Research Article

Solitary and Periodic Wave Solutions of Sasa–Satsuma Equation and Their Relationship with Hamilton Energy

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In this paper, we study the exact solitary wave solutions and periodic wave solutions of the S-S equation and give the relationships between solutions and the Hamilton energy of their amplitudes. First, on the basis of the theory of dynamical system, we make qualitative analysis on the amplitudes of solutions. Then, by using undetermined hypothesis method, the first integral method, and the appropriate transformation, two bell-shaped solitary wave solutions and six exact periodic wave solutions are obtained. Furthermore, we discuss the evolutionary relationships between these solutions and find that the appearance of these solutions for the S-S equation is essentially determined by the value which the Hamilton energy takes. Finally, we give some diagrams which show the changing process from the periodic wave solutions to the solitary wave solutions when the Hamilton energy changes.

1. Introduction

Sasa–Satsuma equation (called S-S equation for short)

\[ i\partial_t z + \frac{1}{2} \partial_{xx} z + |z|^2 z + i\varepsilon \left[ \partial_{xxx} z + 6|z|^2 \partial_x z + 3 \left( |z|^2 \right)_x \right] = 0 \]  

(1)

was proposed by Sasa and Satsuma in studying the integrability of the Schrödinger equation with higher-order nonlinear terms, and it is a typical equation which describes the propagation of short pulses in an optical fiber [1–4]. The 1-soliton solutions and the N-soliton solutions of equation (1) were also given by Sasa and Satsuma in [1].

In the past 30 years, with the development of nonlinear system, different methods have been found and developed, such as the inverse scattering method, the bilinear method, the Darboux transformation, the Backlund transformation, the extended exploratory equation method, and the auxiliary equation method [4–28]. At the same time, the S-S equation has attracted interests of many scholars because of its importance [4–16]. They have worked out some solitary wave solutions and periodic wave solutions of the form

\[ z(x, t) = u(\xi) e^{i(k(x-c)t)}, \quad \xi = x - ct, \]  

(2)

for equation (1) by the inverse scattering method [4], the bilinearization method [5], the Darboux transformation [7–10], the Backlund transformation [11], and some other methods, where \( u(\xi), \xi', \) and \( c \) denote the amplitude, envelope, and wave velocity, respectively.

By using transformations \( z(x, t) = u(x, t) e^{i(x+\varepsilon t/6)}, \) \( T = t, \) and \( X = (x + \varepsilon/2t), \) [5–10] simplified the S-S equation into the following form:

\[ u_t + \frac{1}{6\varepsilon} \left( u_{xxx} + 6|u|^2 u_x + 3 \left( |u|^2 \right)_x \right) = 0. \]  

(3)

In [5, 6, 11, 12], the solitary wave solutions for the S-S equation in the form of

\[ z(x, t) = \frac{A_1}{D + \cosh[B(mX + ct)]} e^{i(kX - ct)} \]  

(4)

were obtained by the Hirota bilinear method, the Backlund transformation, the Darboux transformation, and the extended exploratory equation method, respectively.

In [12], they obtained the dark soliton solution of equation (1) by the generalized Kudryashov method. And by
using the extended exploratory equation method, they obtained the unbounded solution for equation (1) with amplitude
\[
u(X, T) = \frac{4A^2(\alpha_2 - \alpha_1)\tau_1}{4A^2 - [(\alpha_1 - \alpha_2)(mx + nt)]^2}.
\]
and periodic solutions for equation (1) with amplitude
\[
u(X, T) = \frac{A_1}{M + N\sin^2(\varphi, l)}.
\]
In recent literature [16], they studied the nonlocal S-S equation in the case of inverse space-time and obtained its solitary wave solutions.

Although the S-S equation has been solved by various methods in many literatures, there are still some solitary wave solutions and periodic wave solutions for the S-S equation that have not been solved. In this paper, we will study solutions of the form (2) for the S-S equation (1) and expect to find its new periodic wave solutions or solitary wave solutions. At the same time, we will study the relationships between solutions and Hamilton energy for the S-S equation.

First, by employing the theory and method of planar dynamical system, we make detailed qualitative analysis on the amplitudes of solutions for the S-S equation and give the global phase portraits under different parameters. We also obtain the conditions for the existence of solitary wave solutions and periodic wave solutions. In Section 3, according to the conclusions of the qualitative analysis, we use the undetermined hypothesis method to obtain two bell-shaped solitary wave solutions for the S-S equation, where one is new, bounded, and rational [17, 18]. In Section 4, by the first integral method and appropriate transformation, we obtain six periodic wave solutions for equation (1), and the amplitudes of two solutions are in the fractional form of the Jacobi elliptic function \(sn^2(\varphi, \xi)\), which are similar as the amplitudes of the periodic wave solutions found in [13]. However, for the rest four kinds of periodic wave solutions, their amplitudes are composed by \(dn(\varphi, k), cn(\varphi, k),\) and \(\cos \varphi\), which have not been obtained in the previous literatures. In Section 5, we study the evolutionary relationships between the solitary wave solutions, periodic wave solutions, and the Hamilton energy for the S-S equation and show the reason of the appearance of different kinds of solutions is that the corresponding Hamilton energy \(h\) takes different values. When \(h\) tends to be \(H(x_1, 0)\), the periodic wave solutions will take the solitary wave solutions as the limit. At last, we will show this evolutionary process in the diagrams.

It is worth to point out that (1) by combining the qualitative analysis with the first integral method, we not only obtain the new bell-shaped solitary wave solutions (unimodal) and periodic wave solutions for equation (1), but also show the mathematical meaning for the solutions by establishing the corresponding relationships between the solutions and the bounded trajectories in the phase portraits. (2) In the previous literatures [12, 19–21] for solving the solitary wave and periodic wave solutions of the nonlinear evolutionary equations, the relationships between solutions and the energy of system are rarely described. In this paper, we not only obtain the solutions of equation (1) but also show the relationships between the amplitudes of the solutions and the Hamilton energy for the corresponding system. Then, we have the conclusion that when Hamilton energy \(h\) takes different values, equation (1) has different solutions in the form of (2). For a nonlinear complex system, it is important to understand the essential reasons for the various and complex phenomena in this system and only by this way can people really control and apply the system which corresponds to the S-S equation.

2. Qualitative Analysis

In this section, we will employ the theory and method of planar dynamical system [29, 30] to make qualitative analysis on the traveling wave solutions of the form (2) for the S-S equation (1) and give the global phase portraits for the amplitudes of these solutions under different parameters. Then, several conclusions for the existence of these solutions can be obtained.

