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Research Article

Finite-Time Stabilization and Destabilization Analysis of Quaternion-Valued Neural Networks with Discrete Delays

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In this paper, the finite-time stabilization and destabilization of a class of quaternion-valued neural networks (QVNNs) with discrete delays are investigated. In order to surmount the difficulty of noncommutativity of quaternion, a new vector matrix differential equation (VMDE) is proposed by employing decomposition method. And then, a nonlinear controller is designed to stabilize the VMDE in a finite-time interval. Furthermore, under that controller, the finite-time stability and instability of the QVNNs are analyzed via Lyapunov function approach, and two criteria are derived, respectively; furthermore, the settling time is also estimated. At last, by two illustrative examples we verify the correctness of the conclusions.

1. Introduction

In 1961, in order to investigate the transient performance of the system, Perter Dorato gave a definition of short-time stability, which was also called finite-time stability later [1]. There are some differences between finite-time stability and classical stability theory, Lyapunov stability. Actually, the finite-time stability mainly reveals the transient dynamic characteristics of the system in a short and desired time interval; however, the Lyapunov stability mainly reveals dynamical behavior of the system in an infinite time interval [2–4]. For a long time, the research concerning the finite-time stability only focused on the stability analysis. However, very limited references considered the problem of controllability due to the difficulty in designing the control strategy [3,5–8]. In fact, many practical systems are required to reach their desired state quickly, such as flight control system, communication network system, and robot control [9–16]. Therefore, lots of scholars are devoted to the controllability of finite-time stability, and some interesting and meaningful results have been reported [2,4,9,17–28]. Nersesov et al. extended the finite-time stability theory and gave a control strategy to reach finite-time stability [2]. For the delayed complex-valued memristive neural networks, a new nonlinear delayed controller was designed to get the finite-time stabilization [4]. When discussing scalar linear systems, a finite-time controller was proposed in [22]. Based on state and output feedback, several especial finite-time controllers were firstly proposed for the stochastic system in [23]. On the other hand, it is also interesting to destabilize a stable system in a finite-time interval, such as preventing eavesdropping and signal encryption. Wang and Shen proposed some finite-time destabilization algebraic criteria for memristive neural networks, and a more general controller was designed to realize the finite-time destabilization for delayed complex-valued memristive neural networks [24]. However, the controllers designed in existing references are invalid to QVNNs because of the noncommutativity of quaternion. And many effective methods for studying the finite-time stability of QVNNs are yet to be discovered, which stimulates us to do this research.

Like $x = c + di + ej + fk$, $c, d, e, f \in \mathbb{R}$, we call number x a quaternion proposed in 1843, and it satisfies the following rule:

$$\begin{aligned} i \times i = j \times j = k \times k = -1, i \times j = -j \times i = k, \\ j \times k = -k \times j = i, k \times i = -i \times k = j. \end{aligned} \tag{1}$$

Quaternion has been widely used in space control, computer 3D image processing, and attitude control of spacecraft [29]. Up to now, the neural network has obtained great development in many fields, such as signal processing, artificial intelligence, and optimization. Particularly, for the real-valued neural networks (RVNNs), many researchers have carried out a lot of work [30–33], as well as complex-valued neural networks (CVNNs) [3,4,34–37]. Since there are three imaginary parts of quaternion, combined with many advantages of neural network, QVNNs have many properties that RVNNs and CVNNs do not have and have been applied in many practical fields, such as high-dimensional data processing, image compression, pattern recognition, and optimization. While, much fewer attentions are given to the dynamical behavior of QVNNs [20,38–47]. Li and Zheng investigated the globally exponential passivity of quaternion-valued memristor-based neural networks with time delays [29]. Tu et al. investigated the globally asymptotical stability and exponential stability of a class of QVNNs with mixed delays via nonseparating technologies [42]. Based on fractional-order QVNNs, quasi-synchronization and bifurcation were also considered [43]. Nevertheless, according to our knowledge, it is still open and significative to study the finite-time stabilization of delayed QVNNs, such as how to carry out the finite-time stabilization of QVNNs and how to design the controller to stabilize the instable systems remain unresolvable. Some new theory and methods should be explored to resolve those problems. We mainly want to discuss the finite-time stability of QVNNs in this paper. By constructing a new vector Lyapunov candidate function and designing a nonlinear vector-matrix controller, both finite-time stabilization and destabilization of delayed QVNNs are analyzed. Furthermore, we only need to adjust the appropriate parameters, and the finite-time stabilization and destabilization can be realized. We sort out the chief contributions of this article as follows:

(1) It is the first time that the finite-time stabilization and destabilization of QVNNs with discrete delays are studied. A new vector Lyapunov function is constructed and a new nonlinear vector-matrix controller is designed to investigate the aforementioned problem.

(2) Based on the new developed method, some easily checked results for the finite-time stabilization and destabilization of QVNNs are provided, respectively. Compared to [4], the obtained criteria are more concise and natural.

(3) The influence of initial condition of the system and parameter of the designed controller to the settling time is analyzed in detail.

The remaining sections of this article will be arranged as follows. In Section 2, an equivalent VMDE of QVNNs is established and several correlative definitions, lemmas, and assumptions are presented. In Section 3, a new nonlinear vector-matrix controller is given, and both the finite-time destabilization and stabilization of QVNNs with discrete delays is analyzed. In Section 4, the validity of our proposed criteria is checked by two illustrative examples. In Section 5, a summary of the paper is given and some thoughts on the future work of finite-time problems are conceived.

Notations. The symbol $\mathbb{R}$ expresses the real number set, the symbol $\mathbb{C}$ expresses complex number set, and the symbol $\mathbb{Q}$ expresses quaternion set. We call $\mathbb{R}^{m\times l}$ and $\mathbb{Q}^{m\times l}$ all $m \times l$ real matrices set and quaternion matrices set, respectively. $\mathbb{Q}^l$ is said to be l-dimensional quaternion space. A continuous mapping from $[t_0 - \tau, t_0]$ to $\mathbb{Q}^l$ is $\phi \in C([t_0 - \tau, t_0]; \mathbb{Q}^l)$. The transpose of B is noted by symbol B^T. We can use $B > 0$ $(B < 0)$ to represent a positive definite (negative definite) matrix, respectively. A vector $y = (y_1, y_2, \ldots, y_l)^T \in \mathbb{R}^l < 0$ means that $y_i < 0$, $i = 1, \ldots, l$. The 1-norm of vector $Q \in \mathbb{R}^l$ is written as $\|Q\| = \sum_{i=1}^{l} |Q|_i$. When $b(t) = (b_1(t), b_2(t), \ldots, b_l(t))^T \in \mathbb{R}^l$ and $\gamma \in \mathbb{R}$, $|b(t)|^\gamma = (|b_1(t)|^\gamma, |b_2(t)|^\gamma, \ldots, |b_l(t)|^\gamma)^T$, $\operatorname{sgn}(b(t)) = (\operatorname{sgn}(b_1(t)), \operatorname{sgn}(b_2(t)), \ldots, \operatorname{sgn}(b_l(t)))^T$. $|B| = (|b_{ij}|) \in \mathbb{R}^{l\times l}$, where $B = (b_{ij}) \in \mathbb{R}^{l\times l}$. A continuous function α: $[0, a) \longrightarrow [0, +\infty)$ is a class $\mathscr{K}$ function if it is strictly increasing and $\alpha(0) = 0$. $I = (1, 1, \ldots, 1)^T \in \mathbb{R}^l$. E is an identity matrix.

