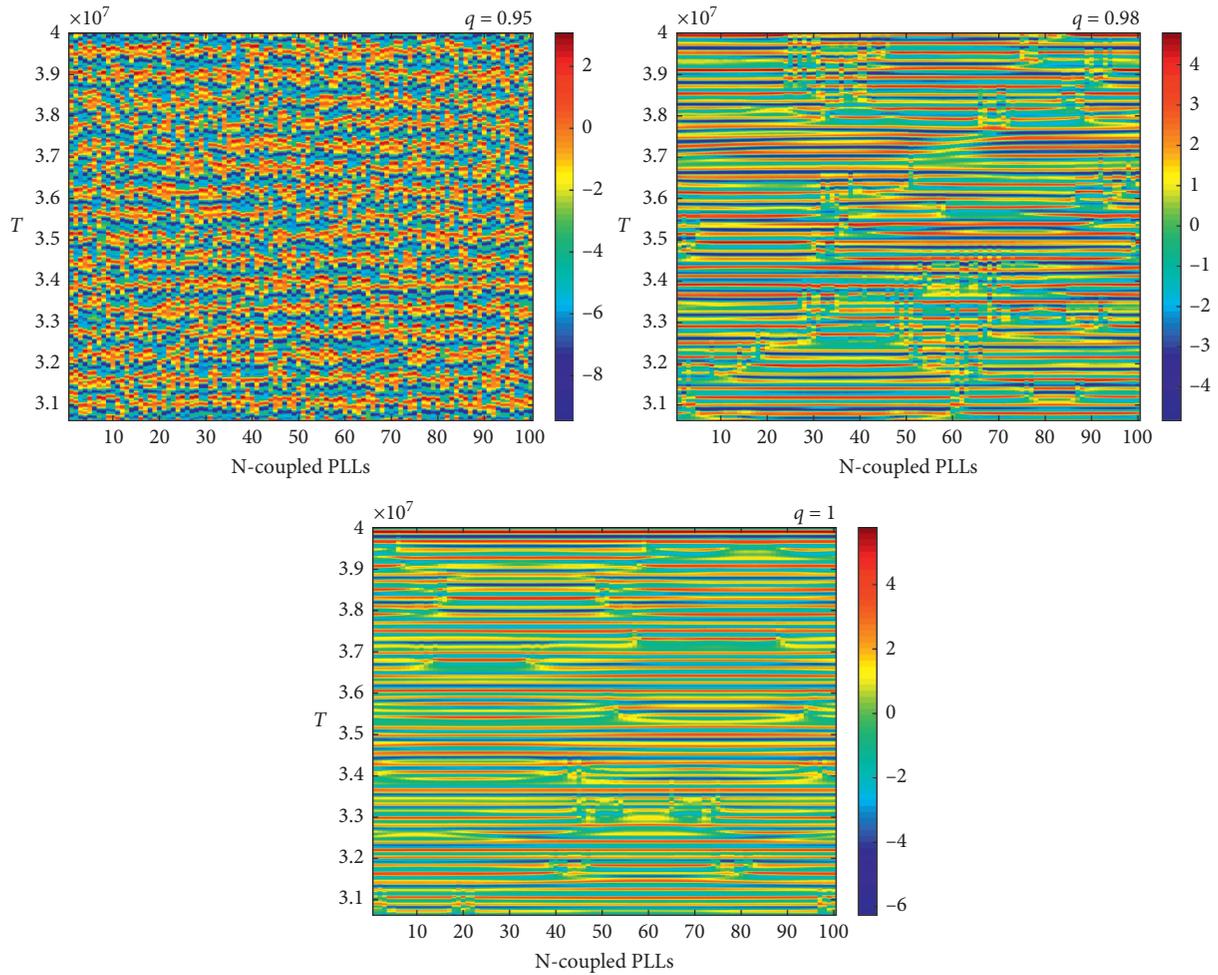


FIGURE 10: Spatiotemporal behaviour of the FOPLL network for different fractional orders. We have used random initial conditions with coupling strength $D=1$ and parameters $K=2.6$ and $G=1$.

attractors, the spatial behaviour of the network depends on the initial conditions which is randomly chosen between $[-\pi, \pi]$.

In our final discussion, we now choose the value of fractional order close to integer order; we could see that the network now goes into coherency with few noncoherent FOPLLs in the network. These noncoherent FOPLLs tend to synchronise as we move close to the integer-order value $q=1$ as seen from Figure 10.

5. Results and Discussion

In this paper, we have investigated a fractional-order PLL with three differential equations. Most of the earlier PLL models were investigated for only chaotic attractors for certain initial conditions but in this paper, we have discussed the infinite coexisting attractors. The FOPLL system has two equilibrium points with the first point at $[\pi, 0, 0]$ which is always unstable for allowed parameter values while the second equilibrium point at origin has both stable and unstable regions depending on the parameter and fractional-order values. The bifurcation diagram reveals a wide range of complex behaviour with the

fractional order and in parameter space. To understand the spatiotemporal behaviour of the FOPLL, we have constructed a lattice array with FOPLL-based local nodes. The constructed square network is given random initial conditions with nonlocally coupled FOPLLs. For lower values of fractional order, the network shows a complete incoherent behaviour. When the fractional order is increased from $q=0.9$, we could see that the nodes organise and enter into states of cluster synchronisation. We could also show that the network attains complete synchronisation for $q=0.85$ even though the FOPLLs are in chaotic regime for the fractional order. Thus, analysing such real circuits using fractional order will help us to understand some unexplored complex behaviours.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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