Assuming that the S-S equation (1) has solutions in the form of (2), by substituting (2) into equation (1), we have

\[
-icu' + c_1u + \frac{1}{2}(u' + 2ik_1u' - k_1^2u) + |u|^2u + i\varepsilon(u'' + 3ik_1u' - 3k_1^2u - ik_1^3u)
\]

\[+ i\varepsilon\{6|u|^2(u' + ik_1u) + 3u(|u|^2)\} = 0.
\]

(7)

Then, taking the real and imaginary parts of (7) as zero, we can get

\[
\left( c_1 - \frac{1}{2}k_1^2 + \varepsilon k_1^3 \right) u + \left( \frac{1}{2} - 3\varepsilon k_1 \right) u'' + (1 - 6\varepsilon k_1)|u|^2u = 0,
\]

(8)

In (8), we set the coefficients of \(u, u'', \) and \(|u|^2u\) as zero and obtain
Complexity

\[
\begin{cases}
    c_1 - \frac{1}{2} k_1^2 + \varepsilon k_1^3 = 0, \\
    \frac{1}{2} - 3\varepsilon k_1 = 0, \\
    1 - 6\varepsilon k_1 = 0.
\end{cases}
\] (10)

According to (10), we have

\[
\begin{cases}
    k_1 = \frac{1}{6\varepsilon}, \\
    c_1 = \frac{1}{108\varepsilon^2}.
\end{cases}
\] (11)

By substituting (11) into (9), we have

\[
\left(\frac{1}{12\varepsilon} - c\right)u'' + \varepsilon u' + 6\varepsilon |u|^2 u' + 3\varepsilon u(\varepsilon |u|^2)' = 0. \tag{12}
\]

Then, integrating on both sides of equation (12) with respect to \(\xi\), we can obtain

\[
\left(\frac{1}{12\varepsilon} - c\right)u + \varepsilon u' + 4\varepsilon u^3 = g_0, \tag{13}
\]

where \(g_0\) is a constant. According to (13), we have

\[
u'' + 4(u^3 + pu + q) = 0, \tag{14}
\]

where \(p = ((1 - 12\varepsilon)/48\varepsilon^2)\) and \(q = (-g_0/4\varepsilon)\).

From the above discussions, we can deduce that for studying the solutions of equation (1), we can start from equation (14).

Let \(x = u(\xi)\) and \(y = u'(\xi)\), equation (14) is equivalent to the following planar dynamic system:

\[
\begin{cases}
    \frac{dx}{d\xi} = P(x, y), \\
    \frac{dy}{d\xi} = -4(x^3 + px + q) = Q(x, y).
\end{cases} \tag{15}
\]

Since system (15) is a Hamiltonian system, it has the following first integral:

\[
H(x, y) = \frac{y^2}{2} + (x^4 + 2px^2 + qx) = h, \tag{16}
\]

where \(h = H(x, y)\) is the Hamilton energy at the point \((x, y)\) for system (15).

On the plane \((x, y)\), the number of finite singularities of system (15) depends on the number of the real roots for equation \(f(x) = x^3 + px + q = 0\). Here, we note the discriminant of \(f(x) = 0\) as

\[
\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 - \frac{1}{4} \left[27q^2 + 4p^3\right].
\]

and then we can get the following conclusions:

(a). If \(\Delta > 0\), \(f(x) = 0\) has one real root and two complex roots.

(b). If \(\Delta = 0\), \(f(x) = 0\) has three real roots (two of them are equal):

\[
x_1 = -2\sqrt{-p/3}, \\
x_2 = x_3 = \sqrt{-p/3}. \tag{18}
\]

(c). If \(\Delta < 0\), \(f(x) = 0\) has three different real roots:

\[
x_1 = 2\sqrt{-q/3} \cos \frac{\theta}{3}, \\
x_2 = 2\sqrt{-q/3} \cos \left(\frac{\theta}{3} - \frac{2\pi}{3}\right), \\
x_3 = 2\sqrt{-q/3} \cos \left(\frac{\theta}{3} - \frac{4\pi}{3}\right). \tag{19}
\]

where \(\cos \theta = (3\sqrt{3}q/2p - \sqrt{-p})\).

In this paper, we mainly discuss the bounded traveling wave solutions of equation (1), so we always assume that \(p < 0\) and \(\Delta \leq 0\) in the following. Under this assumption, there exist three finite singular points \(P_i(x_i, 0)\) \((i = 1, 2, 3)\) in system (15), where \(x_i\) \((i = 1, 2, 3)\) is the real root of \(f(x) = 0\) and satisfies

\[
\begin{align*}
    x_1 + x_2 + x_3 &= 0, \\
x_1x_2 + x_1x_3 + x_2x_3 &= p, \\
x_1x_2x_3 &= -q.
\end{align*} \tag{20}
\]

Here we assume \(x_1 \leq x_2 \leq x_3\), and the Jacobi matrix at \(P_i\) for system (15) can be expressed as follows:

\[
f(x_i, 0) = \begin{pmatrix} 0 & 1 \\ -4f'(x_i) & 0 \end{pmatrix}, \tag{21}
\]

where \(f'(x_i) = 3x_i^2 + p (i = 1, 2, 3)\).

(1) In the case of \(q = 0\), we have \(\Delta < 0\) now, and equation \(f(x) = 0\) has three real roots \(x_1 = -\sqrt{-p}, x_2 = 0, \) and \(x_3 = \sqrt{-p}\), which correspond to the three different finite singular points \(P_i(x_i, 0)\) \((i = 1, 2, 3)\) of system (15). From (21), we have the determinants of the Jacobi matrix \(f(x_i, 0)\) \((i = 1, 2, 3)\) as follows:

\[
\begin{align*}
    \det f(x_1, 0) &= 4 \left(3x_1^2 + p\right) = 4(-3p + p) = -8p > 0, \\
    \det f(x_2, 0) &= 4 \left(3x_2^2 + p\right) = 4p < 0, \\
    \det f(x_3, 0) &= 4 \left(3x_3^2 + p\right) = 4(-3p + p) = -8p > 0.
\end{align*} \tag{22}
\]

So \(P_1\) and \(P_3\) are centers, and \(P_2\) is a saddle point.

(2) In the case of \(q < 0\).

(I) When \(\Delta = 0\), the three real roots \(x_1 = x_2 < x_3\) for the equation \(f(x) = 0\) correspond to two
different finite singular points $P_1(x_{1,2},0)$ and $P_3(x_3,0)$ of system (15). From (20), we have

$$\begin{align*}
2x_1 + x_3 &= 0, \\
x_1^2 + 2x_1x_3 &= p, \\
x_1^3x_3 &= -q.
\end{align*}$$

(23)

Then, we obtain the determinants of the Jacobi matrix $J(x_i,0)$ at the singular points $P_{1,2}$ and $P_3$ as follows:

$$\begin{align*}
\det J(x_1,0) &= 4(3x_1^3 + p) = 4(4x_1^3 + 2x_1x_3) \\
\det J(x_2,0) &= 4(3x_2^3 + p) = 4(2x_2^3 + x_1x_3) \\
\det J(x_3,0) &= 4(3x_3^3 + p) > 0.
\end{align*}$$

(24)

So $P_{1,2}$ is a cusp and $P_3$ is a center. (II) When $\Delta < 0$, there exist three different real roots in the equation $f(x) = 0$ which satisfy $x_1 < x_2 < 0 < -x_3 < -x_1 < x_3$, and the corresponding system (15) has three different finite singular points $P_i(x_i,0) (i = 1, 2, 3)$. Then, from (21), we get the determinants of the Jacobi matrix $J(x_i,0) (i = 1, 2, 3)$ at $P_i(x_i,0) (i = 1, 2, 3)$ as follows:

$$\begin{align*}
\det J(x_1,0) &= 4(3x_1^3 + p) = 4(4x_1^3 + 2x_1x_3) = 0, \\
\det J(x_2,0) &= 4(3x_2^3 + p) = 4(2x_2^3 + x_1x_3) < 4(x_1x_2 + x_2x_3) = 0, \\
\det J(x_3,0) &= 4(3x_3^3 + p) = 4(2x_3^3 + x_1x_3) > 0.
\end{align*}$$

(25)

So, $P_1$ and $P_3$ are centers, and $P_2$ is a saddle point. (3) In the case of $q > 0$.