2. Preliminaries

Based on the following QVNNs model with discrete time-varying delays, we will analyze how to stabilize and destabilize the QVNNs in a finite- and short-time interval:

$$\dot{x}(t) = -Cx(t) + Mg(x(t)) + Ng(x(t - \tau(t))) + I(t), \tag{2}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_l(t)) \in \mathbb{Q}^l$ is called a l-dimensional state variable at time t, $C = \operatorname{diag}\{c_1, c_2, \ldots, c_l\} \in \mathbb{R}^{l\times l}$ is called a self-feedback link weight matrix with $c_i > 0$, $i = 1, 2, \ldots, l$, M, $N \in \mathbb{Q}^{l\times l}$ denote link weight matrices, $g(x(\cdot)) = (g_1(x_1(\cdot)), g_2(x_2(\cdot)), \ldots, g_l(x_l(\cdot)))^T \in \mathbb{Q}^l$ is activation function, $\tau(t)$ satisfies $0 < \tau(t) < \tau$, $0 < \tau < +\infty$, which is the time-varying delay, and $I(t) = (I_1(t), I_2(t), \ldots, I_l(t))^T \in \mathbb{Q}^l$ denotes outer input vector which will be designed later. The initial condition is given by $x(s) = \psi(s) \in \mathbb{Q}^l, s \in [t_0 - \tau, t_0]$, where $\psi(s) = \psi^{(r)}(s) + \psi^{(i)}(s)i + \psi^{(j)}(s)j + \psi^{(k)}(s)k$.

Let

$$\begin{aligned} x(t) &= x^{(r)}(t) + x^{(i)}(t)i + x^{(j)}(t)j + x^{(k)}(t)k, \\ M &= M^{(r)} + M^{(i)}i + M^{(j)}j + M^{(k)}k, \\ N &= N^{(r)} + N^{(i)}i + N^{(j)}j + N^{(k)}k, \\ g(x(t)) &= g^{(r)}\left(x^{(r)}(t)\right) + g^{(i)}\left(x^{(i)}(t)\right)i \\ &\quad + g^{(j)}\left(x^{(j)}(t)\right)j + g^{(k)}\left(x^{(k)}(t)\right)k, \end{aligned} \tag{3}$$

where $x^{(p)}(t)$, $g^{(p)}(x^{(p)}(t)) \in \mathbb{R}^l$ and $M^{(p)}$, $N^{(p)} \in \mathbb{R}^{l\times l}$, $p = r, i, j, k$.

Remark 1. In general, let $x = x^{(r)} + x^{(i)}i + x^{(j)}j + x^{(k)}k$, and the activation function $g(x)$ should be written as follows:

$$\begin{aligned} g(x) = {} & g^{(r)}\left(x^{(r)}, x^{(i)}, x^{(j)}, x^{(k)}\right) + g^{(i)}\left(x^{(r)}, x^{(i)}, x^{(j)}, x^{(k)}\right)i \\ & + g^{(j)}\left(x^{(r)}, x^{(i)}, x^{(j)}, x^{(k)}\right)j \\ & + g^{(r)}\left(x^{(r)}, x^{(i)}, x^{(j)}, x^{(k)}\right)k. \end{aligned} \tag{4}$$

However, in this paper, to reduce the difficulty of research and simplify the results of finite-time stability of QVNNs, we employ a special activation function introduced above, such as the activation functions of illustrative examples later.

By means of decomposition methods as those used in [41,47], we decompose QVNNs (2) into four RVNNs equally and combine them into a equivalent VMDE as follows

$$\dot{Q}(t) = -\widehat{C}Q(t) + \widehat{A}\widehat{g}(Q(t)) + \widehat{B}\widehat{g}(Q(t-\tau(t))) + \widehat{I}(t), \tag{5}$$

$$Q(s) = \Psi(s), s \in \left[t_0 - \tau, t_0\right], \tag{6}$$

where

$$\begin{aligned} \widehat{C} &= \operatorname{diag}\{C, C, C, C\} \in \mathbb{R}^{4l\times 4l}, \\ \widehat{A} &= \begin{pmatrix} M^{(r)} & -M^{(i)} & -M^{(j)} & -M^{(k)} \\ M^{(i)} & M^{(r)} & -M^{(k)} & M^{(j)} \\ M^{(j)} & M^{(k)} & M^{(r)} & -M^{(i)} \\ M^{(k)} & -M^{(j)} & M^{(i)} & M^{(r)} \end{pmatrix} \in \mathbb{R}^{4l\times 4l}, \widehat{B} = \begin{pmatrix} N^{(r)} & -N^{(i)} & -N^{(j)} & -N^{(k)} \\ N^{(i)} & N^{(r)} & -N^{(k)} & N^{(j)} \\ N^{(j)} & N^{(k)} & N^{(r)} & -N^{(i)} \\ N^{(k)} & -N^{(j)} & N^{(i)} & N^{(r)} \end{pmatrix} \in \mathbb{R}^{4l\times 4l}, \\ Q(t) &= \left(x^{(r)}(t)^T, x^{(i)}(t)^T, x^{(j)}(t)^T, x^{(k)}(t)^T\right)^T \in \mathbb{R}^{4l}, \\ \Psi(s) &= \Psi = \left(\psi^{(r)}(t)^T, \psi^{(i)}(t)^T, \psi^{(j)}(t)^T, \psi^{(k)}(t)^T\right)^T \in \mathbb{R}^{4l}, \\ \widehat{I}(t) &= \left(\left(I^{(r)}(t)\right)^T, \left(I^{(i)}(t)\right)^T, \left(I^{(j)}(t)\right)^T, \left(I^{(k)}(t)\right)^T\right)^T \in \mathbb{R}^{4l}, \\ \widehat{g}(Q(t)) &= \left(\left(g^{(r)}\left(x^{(r)}(t)\right)\right)^T, \left(g^{(i)}\left(x^{(i)}(t)\right)\right)^T, \left(g^{(j)}\left(x^{(j)}(t)\right)\right)^T, \left(g^{(k)}\left(x^{(k)}(t)\right)\right)^T\right)^T \in \mathbb{R}^{4l}, \\ \widehat{g}(Q(t-\tau(t))) &= \left(\left(g^{(r)}\left(x^{(r)}(t-\tau(t))\right)\right)^T, \left(\left(g^{(i)}\left(x^{(i)}(t-\tau(t))\right)\right)^T, \left(\left(g^{(j)}\left(x^{(j)}\left(t-\tau(t)\right)\right)\right)^T, \right.\right.\right. \\ & \qquad g^{(k)}\left(x^{(k)}(t-\tau(t))\right)\Big)^T \in \mathbb{R}^{4l}. \end{aligned} \tag{7}$$

Remark 2. In fact, system (5) is a real-valued system. Evidently, the dynamic characteristics of QVNNs (2) are in accord with those of system (5) by considering that $x(t) = x^{(r)}(t) + x^{(i)}(t)i + x^{(j)}(t)j + x^{(k)}(t)k$ corresponds to $Q(t)$. Therefore, one only needs to analyze system (5)'s dynamical characteristics instead of system (2), and the difficulty of noncommutativity of quaternion can be overcome.