(I) When $\Delta = 0$, the three real roots $x_1 = -2\sqrt{-p/3}$ and $x_2 = x_3 = \sqrt{-p/3}$ of the equation $f(x) = 0$ correspond to the finite singular points $P_1(x_1,0), P_2(x_2,0) = P_3(x_3,0)$ of system (15), so we get the determinants of the Jacobi matrix $J(x_i,0) (i = 1, 2, 3)$ at $P_i(x_i,0) (i = 1, 2, 3)$ as follows:

$$\begin{align*}
\det J(x_1,0) &= 4(3x_1^3 + p) = 4(-4p + p) = -12p > 0, \\
\det J(x_2,0) &= \det J(x_3,0) = 4(-p + p) = 0.
\end{align*}$$

(26)

So $P_1$ is a center and $P_{2,3}$ is a cusp. (II) When $\Delta < 0$, the three different real roots of the equation $f(x) = 0$ satisfy $x_1 < -x_3 < -x_2 < 0 < x_2 < x_3$, and corresponding system (15) has three different finite singular points $P_i(x_i,0) (i = 1, 2, 3)$.

From (21), we get the determinants of the Jacobi matrix $J(x_i,0) (i = 1, 2, 3)$ at $P_i(x_i,0) (i = 1, 2, 3)$ as follows:

$$\begin{align*}
\det J(x_1,0) &= 4(3x_1^3 + p) = 4(2x_1^3 + x_2x_3) > 0, \\
\det J(x_2,0) &= 4(3x_2^3 + p) = 4(2x_2^3 + x_1x_3) < 4(x_1 + x_2 + x_3) = 0, \\
\det J(x_3,0) &= 4(3x_3^3 + p) = 4(2x_3^3 + x_1x_2) > 4(x_1 + x_3 + x_2x_3) = 0.
\end{align*}$$

(27)

So, $P_1$ and $P_3$ are centers and $P_2$ is a saddle point.

According to the above qualitative analysis, we can obtain five global phase portraits of system (15) under different parameters (see Figures 1(a)–1(e)). Then, we can get the following propositions from the above five global phase portraits.

**Proposition 1.** Let $\Delta \leq 0$ and $p < 0$.

(1) When $\Delta < 0$, no matter what value $q$ takes, there are two homoclinic orbits and numerous closed orbits in system (15) (see Figures 1(a)–1(c)).

(2) When $\Delta = 0$, there exist one homoclinic orbit and numerous closed orbits in system (15) in the case of $q < 0$ or $q > 0$ (see Figures 1(d) and 1(e)).

Because the homoclinic orbits correspond to the bell-shaped solutions of equation (14), and the periodic trajectories correspond to the periodic solutions of equation (14); on the basis of Proposition 1, we can obtain the following theorem.

**Theorem 1.** Assume $\Delta \leq 0$ and $p < 0$.

(1) When $\Delta < 0$ in the case of $q < 0, q = 0$, or $q > 0$, there exist two bell-shaped solitary wave solutions (corresponding to the homoclinic orbits in Figures 1(a)–1(c)) and numerous periodic traveling wave solutions (corresponding to closed orbits in Figures 1(a)–1(c)) of the form (2) for equation (1)

(2) When $\Delta = 0$, in the case of $q < 0$ or $q > 0$, there exist one bell-shaped solitary wave solution (corresponding to the homoclinic orbit in Figures 1(d) and 1(e)) and numerous periodic traveling wave solutions (corresponding to closed orbits in Figures 1(d) and 1(e)) of the form (2) for equation (1)

**3. Solitary Wave Solutions of Equation (1)**

In this section, we use the undetermined hypothesis method to obtain two kinds of solitary wave solutions of the form (2) for equation (1). Since equation (1) can be transformed into equation (14), we only need to consider the bounded solutions for equation (14) [31, 32]. Here, we assume that equation (14) has solutions with the following forms:
\[
\begin{aligned}
\mathbf{A} \mathbf{m} & \quad (\xi) \mathbf{B} \\
\mathbf{e} & \quad \mathbf{m} \\
\mathbf{B} & \quad \mathbf{m} \\
\mathbf{C} & \quad \mathbf{m} \\
\mathbf{D} & \quad \mathbf{m}
\end{aligned}
\]

where \(A, B, D, m\) are undetermined parameters. Substituting (28) and (29) into (14), we can get the solutions of equation (14), and then, we have the solitary wave solutions of equation (1). So the following theorem can be obtained.

**Theorem 2.** Assuming \(p < 0\) and \(\Delta \leq 0\), here we set \(x_0\) as the real solution for the equation \(f(x_0) = x_0^3 + px_0 + q = 0\).

1. When \(p + 3x_0^2 < 0\), equation (1) has bell-shaped solitary wave solution:

\[
\begin{aligned}
u(\xi) &= \frac{Ae^{m\xi}}{(1 + e^{m\xi})^2 + Be^{m\xi}} + D, \quad (28) \\
u(\xi) &= \frac{A}{B + m\xi} + D, \quad (29)
\end{aligned}
\]

Besides, (31) can be equivalently expressed as

\[
\begin{aligned}
u_{31}^+ (\xi) &= x_0 - \frac{\sqrt{2} \left( p + 3x_0^2 \right) \sqrt{-p - x_0^2}}{2 + \left( -1 \pm \frac{\sqrt{2} x_0 \sqrt{-p - x_0^2}}{\sqrt{-p - 3x_0^2}} \right) \sech^2 \left( \sqrt{-p - 3x_0^2} \xi \right)}
\end{aligned}
\]

2. When \(p < 0\) and \(\Delta = 0\), equation (1) has bell-shaped solutions:

\[
\begin{aligned}
u_{32}^+ (\xi) &= x_0 - \frac{\sqrt{2} \left( p + 3x_0^2 \right)}{\sqrt{2} x_0 \pm \sqrt{-p - x_0^2} \cosh \left( \frac{\sqrt{-p - 3x_0^2}}{2} \xi \right)}
\end{aligned}
\]
\[
\pm \sqrt{\frac{-p}{3} - \frac{4\sqrt{-3p}}{9} \xi^2 - 8p \xi^3},
\]
(34)

Note 1

1. \(u_{\xi}^\pm\) and \(u_{\xi}^\pm\) (\(i = 1, 2\)) represent the solutions with − and + before radical sign, respectively.

2. When \(q = 0\) and \(x_o = 0\), \(u_{\xi}^\pm\) can be simplified as \(u_{\xi}^\pm(\xi)\), which can be expressed as follows:

\[
\pm \sqrt{-2} \operatorname{sech}(\sqrt{-4p \xi}).
\]
(35)

So when \(q = 0\), equation (1) has bell-shaped solutions:

\[
z_{1}^\pm(\xi) = u_{\xi}^\pm(\xi)e^{(ix0)\xi - (7708c^2i)}.
\]
(36)

3. Equation \(f(x) = x^3 + px + q = 0\) has three real roots which satisfy \(x_1 < x_2 < x_3\), since we set \(x_3\) as the real root for equation \(f(x) = 0\) and it satisfies \(p + 3x_1^2 < 0\), according to the discussions in Section 3, we know that here \(x_0 = x_2\).

Here, we point out that

1. When \(q \neq 0\) and \(\Delta < 0\), the amplitude \(u_{\xi}^\pm(\xi)\) of bell-shaped solution \(z_{1}^\pm(\xi)\) for equation (1) corresponds to the homoclinic trajectory with \(P_2\) as a saddle point and \(P_3\) as a center in the global phase portraits Figures 1(b) and 1(c). The amplitude \(u_{\xi}^\pm(\xi)\) of bell-shaped solution \(z_{1}^\pm(\xi)\) of equation (1) corresponds to the homoclinic trajectory with \(P_2\) as a saddle point and \(P_1\) as a center in the global phase portraits Figures 1(b) and 1(c).

2. When \(q = 0\) and \(\Delta < 0\), the amplitudes \(u_{\xi}^\pm(\xi)\) of bell-shaped solutions \(z_{1}^\pm(\xi)\) of equation (1) correspond to the homoclinic trajectories in the global phase portraits Figures 1(d) and 1(e), respectively. Here, \(u_{\xi}^\pm(\xi)\) are bounded and rational [17, 18].

4. Periodic Wave Solutions of Equation (1)

Similar to the solving process in the last section, we need to solve the periodic wave solutions for equation (14) if we want to get the periodic wave solutions in the form of (2) for equation (1).

Equation (14) corresponds to the planar dynamic system (15), and its first integral (16) is the general expression of the trajectories which correspond to the solutions for equation (14). Therefore, we can use the first integral (16) to obtain the periodic wave solutions for equation (14), and then, we can get the periodic wave solutions for the S-S equation (1). In this section, we always assume that \(p < 0\) and \(\Delta \leq 0\) and three real roots \(x_{1,2,3}\) of \(f(x) = 0\) satisfy \(x_1 < x_2 < x_3\).

From the first integral (16) of system (15), we can obtain

\[
y^2 = -2(x^2 + 2px^2 + 4qx - h).
\]
(37)

From \(y = \dot{x}\), we have

\[
\frac{dx}{d\xi} = \dot{x} = y = \pm \sqrt{-2(x^2 + 2px^2 + 4qx - h)}.
\]
(38)

By the separate variable method, we can translate the problem of solving equation (14) into the following integral:

\[
\int \frac{dx}{\sqrt{-F_h(x)}} = \pm \sqrt{2} \xi,
\]
(39)

where \(F_h(x) = x^4 + 2px^2 + 4qx - h\).

Note 2. In Theorem 2, by taking the Hamilton energy \(h = H(x_0, 0)\) in (39) and integrating on it, the amplitude \(u_{\xi}^\pm(\xi)\) of the bell-shaped solutions \(z_{1}^\pm(\xi)\) for equation (1) can be obtained. Similarly, we can obtain the amplitude \(u_{\xi}^\pm(\xi)\) of the bell-shaped solutions \(z_{1}^\pm(\xi)\) for equation (1) in the case of \(\Delta = 0\) and \(q \neq 0\).

4.1. Periodic Wave Solutions Corresponding to Periodic Trajectories Surrounded by Symmetric Homogeneous Orbits.

When \(q = 0\) and \(\Delta < 0\), there exist symmetric homoclinic orbits in system (15). Since the Hamilton energy on the same periodic trajectory is equal, we can set the Hamiltonian energy on the same periodic trajectory as follows:

\[
H(x, y) = H(\eta_1, 0) = (n_1^4 + 2p\eta_1^2) = (\eta_1^2 + p) - p^2 = h_1,
\]
(40)

where \((\eta_1, 0)\) is the intersection of the periodic trajectory and the \(x\)-axis and satisfies \(|\eta_1| < \sqrt{-p}\). According to (16), we get

\[
H(x, y) = \frac{1}{2} y^2 + (x^2 + p)^2 - p^2 = h_1.
\]
(41)

Substituting three solutions \(x_1 = -\sqrt{-p}\), \(x_2 = 0\), and \(x_3 = \sqrt{-p}\) of the equation \(f(x) = x^3 + px = x(x^2 + p) = 0\) into \(H(x, 0)\), we have \(H(x_1, 0) = H(x_3, 0) < H(x_2, 0) = 0\). Then, we have the \(-F_h(x)\) function curve and periodic trajectories when \(h_1 \in (-p^2, 0)\) as shown in Figure 2.

Then, (39) can be transformed into

\[
\pm \sqrt{2d} \xi = \frac{dx}{\sqrt{p^2 + h_1 - (x^2 + p)^2}}.
\]
(42)

\[
\sqrt{p^2 + h_1 - (x^2 + p)^2} = \left(\sqrt{x^2 + p - \sqrt{p^2 + h}}\right)\left(x^2 + p - \sqrt{p^2 + h_1}\right).
\]
(43)

In (42), we let \(A_1^2 = \sqrt{p^2 + h_1} - p\) and \(B_1^2 = \sqrt{p^2 + h_1} - p\) and make transformation \(t = x/A_1 = u(\xi)/A_1\); then, we have
When \( u \in (x_1', x_2') \), let \( u = x_1' - (x_4' - x_3') (x_4' - x_3')/((x_4' - x_3')^2 + (x_4' - x_3') \sin \theta \cos \theta) \), we get

\[
F_h(x) = (x - x_1') (x - x_2') (x - x_3') (x - x_4')
\]

\[
= -\left((x_4' - x_3')^2 (x_4' - x_3')^2 (x_4' - x_3')^2 \sin^2 \theta \cos^2 \theta \right)
\]

\[
\left((x_4' - x_3') + (x_4' - x_3') \sin^2 \theta \right)
\]

\[
(51)
\]