In order to explicitly present main results, some definitions, assumptions, and lemmas should be introduced firstly.

Assumption 1. g: $\mathbb{R}^l \longrightarrow \mathbb{R}^l$ (or $g = (g_1, g_2, \ldots, g_l)^T$), which is a continuous function, is called a function of class $\Delta\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$; if $g(x)$ satisfies $g_i(0) = 0$ and for each $a, b \in \mathbb{R}$, $a \neq b$, there exist $\alpha_i > 0$ such that

$$0 \leq \frac{g_i(a) - g_i(b)}{a - b} \leq \alpha_i, \quad i = 1, 2, \ldots, l, \tag{8}$$

and let $\Delta = \operatorname{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$.

Definition 1 (see [7]). System (5)can reach a stable state in a finite time if a initial condition Ψ is given such that the system (5) is Lyapunov stable and any solution $Q(t, \Psi)$ of (5) satisfies $Q(t, \Psi) = 0, \quad \forall t \geq T(\Psi),$ where $T(\Psi)$: $R^{4l} \longrightarrow R^+ \cup \{0\}$ is the settling time function.

Remark 3. The convergence time interval of finite-time stability must be given in advance, but it is difficult to estimate the upper boundary of the time interval. In this paper, some new vector-matrix analysis techniques are developed to derive the upper boundary, and the vector-matrix techniques can be used to investigate the finite-time synchronization of QVNNs in future work.

Assumption 2. If Assumption 1 holds, one obtains $\widehat{g} = ((g^{(r)})^T, (g^{(i)})^T, (g^{(j)})^T, (g^{(k)})^T)^T = (\widehat{g}_1, \widehat{g}_2, \cdots, \widehat{g}_{4l})^T$: $\mathbb{R}^{4l} \longrightarrow \mathbb{R}^{4l}$, $\widehat{g} \in \widehat{\Delta}\{\alpha_1, \alpha_2, \ldots, \alpha_l, \alpha_1, \alpha_2, \ldots, \alpha_l, \alpha_1, \alpha_2, \ldots, \alpha_l, \alpha_1, \alpha_2, \ldots, \alpha_l\}$ and $\widehat{\Delta} = \operatorname{diag}\{\Delta, \Delta, \Delta, \Delta\}$.

Lemma 1 (see [48]). *The system VMDE (5) is called to be finite-time stable; if under Assumption 1 and the initial* *condition $\psi \in \Omega$, a continuous function V: $[0, +\infty) \times \Omega \longrightarrow R^{+}$ ($\alpha, r \in \mathscr{K}$) can satisfy:*

(1) $V(t, 0) = 0, \alpha(\|\psi\|) \leq V(t, \psi), t \in [0, +\infty)$.

(2) $D^{+}V(t, \psi) \leq -r(V(t, \psi))$ with $\int_0^{\varepsilon}(dz/r(z)) < +\infty$, for all $\varepsilon > 0, \psi \in \Omega$.

And the settling time is estimated to be $T \leq \int_0^{V(0,\psi)}(dz/r(z))$. Moreover, when $r(V) = kV^{\sigma}$ $(k > 0, 0 < \sigma < 1)$, the settling time can be estimated by the following inequality:

$$T \leq \int_0^{V(0,\psi)} \frac{dz}{r(z)} = \frac{V^{1-\sigma}(0, \psi)}{k(1-\sigma)}. \tag{9}$$

Lemma 2 (see [49]). *Let $Q_j \geq 0$ for $j = 1, 2, \ldots, l$, and $0 < a \leq 1$, $b > 1$; then, the following inequalities hold:*

$$\left(\sum_{j=1}^{l} Q_j\right)^a \leq \sum_{j=1}^{l} Q_j^a,\ l^{1-b}\left(\sum_{j=1}^{l} Q_j\right)^b \leq \sum_{j=1}^{l} Q_j^b. \tag{10}$$

Lemma 3 (see [48]). *If system (5) can reach a finite-time stable state, then we can find a function $r \in \mathscr{K}$, which is a continuous and positive definite, such that, for all Lyapunov functions $V(t, \psi)$ ($V(t, \psi)$ is the same as $V(t, \psi)$ in Lemma 1),*

$$D^{-}V(t, \psi) \geq -r(V(t, \psi)), \tag{11}$$

$$\int_0^{\varepsilon} \frac{1}{r(z)} dz < +\infty, \tag{12}$$

always hold.

Remark 4. Lemma 1 is a sufficient condition for judging finite-time stability, and Lemma 3 is a necessary condition about finite-time stability. Lemma 3 can be used when we judge finite-time instability of that QVNN. Lemma 2 will be used to derive $D^{+}V(t, \psi) \leq -r(V(t, \psi))$ and $D^{-}V(t, \psi) \geq -r(V(t, \psi))$ in the proof of Theorems 1 and 2 later.

3. Main Results

In this section, by designing several suitable nonlinear controllers, some criteria are proposed to carry out stabilization and destabilization of system (5) in a finite time. The following controllers are designed:

$$\begin{aligned}
I^{(r)}(t) &= -\lambda_1^{(r)} x^{(r)}(t) - \lambda_2^{(r)}\left(\left|x^{(r)}(t)\right|^{\sigma_1}\right)^T \operatorname{sgn}\left(x^{(r)}(t)\right) - \theta^{(r)}\left(\left|x^{(r)}(t-\tau(t))\right|\right)^T \operatorname{sgn}\left(x^{(r)}(t)\right),\\
I^{(i)}(t) &= -\lambda_1^{(i)} x^{(i)}(t) - \lambda_2^{(i)}\left(\left|x^{(i)}(t)\right|^{\sigma_1}\right)^T \operatorname{sgn}\left(x^{(i)}(t)\right) - \theta^{(i)}\left(\left|x^{(i)}(t-\tau(t))\right|\right)^T \operatorname{sgn}\left(x^{(i)}(t)\right),\\
I^{(j)}(t) &= -\lambda_1^{(j)} x^{(j)}(t) - \lambda_2^{(j)}\left(\left|x^{(j)}(t)\right|^{\sigma_1}\right)^T \operatorname{sgn}\left(x^{(j)}(t)\right) - \theta^{(j)}\left(\left|x^{(j)}(t-\tau(t))\right|\right)^T \operatorname{sgn}\left(x^{(j)}(t)\right),\\
I^{(k)}(t) &= -\lambda_1^{(k)} x^{(k)}(t) - \lambda_2^{(k)}\left(\left|x^{(k)}(t)\right|^{\sigma_1}\right)^T \operatorname{sgn}\left(x^{(k)}(t)\right) - \theta^{(k)}\left(\left|x^{(k)}(t-\tau(t))\right|\right)^T \operatorname{sgn}\left(x^{(k)}(t)\right),
\end{aligned} \tag{13}$$

where $\sigma_1 > 0$, and $\lambda_1^{(p)}, \lambda_2^{(p)}, \theta^{(p)} \in \mathbb{R}$, $p = r, i, j, k$, and the vector form

$$\widehat{I}(t) = -\Lambda_1 Q(t) - \Lambda_2 Q_t^{\sigma_1} \operatorname{sgn}(Q(t)) - \Theta Q_{t-\tau} \operatorname{sgn}(Q(t)), \tag{14}$$