From (39) and (51), we have

\[
\int \frac{d\theta}{\sqrt{-F_h(x)}}
\]

\[
= -2 \int \frac{2d\theta}{\sqrt{(x_4' - x_3') (x_4' - x_3') - (x_4' - x_3') (x_4' - x_3') \sin^2 \theta}}
\]

\[
= - \frac{2}{\sqrt{1 - k_1^2 \sin^2 \theta}} \int \frac{2d\theta}{\sqrt{1 - k_1^2 \sin^2 \theta}} = \sqrt{2\xi}
\]

\[
(52)
\]
where \( k^2 = ((x_2' - x_1')(x_2' - x_3')/(x_1' - x_3'))(x_2' - x_4')). \)

According to the definition of Jacobi elliptic function [33, 34], we get

\[
\sin \theta = \text{sn}\left(\frac{1}{2}\sqrt{2(x_1' - x_3')(x_4' - x_3') \xi, k^2}\right). \tag{53}
\]

Then, by substituting (53) into \( u = x_1' - ((x_1' - x_2')/(x_2' - x_4') + (x_1' - x_3')\sin^2 \theta) \), we get the bounded solution of equation (14):

\[
u_{p2}(\xi) = x(\xi) = x_1'
\]

\[- \frac{(x_1' - x_3')(x_2' - x_4')}{(x_2' - x_4') + (x_1' - x_3')\sin^2 \theta} (x_1' - x_3') + (1/2) \sqrt{2(x_1' - x_3')(x_4' - x_3') \xi, k^2} \tag{54}
\]

(2) When \( x \in (x_3', x_4') \), let \( u = x_1' + ((x_1' - x_1')(x_1' - x_2')/(x_2' - x_4') + (x_1' - x_3')\sin^2 \theta) \), we have

\[
F_h(x) = (x - x_1')(x - x_2')(x - x_3')(x - x_4')
\]

\[= (x_1' - x_1')^2(x_2' - x_4')^2(x_3' - x_4')^2 \sin^2 \theta \cos^2 \theta
\]

\[\frac{[((x_1' - x_1')(x_1' - x_2')(x_1' - x_3')(x_1' - x_4')]}{((x_1' - x_1') + (x_1' - x_3')\sin^2 \theta)^4}\] \tag{55}

From (39) and (55), we have

\[
\int \frac{d\theta}{\sqrt{-F_h(\xi)}}
\]

\[= \int \frac{2d\theta}{\sqrt{-(x_1' - x_1')(x_4' - x_3') \sin^2 \theta + (x_1' - x_3')(x_4' - x_2')}}
\]

\[= \sqrt{2} \xi, \tag{56}
\]

where \( k^2 = ((x_2' - x_1')(x_2' - x_3')/(x_1' - x_3'))(x_2' - x_4')). \) Then we get

\[
\sin \theta = \text{sn}\left(\frac{1}{2}\sqrt{2(x_1' - x_3')(x_4' - x_3') \xi, k^2}\right). \tag{57}
\]

By substituting (57) into \( u = x_1' + ((x_1' - x_1')(x_1' - x_2')/(x_2' - x_4') + (x_1' - x_3')\sin^2 \theta) \), we obtain the bounded solution of equation (14):

\[
u_{p3}(\xi) = x_1'
\]

\[+ \frac{(x_1' - x_1')(x_4' - x_3')}{(x_2' - x_4') + (x_1' - x_3')\sin^2 \theta} (1/2) \sqrt{2(x_1' - x_3')(x_4' - x_3') \xi, k^2} \tag{58}
\]

From the above discussions, we have the following lemma.

**Lemma 2.** Assuming \( q \neq 0 \) and \( \Delta < 0 \), when Hamilton energy \( h \) satisfies (48), equation (14) has two periodic wave solutions \( u_{p2}(\xi) \) and \( u_{p3}(\xi) \), where \( u_{p2}(\xi) \in (x_3', x_4') \) and \( u_{p3}(\xi) \in (x_3', x_4') \).

Then, according to Lemma 2 and equation (11), we can get the following theorem.

**Theorem 4.** Let \( q \neq 0 \) and \( \Delta < 0 \), when Hamilton energy \( h \) satisfies (48), equation (1) has two periodic wave solutions:

\[
z_{p2}(\xi) = u_{p2}(\xi)e^{i((x/d\epsilon) - (\sigma/10\epsilon^2))}, \tag{59}
\]

\[
z_{p3}(\xi) = u_{p3}(\xi)e^{i((x/d\epsilon) - (\sigma/10\epsilon^2))}, \tag{60}
\]

where \( u_{p2}(\xi) \) and \( u_{p3}(\xi) \) are given by (54) and (58), respectively.

Here, we point out that, in Theorem 4, \( u_{p2}(\xi) \) corresponds to the periodic trajectory contained in the homoclinic trajectory with \( P_1 \) as the center, and \( u_{p3}(\xi) \) corresponds to the periodic trajectory contained in the homoclinic trajectory with \( P_3 \) as the center in Figures 1(b) and 1(c).
4.3. Periodic Wave Solutions Corresponding to the Remaining Periodic Trajectories. Except the periodic solutions $\gamma_{pi}(\xi)(i = 1, 2, 3)$ found in Sections 4.1 and 4.2, there exist other periodic solutions for equation (1), which correspond to the periodic trajectories surrounded by the homoclinic trajectory in Figures 1(b)–1(e). Here, we set

$$H(x, y) = H(\eta_3, 0) = h_3,$$

where $(\eta_3, 0)$ is the intersection of the periodic trajectory and the $x$-axis.

In the case of $\Delta < 0$ and $q < 0$, when $h_3$ satisfies $h_3 > H(x, 0)$ or $H(x, 0) < h_3 < H(x, 0)$, the corresponding $Y = -F_h(x)$ function curve and periodic trajectories are shown in Figure 4. In the case of $\Delta < 0$ and $q > 0$, when $h_3 > H(x, 0)$ or $H(x, 0) < h_3 < H(x, 0)$, the corresponding $Y = -F_h(x)$ function curve and periodic trajectories have the relationships which are similar to Figure 4.

In addition, in the case of $\Delta = 0$ and $q > 0$, we have $H(x, 0) < h_3 < H(x, 0)$, which are shown in Figure 5 (note: in the case of $\Delta = 0$ and $q < 0$, we can give the similar diagrams).

As shown in Figures 4 and 5, there is a common feature for the above four cases, i.e., the function

$$\pm F_h(x) = \pm \left(x^4 + 2px^2 + 4qx - h\right)$$

has only two different intersections $s_1, s_2 (s_1 < s_2)$ with the $x$-axis. Then, we can express $F_h(x)$ as

$$F_h(x) = (x - s_1)(x - s_2)(x - \alpha)^2 + \beta^2,$$

where $2\alpha + s_1 + s_2 = 0, \beta > 0$. Then, we divide (63) into the following parts:

$$(x - \alpha)^2 + \beta^2 = \varepsilon_1(x - r_1)^2 + (1 - \varepsilon_1)(x - r_2)^2,$$

$$\varepsilon - s_1(x - s_2) = \varepsilon_2(x - r_1)^2 + (1 - \varepsilon_2)(x - r_2)^2,$$

where $r_i = \lambda(1-\lambda_i/1+\lambda_i), (i = 1, 2), \varepsilon_1 = (\lambda_1(1+\lambda_i)/\lambda_2-\lambda_1), \varepsilon_2 = (1+\lambda_i/\lambda_2-\lambda_1)$, and $\lambda_1, \lambda_2$ are two real roots of the equation

$$(s_1 - s_2)^2 \lambda^2 - 4(\beta^2 + (\lambda - s_1)(\alpha - s_2)) \lambda - 4\beta^2 = 0.$$  

Since $x = u(\xi), y = u'(\xi)$, by substituting $s = (x - r_1/x - r_2) = (u(\xi) - r_1/u(\xi) - r_2)$ into (39), we get

$$\pm \sqrt{2\xi} = \frac{1}{(r_1 - r_2)\varepsilon_1\varepsilon_2}\int \frac{ds}{\sqrt{(s^2 + (1 - \varepsilon_1/\varepsilon_2))(s^2 - (1 + \varepsilon_2/\varepsilon_2))}}$$

Then, let $s^2 = (1 + \varepsilon_1/e_2 (1 - t^2))$ and $m^2 = (\varepsilon_2(1 - \varepsilon_1)/\varepsilon_1 + e_2)$, we have

$$\pm \sqrt{2(\varepsilon_1 + e_2)(r_1 - r_2)}\xi = \int \frac{dt}{\sqrt{(1 - t^2)(1 - m^2t^2)}}$$

By using the Jacobi elliptic function integral [33, 34], we get

$$u(\xi) = r_1 - \frac{r_1 - r_2}{1 \pm (\varepsilon_2 + 1 + e_2)\varepsilon_1 \sqrt{(2(\varepsilon_1 + e_2)(r_1 - r_2))\xi, m}}.$$  

After simplification, we have the bounded solutions of equation (14) as follows:

$$u^+_{P_1}(\xi) = \frac{1 - \lambda_1}{1 + \lambda_1} \alpha - \frac{\alpha}{2\pm 2\alpha \sqrt{(1 + \lambda_1)/(1 + \lambda_2)} \varepsilon_1 \sqrt{(2\sqrt{2\xi, m})}},$$

$$u^-_{P_1}(\xi) = \frac{1 - \lambda_1}{1 + \lambda_1} \alpha - \frac{\alpha}{2\pm 2\alpha \sqrt{(1 + \lambda_1)/(1 + \lambda_2)} \varepsilon_1 \sqrt{(2\sqrt{2\xi, m})}}.$$
where \( \sigma = \sqrt{[\beta^2 + (\alpha - s_1)^2][\beta^2 + (\alpha - s_2)^2]} \), \( \lambda_1 < \lambda_2 \), and \( m \) satisfies

\[
m^2 = \frac{1}{2} - \frac{\beta^2 + (\alpha - s_1)(\alpha - s_2)}{2\left[\beta^2 + (\alpha - s_1)^2]\left[\beta^2 + (\alpha - s_2)^2\right]\right.}
\]

(70)

Here, we point out that (1). In the case of \( \Delta \leq 0 \) and \( q \neq 0 \), we have

\[
x^1 + 2px^2 + 4qx - h_3 = (x-s_1)(x-s_2)(x-\alpha)^2 + \beta^2),
\]

(71)

from (63). Then we expand the right side of the above equation, and let the coefficients of \( x^4, x^3, x^2, x \) be the same, we have

\[
s_1 + s_2 + 2\alpha = 0,
\]

\[
s_1s_2 + 2s_1\alpha + 2s_2\alpha + \alpha^2 + \beta^2 = 2p,
\]

(72)

\[
2s_1s_2\alpha + s_1\alpha^2 + s_2\alpha^2 + 2\beta^2 = 4q.
\]

(a) When \( m^2 > (1/2) \), which is equivalent to \( \beta^2 + (\alpha - s_1)(\alpha - s_2) < 0 \), since \( \beta^2 + (\alpha - s_1)(\alpha - s_2) = \beta^2 + 3\alpha^2 + s_1s_2 = 2p + 6\alpha^2 < 0 \), we have \( \alpha^2 = (s_1 + s_2/2)^2 < (p/3) \), where \( s_i (i = 1, 2) \) and \( x_i (i = 1, 2, 3) \) satisfy

\[
s_1 < x_1 = \frac{-x_0 + \sqrt{-3x_0 - 4p}}{2} < x_2 = x_0
\]

\[
< x_3 = \frac{-x_0 + \sqrt{-3x_0 - 4p}}{2} < s_2.
\]

(73)

At this time, there exist periodic trajectories surrounding the homoclinic trajectories in system (15).

(b) When \( m^2 < 1/2 \), which is equivalent to \( \beta^2 + (\alpha - s_1)(\alpha - s_2) > 0 \), we can get \( (s_1 + s_2/2)^2 > (p/3) \). Now \( s_i (i = 1, 2) \) and \( x_i (i = 1, 2, 3) \) have the following relationship:

\[
x_1 = x_0 < x_2 = \frac{-x_0 + \sqrt{-3x_0 - 4p}}{2} < s_1 < s_2
\]

\[
< x_3 = \frac{-x_0 + \sqrt{-3x_0 - 4p}}{2},
\]

(74)

and there exist periodic trajectories contained in the homoclinic trajectories in system (15).

(2). In the case of \( \Delta < 0 \) and \( q < 0 \), when \( h_3 > H(x_1, 0) = 0 \), we have

\[
-F_h(x) = \left(x^2 + p + \sqrt{p^2 + h_3}\right)\left(x^2 - \left(\sqrt{p^2 + h_3} - p\right)\right)
\]

(75)

Since \( p + \sqrt{p^2 + h_3} > 0 \), equation \( x^2 + p + \sqrt{p^2 + h_3} = 0 \) has no real root. By substituting (75) into (39), we get

\[
\sqrt{2}\xi = \int \frac{dx}{\sqrt{(x^2 + p + \sqrt{p^2 + h_3})(x^2 - (\sqrt{p^2 + h_3} - p))}}
\]

(76)

Set \(-\sqrt{p^2 + h_3} - p = p_1, \sqrt{p^2 + h_3} - p = p_2, \) by using transformation \( x^2 = (p_2/1 - t^2) \), we have

\[
\sqrt{2}\xi = \int \frac{dt}{\sqrt{p_1(p_2 - t^2)}(1 - (p_2/(p_1 + p_2))t^2)}
\]

(77)

According to the Jacobi elliptic function integral [33, 34], we have the bounded periodic solution of equation (14) as follows:

\[
\tilde{u}_{p_4} \pm = \pm \frac{-2pk^2}{p_1 - 1} \sin \left(\frac{-4p}{2k^2 - 1}\xi, k\right),
\]