$$\begin{aligned}
\Lambda_1 &= \operatorname{diag}\left\{\lambda_1^{(r)}, \ldots, \lambda_1^{(r)}, \lambda_1^{(i)}, \ldots, \lambda_1^{(i)}, \lambda_1^{(j)}, \ldots, \lambda_1^{(j)}, \lambda_1^{(k)}, \ldots, \lambda_1^{(k)}\right\} \in \mathbb{Q}^{4l\times 4l},\\
\Lambda_2 &= \operatorname{diag}\left\{\lambda_2^{(r)}, \ldots, \lambda_2^{(r)}, \lambda_2^{(i)}, \ldots, \lambda_2^{(i)}, \lambda_2^{(j)}, \ldots, \lambda_2^{(j)}, \lambda_2^{(k)}, \ldots, \lambda_2^{(k)}\right\} \in \mathbb{Q}^{4l\times 4l},\\
\Theta &= \operatorname{diag}\left\{\theta^{(r)}, \ldots, \theta^{(r)}, \theta^{(i)}, \ldots, \theta^{(i)}, \theta^{(j)}, \ldots, \theta^{(j)}, \theta^{(k)}, \ldots, \theta^{(k)}\right\} \in \mathbb{Q}^{4l\times 4l},\\
Q_t &= \operatorname{diag}\left\{\left|x_1^{(r)}(t)\right|, \ldots, \left|x_l^{(r)}(t)\right|, \left|x_1^{(i)}(t)\right|, \ldots, \left|x_l^{(i)}(t)\right|, \left|x_1^{(j)}(t)\right|, \ldots, \left|x_l^{(j)}(t)\right|, \left|x_1^{(k)}(t)\right|, \ldots, \left|x_l^{(k)}(t)\right|\right\} \in \mathbb{Q}^{4l\times 4l},\\
Q_{t-\tau} &= \operatorname{diag}\left\{\left|x_1^{(r)}(t-(\tau)t)\right|, \ldots, \left|x_l^{(r)}(t-(\tau)t)\right|, \left|x_1^{(i)}(t-(\tau)t)\right|, \ldots, \left|x_l^{(i)}(t-(\tau)t)\right|, \left|x_1^{(j)}(t-(\tau)t)\right|, \ldots, \left|x_l^{(j)}(t-(\tau)t)\right|,\right.\\
&\qquad \left.\left|x_1^{(k)}(t-(\tau)t)\right|, \ldots, \left|x_l^{(k)}(t-(\tau)t)\right|\right\} \in \mathbb{Q}^{4l\times 4l},\\
\operatorname{sgn}(Q(t)) &= \left(\operatorname{sgn}\left(x^{(r)}(t)\right)^T, \operatorname{sgn}\left(x^{(i)}(t)\right)^T, \operatorname{sgn}\left(x^{(j)}(t)\right)^T, \operatorname{sgn}\left(x^{(k)}(t)\right)^T\right)^T \in \mathbb{Q}^{4l}.
\end{aligned} \tag{15}$$

Theorem 1. *When Assumptions 1 and 2 hold,* $0 < \sigma_1 < 1$ *and* $\Lambda_2 > 0$*, given positive diagonal matrices* Λ_1 *and* Θ *such that*

$$\begin{aligned} I^T\left[-\left(\widehat{C}+\Lambda_1\right)+|\widehat{A}|\widehat{\Delta}\right] &< 0, \\ I^T\left(|\widehat{B}|\widehat{\Delta}-\Theta\right) &< 0, \end{aligned} \quad (16)$$

then under controller (14), the VMDE (5) will reach a stable state in a finite-time interval. T is the settling time and can be prescribed by $T \leq (1/\lambda_{2\min}(1-\sigma_1))V(0)^{1-\sigma_1}$*, where* $\lambda_{2\min} = \min\{\lambda_2^{(\mu)}\}$*,* $\mu = r, i, j, k$*.*

Proof. The following Lyapunov candidate functional will be considered by us:

$$V(t) = \|Q(t)\|. \quad (17)$$

Based on the solution trajectories of system (5) to calculate the upper-right Dini derivative of $V(t)$, one obtains

$$\begin{aligned} D^{+}V(t) &= \operatorname{sgn}(Q(t))^T\dot{Q}(t) \\ &= \operatorname{sgn}(Q(t))^T\left[-\widehat{C}Q(t)+\widehat{A}\widehat{g}(Q(t))+\widehat{B}\widehat{g}(Q(t-\tau(t)))-\Lambda_1Q(t)-\Lambda_2Q_t^{\sigma_1}\operatorname{sgn}(Q(t))-\Theta Q_{t-\tau}\operatorname{sgn}(Q(t))\right] \\ &\leq -I^T\left(\widehat{C}+\Lambda_1\right)|Q(t)|+I^T|\widehat{A}||\widehat{g}(Q(t))|+I^T|\widehat{B}||\widehat{g}(Q(t-\tau(t)))|-I^T\Lambda_2Q_t^{\sigma_1}I-I^T\Theta Q_{t-\tau}I \\ &\leq -I^T\left(\widehat{C}+\Lambda_1\right)|Q(t)|+I^T|\widehat{A}|\widehat{\Delta}|Q(t)|+I^T|\widehat{B}|\widehat{\Delta}|Q(t-\tau(t))|-I^T\Theta Q_{t-\tau}I-I^T\Lambda_2Q_t^{\sigma_1}I \\ &\leq -I^T\left(\widehat{C}+\Lambda_1\right)|Q(t)|+I^T|\widehat{A}|\widehat{\Delta}|Q(t)|+I^T\left(|\widehat{B}|\widehat{\Delta}-\Theta\right)Q_{t-\tau}I-I^T\Lambda_2Q_t^{\sigma_1}I \\ &= I^T\left[-\left(\widehat{C}+\Lambda_1\right)+|\widehat{A}|\widehat{\Delta}\right]|Q(t)|+I^T\left(|\widehat{B}|\widehat{\Delta}-\Theta\right)|Q(t-\tau(t))|-I^T\Lambda_2Q_t^{\sigma_1}I, \end{aligned} \quad (18)$$

Here, by Assumption 2, $|\widehat{g}_i(Q_i(t))-\widehat{g}_i(0)| \leq \alpha_i|Q_i(t)-0|$ $(i=1,\ldots,4l)$ is used.

In view of $I^T[-(\widehat{C}+\Lambda_1)+|\widehat{A}|\widehat{\Delta}] < 0$, $I^T(|\widehat{B}|\widehat{\Delta}-\Theta) < 0$, and Lemma 2, the following inequality can be established:

$$\begin{aligned} D^{+}V(t) &\leq -I^T\Lambda_2Q_t^{\sigma_1}I \\ &\leq -\lambda_{2\min}\left(I^TQ_tI\right)^{\sigma_1} \\ &= -\lambda_{2\min}\|Q(t)\|^{\sigma_1} \\ &= -\lambda_{2\min}V^{\sigma_1}(t), \end{aligned} \quad (19)$$

where $\lambda_{2min} = min\{\lambda_2^{(\mu)}\}, \mu = r, i, j, k, \Lambda_2 > 0$.