(78)

where \( k = (-p + \sqrt{p^2 + h_3}/2\sqrt{p^2 + h_3}) \in ((1/2), 1] \). Here, \( \tilde{u}_{p_4} (\xi) \) correspond to the periodic trajectories surrounding the homoclinic trajectory in the global phase portrait Figure 1(a). At this time, equation (1) has periodic wave solutions of the form (2) as follows:

\[
\Xi_{p_4} (\xi) = \tilde{u}_{p_4} (\xi) e^{i\left(y/\delta \right)} + F_{p_4} (x) e^{i\left(X/\delta \right)},
\]

(79)

(3). In the case of \( \Delta < 0 \) and \( q < 0 \), it can be known from Figure 4 that there exist two periodic trajectories centered on \( P_3 \) inside the homoclinic trajectories (correspond to the periodic wave solutions \( u_{p_3} \) and \( u_{p_4} \) of equation (14), respectively), and these two trajectories are divided by the line which corresponds to the Hamilton energy \( h_3 = H(x_1, 0) \). When Hamilton energy \( h = H(x_1, 0) \), since \( H(x_1, 0) = x_1^4 + 2px_1^2 + 4qx_1 \), and the abscissa of singularity \( P(x_1, 0) \) of system (15) satisfies \( x_1^4 + px_1 + q = 0, -F_{h}(x) \) in (39) can be expressed as follows:

\[
-F_{h}(x) = -(x - x_1)^2(x - s_1)(x - s_2),
\]

(80)

where \( s_1 = -x_1 - \sqrt{-2p - 2x_1^2} \) and \( s_2 = -x_1 + \sqrt{-2p - 2x_1^2} \). Substituting (80) into (39), we can obtain the periodic wave solution, which corresponds to the periodic trajectory with Hamilton energy \( h_3 = H(x_1, 0) \) as follows:
Theorem 5.

1. Set $\Delta \leq 0$ and $q \neq 0$. (i) When $\Delta < 0$, the Hamilton energy $h$ satisfies $h \in (H(x_3, 0), H(x_1, 0))$ or $h \in (H(x_2, 0), +\infty)$; (ii) when $\Delta = 0$, the Hamilton energy $h$ satisfies $h \neq H(x_2, 0)$ and $h \neq H(x_1, 0)$. If (i) or (ii) establishes, there will exist periodic wave solutions of equation (1) as follows:

$$u_{p5} = x_4 - \frac{\sqrt{2}(p + 3x_2^2)}{\sqrt{2}x_4 - \sqrt{-p - x_2^2} \cos \left(2\sqrt{p + 3x_2^2} \xi \right)}.$$ (81)

(4). In the case of $\Delta < 0$ and $q > 0$, similar to the analysis in (3), there exist two periodic trajectories centered on $P_4$ inside the homoclinic trajectories in Figure 5, and these two periodic trajectories are divided by the line which corresponds to the Hamilton energy $h_3 = H(x_3, 0)$; then, we get the periodic wave solution corresponding to the periodic trajectory with Hamilton energy $h_3 = H(x_3, 0)$ as follows:

$$u_{p6} = x_3 - \frac{\sqrt{2}(p + 3x_2^2)}{\sqrt{2}x_3 - \sqrt{-p - x_2^2} \cos \left(2\sqrt{p + 3x_2^2} \xi \right)}.$$ (82)

Combining with the above discussions, and according to the relationships between the periodic wave solutions of equations (14) and (20), we can obtain the following theorem.

5. Relationships between the Solitary Wave Solutions and Periodic Wave Solutions of Equation (1)

In this paper, the two forms of bell-shaped solitary wave solutions for equation (1) have been given in Section 3, and the exact periodic wave solutions of equation (1) under different parameters are obtained in Section 4. In this section, we will study the evolutionary relationships between the solitary wave solutions and the periodic wave solutions for equation (1) and reveal the influence by the energy variation of the Hamilton system on the waveform of the solutions.

5.1. The Relationships between Periodic Wave Solutions, Solitary Wave Solutions for Equation (1), and the Value of Hamilton Energy of System (15).

The planar dynamic system (15) is a Hamilton system, and the points on the same trajectory have the same Hamilton energy. When the parameters are fixed, for any $h \in R$, the trajectory $C_h = \{(x, y) \in R \times R \mid H(x, y) = h\}$ with energy $h$ in the global phase portrait is unique for $-F_p(x)$ function curve in (39), where $-F_p(x) = -x^4 - 2px^3 - 4qx + h$.

The global phase portrait Figure 1(b) as an example and discuss the relationships between the traveling wave solutions of the form (2) and the value which the Hamilton energy takes.

We will discuss in the following three cases: $h = H(x_3, 0)$, $h > H(x_3, 0)$, and $h < H(x_3, 0)$.

As shown in Figure 6, these three cases correspond to the three types of trajectories in system (15) in the case of $\Delta < 0$ and $q < 0$: the homoclinic trajectory, the periodic trajectory surrounding the homoclinic trajectory, and the periodic trajectory contained in the same trajectory. From the discussions in Sections 3 and 4, we know that in Figure 6, the homoclinic trajectories correspond to the bell-shaped solutions $u_{p4}^\pm (\xi)$ of equation (14), and the periodic trajectories surrounding the homoclinic trajectories correspond to the periodic solutions $u_{p5}^\pm (\xi)$ (modulo $m^2 < 1/2$) of equation (14), and the periodic trajectories contained in the homoclinic trajectories correspond to the periodic solutions $u_{p2}^\pm (\xi)$ and $u_{p3}^\pm (\xi)$. Then, we can obtain that, in the case of $\Delta < 0$ and $q < 0$, the relationships between the value of Hamilton energy $h$ and the solitary wave solutions, periodic wave solutions of equation (1) are as follows:

1. When $h \leq H(x_3, 0)$, there is no bounded trajectory in system (15). At this time, the S-S equation (1) has no bounded traveling wave solution of the form (2).

2. When $h \in (H(x_3, 0), H(x_1, 0)]$, there is a periodic trajectory centered on $P_3$ and contained in the homoclinic orbit in system (15). At this time, the S-S equation (1) has a cluster of periodic wave solution $z_{p4}^\pm (\xi) = u_{p4}^\pm (\xi)^e^{i(\xi(108\xi^2))}$ of the form (2) (modulo $m^2 < 1/2$), where $u_{p4}^\pm (\xi)$ is given by (69).
For other cases of equation (14). IV. Therefore, we can give the bell-shaped solutions or periodic wave solutions surrounding the homoclinic orbit, and their amplitudes $u_{p1}^\pm(ξ)$ are given by (78).