And for all $\varepsilon > 0$, one has

$$\int_0^{\varepsilon}\frac{1}{\lambda_{2\min}z^{\sigma}}dz = \frac{1}{\lambda_{2\min}(1-\sigma_1)}\varepsilon^{1-\sigma_1} < +\infty. \quad (20)$$

Hence, by Lemma 1, we obtain that system (5) is finite-time stable under controller (14). And the settling time is prescribed by

$$T \leq \int_0^{V(0)}\frac{1}{\left(\lambda_{2\min}z^{\sigma_1}\right)}dz = \frac{1}{\lambda_{2\min}(1-\sigma_1)}V(0)^{1-\sigma_1}. \quad (21)$$

□

Remark 5. Obviously, the settling time is related to the parameters λ_{2min} and $V(0)$ under $0 < \sigma_1 < 1$. The results obtained here is more general; let σ_1 choose some special value, and the exponentially stable and power stable can be obtained. If $\sigma_1 = 1$, the VMDE (5) is exponentially stable. However, when $\sigma_1 > 1$, $t = \int_{V(t)}^{V(0)} 1/\lambda_{2\min}z^{\sigma_1}dz = V(t)^{1-\sigma_1} - V(0)^{1-\sigma_1}/\lambda_{2\min}(\sigma_1-1)$ or $V(t) = [V(0)^{1-\sigma_1}+\lambda_{2\min}(\sigma_1-1)t]^{1/1-\sigma}$; then, we know VMDE (5) is power stable with power rate $(1/1-\sigma_1)$.

Theorem 2. *When Assumptions 1 and 2 hold,* $\sigma_1 > 1$ *and* $\Lambda_2 > 0$*, given negative definite diagonal matrices* Λ_1 *and* Θ*, such that*

$$I^T\left[\widehat{C}+\Lambda_1+|\widehat{A}|\widehat{\Delta}\right] < 0, \quad (22)$$

$$I^T\left[|\widehat{B}|\widehat{\Delta}+\Theta\right] < 0, \quad (23)$$

then under controller (14), the VMDE (5) cannot reach a stable state in a finite time.

Proof. Choose the same Lyapunov candidate function as Theorem 1:

$$V(t) = \|Q(t)\|. \quad (24)$$

Computing the lower-right Dini derivative of V(t) based on the solution trajectories of system (5), one obtains&ecmath;

$$
\begin{aligned}
D^{-}V(t) &= \operatorname{sgn}(Q(t))^{T}\dot{Q}(t)\\
&= \operatorname{sgn}(Q(t))^{T}\left[-\hat{C}Q(t) + \hat{A}\hat{g}(Q(t)) + \hat{B}\hat{g}\right.\\
&\quad \cdot \left(-\Lambda_1 Q(t) - \Lambda_2 Q_t^{\sigma_1}\operatorname{sgn}(Q(t)) - \Theta Q_{t-\tau}\operatorname{sgn}(Q(t))\right)]\\
&\geq -I^{T}\left(\hat{C}+\Lambda_1\right)|Q(t)| - I^{T}|\hat{A}||\hat{g}(Q(t))\big| - I^{T}\hat{B}\big\|\hat{g}\\
&\quad \cdot (Q(t-\tau(t))\big| - I^{T}\Lambda_2 Q_t^{\sigma_1} I - I^{T}\Theta Q_{t-\tau} I\\
&\geq -I^{T}\left(\hat{C}+\Lambda_1\right)|Q(t)| - I^{T}|\hat{A}|\hat{\Delta}|Q(t)| - I^{T}|\hat{B}|\hat{\Delta}|Q\\
&\quad \cdot (t-\tau(t))| - I^{T}\Theta Q_{t-\tau} I - I^{T}\Lambda_2 Q_t^{\sigma_1} I\\
&\geq -I^{T}\left(\hat{C}+\Lambda_1\right)|Q(t)\big| - I^{T}\big|\hat{A}|\hat{\Delta}|Q(t)\big| - I^{T}\left(|\hat{B}|\hat{\Delta} + \Theta\right)\\
&\quad \cdot Q_{t-\tau} I - I^{T}\Lambda_2 Q_t^{\sigma_1} I\\
&= -I^{T}\left[\hat{C} + \Lambda_1 + |\hat{A}|\hat{\Delta}\right]|Q(t)| - I^{T}\left(|\hat{B}|\hat{\Delta} + \Theta\right)|Q\\
&\quad \cdot (t-\tau(t))|\big| - I^{T}\Lambda_2 Q_t^{\sigma_1} I.
\end{aligned}
\tag{25}
$$

Here, by Assumption 2, $|\widehat{g}_i(Q_i(t)) - \widehat{g}_i(0)| \leq \alpha_i |Q_i(t) - 0|\,(i = 1, \ldots, 4l)$ is employed.

And it follows from $\widehat{C} + \Lambda_1 + |\widehat{A}|\widehat{\Gamma} < 0$, $|\widehat{B}|t\widehat{\Gamma}n + q\,\Theta h <_0$, and Lemma 2 that

$$
\begin{aligned}
D^{-}V(t) &\geq -I^{T}\Lambda_2 Q_t^{\sigma_1} I\\
&\geq -(4l)^{(1-\sigma_1)}\lambda_{2\max}\left(I^{T}Q_t I\right)^{\sigma_1}\\
&= -(4l)^{(1-\sigma_1)}\lambda_{2\max}\|Q(t)\|^{\sigma_1},
\end{aligned}
\tag{26}
$$

where $\lambda_{2\max} = \max\left\{\lambda_2^{(p)}\right\}, \quad p = r, i, j, k,\ \Lambda_2 > 0.$

Therefore,

$$
D^{-}V(t) \geq -(4l)^{(1-\sigma_1)}\lambda_{2\max}V^{\sigma_1}(t). \tag{27}
$$

However, by $\sigma_1 > 1$, for all $\varepsilon > 0$,

$$
\begin{aligned}
\int_0^{\varepsilon}\frac{1}{(4l)^{(1-\sigma_1)}\lambda_{2\max}z^{\sigma_1}}dz &= \frac{(4l)^{(\sigma_1-1)}}{\lambda_{2\max}(1-\sigma_1)}\\
&\quad \cdot\left(\varepsilon^{1-\sigma_1} - \lim_{z\longrightarrow 0^{+}} z^{1-\sigma_1}\right) = +\infty.
\end{aligned}
\tag{28}
$$

By Lemma 1, one obtains that system (5) under controller (14) cannot be finite-time stable. □

Remark 6. The time-varying delays of system (5) under controller (14) can be understood as follows. In fact, the third term $-\Theta Q_{t-\tau}\operatorname{sgn}(Q(t))$ in controller (14) and scaling techniques is employed to reduce its influence. And if the time delays are infinite, the system cannot achieve finite-time stabilization; therefore, $\tau(t)$ is supposed to be finite. Furthermore, we cannot ignore time delays' influence when discussing the short-time stability of various dynamical systems. However, fewer literature utilized the time delays in their controllers; hence, this paper attempts to design a nonlinear controller with time delays, which is a meaningful work.

Remark 7. $I^{T}[-(\widehat{C}+\Lambda_1) + |\widehat{A}|\widehat{\Delta}] < 0$, as well as (16), (22), and (23), indicate the column summations of square matrices are negative. And they are algebraic expressions which can be easily checked.

Remark 8. Zhang et al. [4] considers stability and instability of a complex value neural network in a finite time. In this paper, the analysis method of [4] is generalized to the finite-time stability and instability of QVNNs. Compared to [4], though the derivation process of this paper is very brief, it can also explain the stability and instability of QVNNs well. Therefore, the vector-matrix analysis method can be widely used for the other stability analysis of neural networks. Furthermore, there is no result to discuss the finite-time stability and instability of QVNNs with discrete delays. This paper is one of the first to do this attempt.

4. Illustrative Examples

In this section, the validity and superiority of the proposed criteria will be checked via two illustrative examples. And we will show that our vector-matrix methods are more suitable for calculating some problems of high-dimension systems by computer programming.

Example 1. Consider the QVNNs model as follows:

$$
\dot{x}(t) = -Cx(t) + Mg(x(t)) + Ng(x(t-\tau(t))) + I(t), \tag{29}
$$

where

$$
\begin{aligned}
M &= \begin{pmatrix} -6+5i+5j+5k & -4+2i-3j+1k\\ 4-2i+3j+1k & 9+5i+1j+6k \end{pmatrix},\\
N &= \begin{pmatrix} 3+2i+3j+1.3k & 4+4i-4j-2k\\ -4-4i+4j+3k & 2+2i+3j+4k \end{pmatrix},\\
C &= diag\{18, 7\},\ \tau(t) = 0.45\sin t + 0.35,\\
g(x(t)) &= \frac{\left|x^{(r)}(t)+1\right| - \left|x^{(r)}(t)-1\right|}{2}\\
&\quad + \frac{\left|x^{(i)}(t)+1\right| - \left|x^{(i)}(t)-1\right|}{2}i\\
&\quad + \frac{\left|x^{(j)}(t)+1\right| - \left|x^{(j)}(t)-1\right|}{2}j\\
&\quad + \frac{\left|x^{(k)}(t)+1\right| - \left|x^{(k)}(t)-1\right|}{2}k.
\end{aligned}
\tag{30}
$$

Under $I(t) = 0$ and initial condition $x(s) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -3 \\ 2.5 \end{pmatrix} i + \begin{pmatrix} -3 \\ -3.1 \end{pmatrix} j + \begin{pmatrix} 3 \\ 4 \end{pmatrix} k, \quad s \in [0.8, 0]$, the state trajectories of system (29) are shown in Figure 1(a), which shows that system (29) is unstable. By Assumptions 1 and 2, choose $\widehat{\Delta} = \mathrm{diag}\{0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01\}$. To reach the finite-time stable conditions of Theorem 1, by (14), the following controller is designed:

$$\widehat{I}(t) = -\Lambda_1 Q(t) - \Lambda_2 Q_t^{\sigma_1} \mathrm{sgn}(Q(t)) - \Theta Q_{t-\tau} \mathrm{sgn}(Q(t)), \tag{31}$$

where

$$\begin{aligned} \sigma_1 &= 0.5, \\ \Lambda_2 &= \mathrm{diag}\{20, 20, 20, 20, 20, 20, 20, 20\}. \end{aligned} \tag{32}$$

Then, when consider appropriate $\widehat{\Delta}, \Lambda_1$ such that $-\widehat{C} - \Lambda_1 + \widehat{A}\widehat{\Delta} < 0$, the LMI toolbox in MATLAB is used, and then it is easy to check $I^T(-\widehat{C} - \Lambda_1 + \widehat{A}\widehat{\Delta}) < 0$. So, the following feasible solutions of Λ_1 and Θ can be obtained:

$$\begin{aligned} \Lambda_1 &= diag\{417.7830, 417.7830, 417.7830, 417.7830, 417.7830, 417.7830, 417.7830, 417.7830\}, \\ \Theta &= diag\{423.9499, 423.9499, 423.9499, 423.9499, 423.9499, 423.9499, 423.9499, 423.9499\}, \\ I^T\left(-\widehat{C} - \Lambda_1 + |\widehat{A}|\widehat{\Delta}\right) &= \{-434.5630, -423.9430, -434.4430, -423.9030, -434.5030, -423.8330, -434.2130, -423.2530\} < 0, \\ I^T\left(-\Theta + |\widehat{B}|\widehat{\Delta}\right) &= \{-423.3759, -422.9399, -422.9669, -422.8299, -423.1369, -422.8899, -422.9669, -422.8899\} < 0. \end{aligned} \tag{33}$$

Therefore, condition (16) of Theorem 1 can be verified. Hence, by Theorem 1, under controller (31), system (29) can reach the stable state in finite time, and one can estimate the settling time $T \leq 0.9716$. Furthermore, the state trajectories of $x(t)$ of system (29) under controller (31) are shown in Figure 1(b), which shows that any solution of system (29) can converge to zero in a finite-time interval. Therefore, the correctness of Theorem 1 is verified.

Now, we analyze the effect of the parameter Λ_2 and initial condition on the settling time T. When initial condition $x(t) = 0$, obviously, $T = 0$. Fix other values and increase the value $\lambda_{2\min}$; the settling time will decrease, which can be shown in Figure 2. Therefore, the settling time in Theorem 1 is reasonable.

Example 2. Consider the QVNNs model as follows:

$$\dot{x}(t) = -Cx(t) + Mg(x(t)) + Ng(x(t - \tau(t))) + I(t), \tag{34}$$

where

$$\begin{aligned} M &= \begin{pmatrix} -3 + 2i + 2j - 4k & -0.4 - 2i - 3j + 1k \\ 0.4 + 2i + 3j - 1k & 3 + 0.5i + 1j + 1.9k \end{pmatrix}, \\ N &= \begin{pmatrix} 2 + 2i + 3j + 1.3k & -1.4 + 0.4i - 4j - 1.2k \\ 1.4 - 0.4i + 4j + 1.2k & 2 + 2i + 3j + 4k \end{pmatrix}, \\ C &= \mathrm{diag}\{20, 20\}, \\ \tau(t) &= 0.45 \sin t + 0.1, \\ g(x(t)) &= \frac{\left|x^{(r)}(t) + 1\right| - \left|x^{(r)}(t) - 1\right|}{2} \\ &\quad + \frac{\left|x^{(i)}(t) + 1\right| - \left|x^{(i)}(t) - 1\right|}{2} i \\ &\quad + \frac{\left|x^{(j)}(t) + 1\right| - \left|x^{(j)}(t) - 1\right|}{2} j \\ &\quad + \frac{\left|x^{(k)}(t) + 1\right| - \left|x^{(k)}(t) - 1\right|}{2} k. \end{aligned} \tag{35}$$

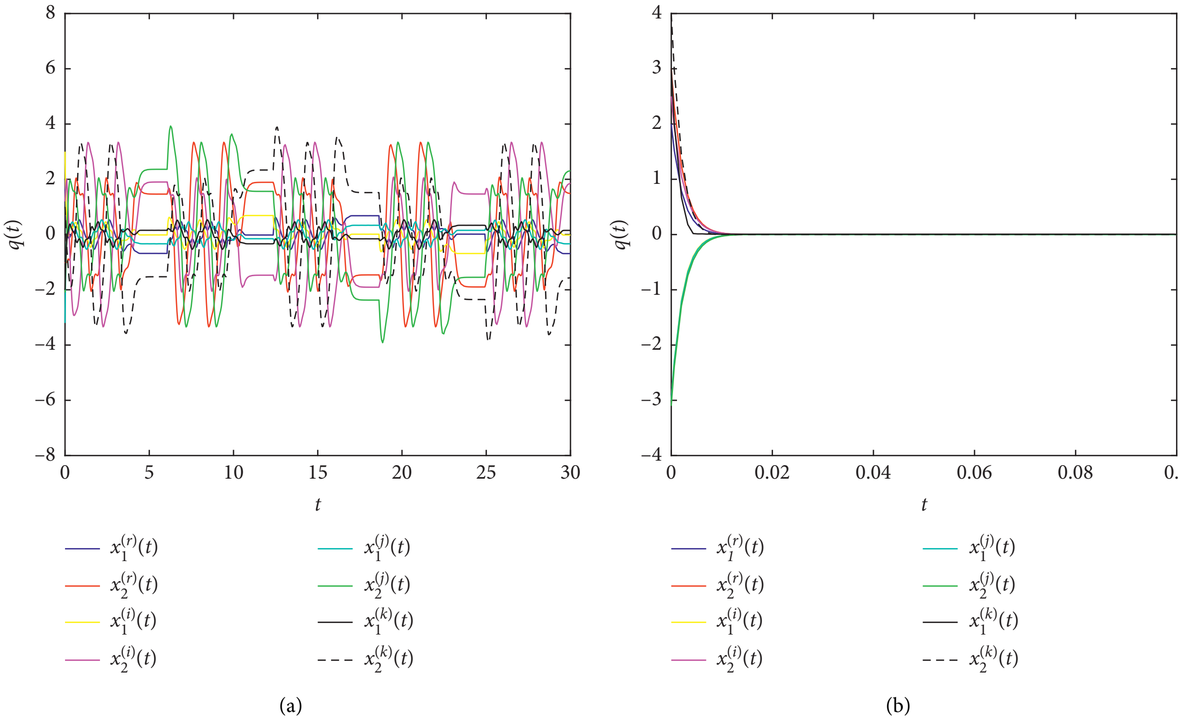


FIGURE 1: The state trajectories of $x^r(t), x^i(t), x^j(t), x^k(t)$ of QVVNs (29). (a) With $\mathrm{I}(t) = 0$. (b) Under controller (31).

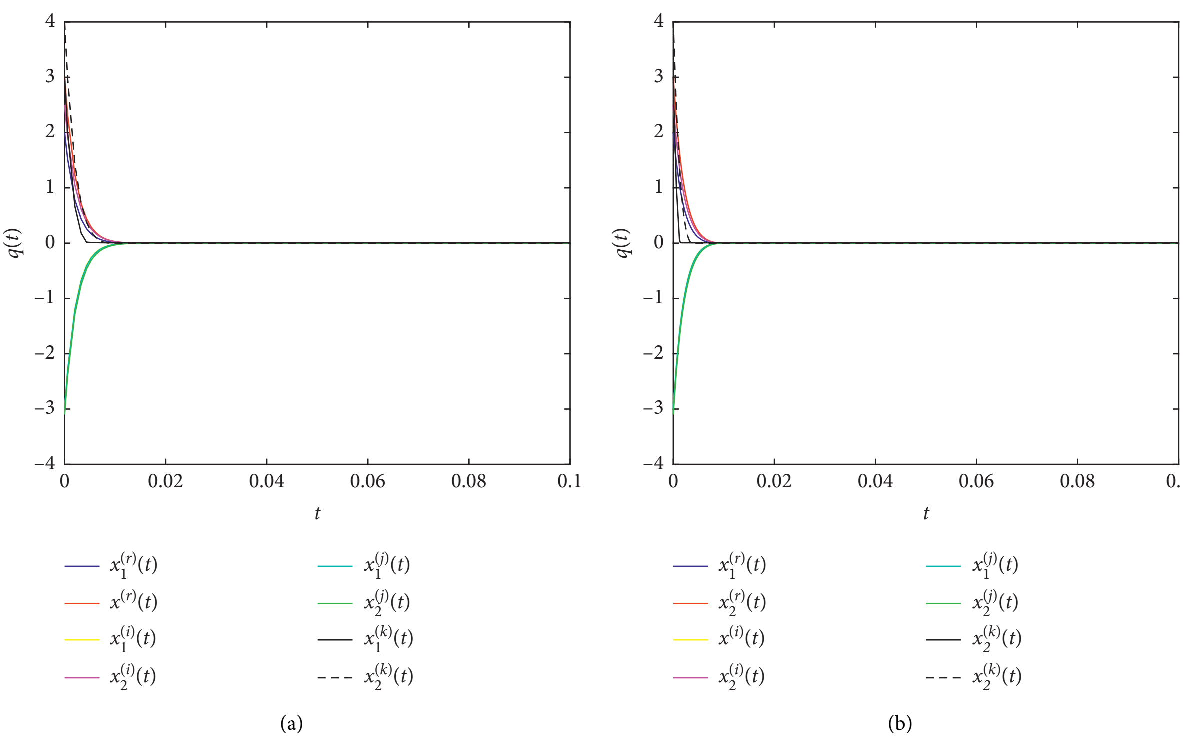
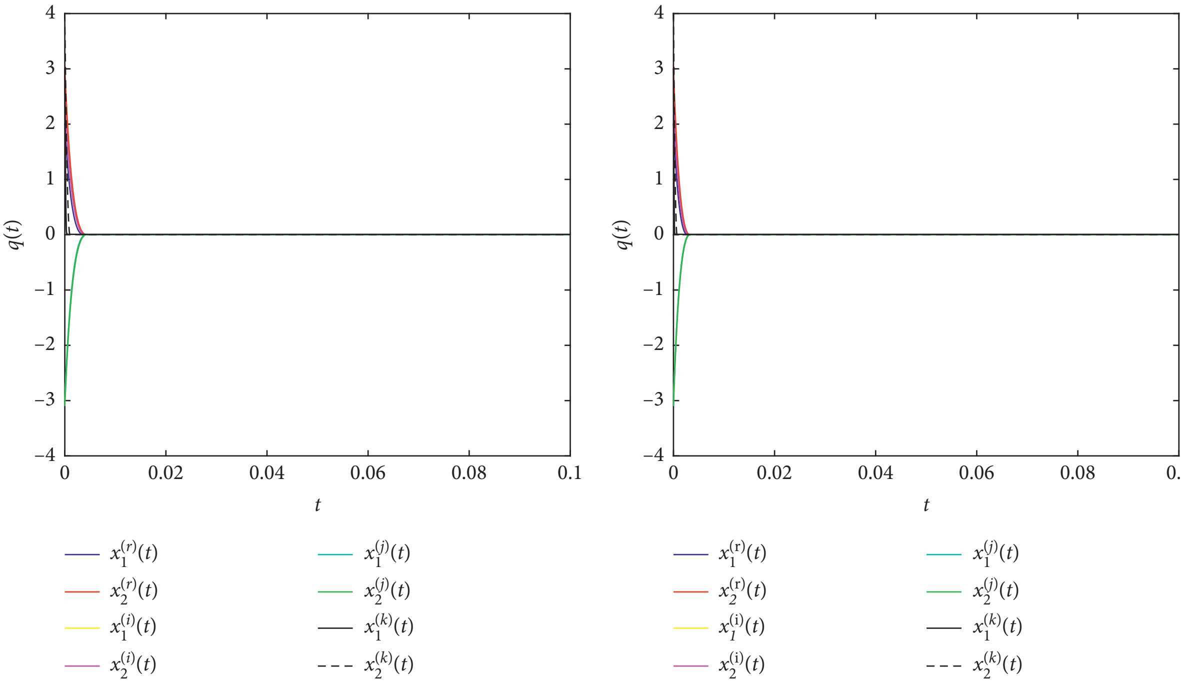
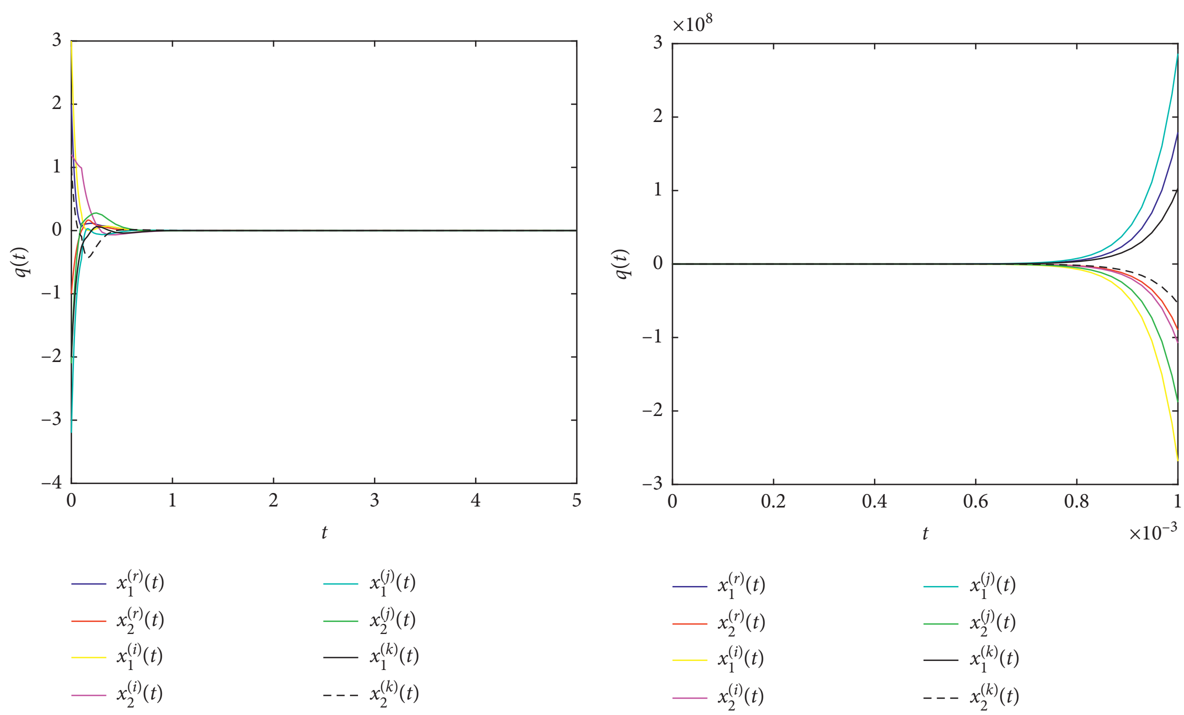


FIGURE 2: Continued.



(c) (d)

FIGURE 2: Effect of the change of Λ_2 on the settling time of QVNNs model (29). (a) $\Lambda_2 = 20 \times \mathrm{E}$, (b) $\Lambda_2 = 100 \times E$, (c) $\Lambda_2 = 500 \times \mathrm{E}$, and (d) $\Lambda_2 = 750 \times \mathrm{E}$.



(a) (b)

FIGURE 3: The state trajectories of $x^{(r)}(t), x^{(i)}(t), x^{(j)}(t)$, and $x^{(k)}(t)$ of QVVNs model (34). (a) With $\mathrm{I}(t) = 0$. (b) Under controller (31).

Under $I(t) = 0$ and initial condition $x(s) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1.2 \end{pmatrix} i + \begin{pmatrix} -3.2 \\ -2.1 \end{pmatrix} j + \begin{pmatrix} -2 \\ 1 \end{pmatrix} k$, $s \in [0.55, 0]$, the state trajectories of system (34) are shown in Figure 3(a), which shows that system (34) is stable. By Assumptions 1 and 2, we let $\widehat{\Delta} = \mathrm{diag}\{5, 5, 5, 5, 5, 5, 5, 5\}$. To reach the finite-time instable conditions of Theorem 1, by (14), the following controller is designed:

$$\widehat{I}(t) = -\Lambda_1 Q(t) - \Lambda_2 Q_t^{\sigma_1} \mathrm{sgn}(Q(t)) - \Theta Q_{t-\tau} \mathrm{sgn}(Q(t)), \tag{36}$$

where

$$\begin{aligned} \sigma_1 &= 1.1, \\ \Lambda_2 &= \mathrm{diag}\{10, 10, 10, 10, 10, 10, 10, 10\}. \end{aligned} \tag{37}$$

Then, similarly, to realize $\widehat{C} + \Lambda_1 + |\widehat{A}|\widehat{\Delta} < 0$, $\Theta + |\widehat{B}|\widehat{\Delta} < 0$, the LMI toolbox in MATLAB is used and the following feasible solutions of Λ_1 and Θ can be obtained:

$$\begin{aligned} \Lambda_1 &= \mathrm{diag}\{-18361, -18361, -18361, -18361, -18361, \\ &\quad -18361, -18361, -18361\}, \\ \Theta &= \mathrm{diag}\{-18337, -18337, -18337, -18337, -18337, \\ &\quad -18337, -18337, -18337\}. \end{aligned} \tag{38}$$

And it so happened that

$$\begin{aligned} I^T\left(\widehat{C} + \Lambda_1 + |\widehat{A}|\widehat{\Delta}\right) &= 10^4 * \{-1.7779, -1.8002, -1.7779, \\ &\quad -1.7897, -1.7874, -1.8040, -1.7931, \\ &\quad -1.7954\} < 0, \\ I^T\left(\Theta + |\widehat{B}|\widehat{\Delta}\right) &= 10^4 * \{-1.8023, -1.7753, -1.7595, \\ &\quad -1.7582, -1.8032, -1.8019, -1.7937, \\ &\quad -1.7924\} < 0, \end{aligned} \tag{39}$$

Therefore, conditions (22) and (23) of Theorem 2 can be verified. The state trajectories of $x(t)$ of system (34) are shown in Figure 3(b), which shows that the state variables of system (34) can become big enough from zero point in a finite time, i.e., system (34) can reach the instable state in a finite-time interval under (36). Hence, the correctness of Theorem 2 is verified.

Remark 9. Through the analysis of these two examples, the advantages of the vector-matrix method processing finite-time stabilization and destabilization of QVNNs are checked, which is easy to calculate by computer programming. Furthermore, this approach is applicable when discussing other high-dimensional systems.

5. Conclusion

In this paper, we analyze two interesting problems, the finite-time stabilization and destabilization of QVNNs with discrete delays, respectively. Utilizing the decomposition method, a new, vector-matrix and suitable nonlinear controller is constructed to carry out the finite-time stabilization and destabilization of the discussed QVNNs, which is used by fewer references. Furthermore, the obtained criteria are compact, effective, and easily checked. Through two numerical examples, the correctness, the convenience, and the applicability of the two criteria are all verified. In addition, the problems of fixed-time stabilization and preassigned-time control of QVNNs are also interesting and challenging, which we will consider in the near future. Moreover, in this paper, the activation functions in model (2) are special functions; hence, we will also discuss the finite-time stability of QVNNs with more general activation functions in future work.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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