When $h_1$ approaches the Hamilton energy $H(x_2, 0) = 0$ which corresponds to the solitary wave solution, we have $\lim_{h_1 \to 0} q^2 = (2\sqrt{p^2/\sqrt{p^2 - p}}) = 1$. According to (45), we can obtain

$$\lim_{h_1 \to 0} u_{p1}^\pm(ξ) = \lim_{k_1 \to 1} -2p dn\left(\sqrt{-4p}\,ξ, k_1\right) = 0,$$

$$\lim_{h_1 \to 0} u_{p1}^\pm(ξ) = \lim_{k_1 \to 1} 2p dn(\sqrt{-4p}ξ, 1) = \pm \sqrt{-2p} sech\left(\sqrt{-4p}ξ\right) = u_{p1}^\pm(ξ).$$

(87)

So we have $\lim_{h_1 \to 0} z_{p1}^\pm(ξ) = z_{s1}^\pm(ξ)$.

This indicates that when $h_1 \to 0^+$, the periodic wave solutions converge to the solitary wave solutions; here, periodic wave solutions correspond to the closed trajectories contained in the symmetric homoclinic trajectories, and solitary wave solutions correspond to the homoclinic trajectories.

According to (78), when $h_3 \to 0^+$, we have $k \to 1$. So

$$\lim_{h_3 \to 0^+} u_{p4}^\pm(ξ) = \lim_{k_1 \to 1} \pm \sqrt{2p} sech\left(\sqrt{-4p}ξ\right) = u_{p4}^\pm(ξ).$$

(88)

Thus, we have $\lim_{h_3 \to 0^+} z_{p4}(ξ) = \pm z_{s1}(ξ)$, the periodic solutions $z_{p4}(ξ)$ take the solitary wave solutions $z_{s1}(ξ)$ as the limit, where $z_{p4}(ξ)$ correspond to the closed trajectories that surround the symmetric homoclinic trajectories and $z_{s1}(ξ)$ correspond to the homoclinic trajectories. Therefore, we have the following theorem.

**Theorem 6.** In the case of $Δ < 0$ and $q < 0$, when the Hamilton energy $h_1$ of system (15) tends to be $0^+$, the periodic trajectories centered on $P_1$ and $P_3$ in Figure 1(a) will expand into homoclinic trajectories, and the corresponding periodic wave solutions $z_{p1}^\pm(ξ)$ of the form (2) for equation (1) will evolve into the bell-shaped solitary wave solutions $z_{s1}^\pm(ξ)$; when the Hamilton energy $h_1$ of the system gradually tends to be $0^+$, the periodic trajectories surrounding the symmetric homoclinic trajectories in Figure 1(a) will shrink into...
homoclinic trajectories, and the corresponding periodic wave solutions $z_{\text{peri}}^p(\xi)$ of the form (2) for equation (1) will evolve into the bell-shaped solitary wave solutions $z_{\text{sol}}^p(\xi)$.

### 5.2.2. The Limit Relationships between Periodic Wave Solutions and Solitary Wave Solutions Corresponding to the Inner and Outer Trajectories of Asymmetric Homoclinic Trajectories

According to the analysis of the global phase portraits under different parameters for system (15) in Section 2, we know that there exist two asymmetric homoclinic trajectories in system (15) when $\Delta < 0, q < 0$ or $\Delta < 0, q > 0$. Since the analyses of the two cases are the same, we only need to consider the case of $\Delta < 0, q < 0$.

When the Hamilton energy $h = H(x_2, 0)$, there exist homoclinic trajectories centered on $P_1$ and $P_3$ in system (15), which correspond to the bell-shaped solutions $z_{\text{peri}}^p(\xi)$ of the form (2) for equation (1), respectively; when $h \in (H(x_1, 0), H(x_2, 0))$, a cluster of periodic trajectories exist inside the homoclinic trajectories centered on $P_1$ or $P_3$, which correspond to the periodic wave solutions $z_{\text{peri}}^p(\xi)$ and $z_{\text{peri}}^q(\xi)$ of the form (2) for equation (1); when $h > H(x_2, 0)$, there exist periodic trajectories surrounding the homoclinic trajectory in system (15), which correspond to the periodic wave solutions $z_{\text{peri}}^p(\xi)$ (for $m > 1$) for the form (2) for equation (1).

Here we first consider the limit relationships between the periodic wave solutions $z_{\text{peri}}^p(\xi)$, $z_{\text{peri}}^q(\xi)$, and the solitary wave solutions $z_{\text{sol}}^p(\xi)$.

From qualitative analysis, we know that system (15) has three different singular points $P_i(x_0, 0) (i = 1, 2, 3)$, where $x_1$, $x_2$, and $x_3$ can be expressed as follows:

$$
x_1 = \frac{-x_0 - \sqrt{-3x_0^2 - 4p}}{2},
$$

$$
x_2 = x_0,
$$

$$
x_3 = \frac{-x_0 + \sqrt{-3x_0^2 - 4p}}{2}.
$$

As shown in Figure 6, when $h = H(x_2, 0)$, the abscissas $e_i (i = 1, 2, 3, 4)$ of the intersections of the function curve $Y = -F_h(x)$ and the $x$-axis can be expressed as follows:

$$
e_1 = -x_2 - \sqrt{-2p - 2x_2^2},
$$

$$
e_2 = e_3 = x_2,
$$

$$
e_4 = -x_2 + \sqrt{-2p - 2x_2^2}.
$$

It is easy to know that $e_1 (i = 1, 2, 3, 4)$ and $x_j (i = 1, 2, 3)$ satisfy $e_1 < x_1 < e_2 = e_3 < x_3 < e_4$, and the abscissas $z_j (i = 1, 2, 3, 4)$ of the intersections of the periodic trajectory contained in the homoclinic trajectory and the $x$-axis in Figure 6 can make the equation

$$
-F_h(x) = -x^4 - 2px^2 - 4qx + h = 0
$$

establish, where $e_1 < x_1 < x_3 < e_4 = e_3 < x_4 < e_4$.

Because the figure of function $Y = -F_h(x)$ is a continuous polynomial curve with respect to $h$, the zero point $x_1$ of $-F_h(x) = 0$ has continuity about $h$. When $h \longrightarrow H^-(x_2, 0)$ and $x_1 \longrightarrow e_1$, we have $x_2^2 = e_2 = x_3$, $x_3^2 = e_3 = x_5$, $x_4^2 = e_4$, and modulo $k \longrightarrow 1$. Thus, according to (54) and (32), we have

$$
\begin{align*}
\lim_{h \longrightarrow H^-} (x_2, 0)u_{p2}(\xi) & = \lim_{k \longrightarrow 1} u_{p2}(\xi) \\
& = e_4 - \frac{(e_4 - e_1)(e_4 - e_2)}{(e_4 - e_2) + (e_4 - e_3) \sin^2 \left( \sqrt{(e_1 - e_3)(e_2 - e_4)} \xi, 1 \right)} \\
& = \frac{\sqrt{2}(p + 3x_2^2)}{2x_2 - \sqrt{-2p - 2x_2^2} \cosh \left( \sqrt{(p + 3x_2^2)} \xi \right)} = u_{q1}(\xi).
\end{align*}
$$

Similarly, we can prove $\lim_{h \longrightarrow H^-} (x_2, 0)u_{p3}(\xi) = u_{q1}(\xi)$. Therefore, when $h \longrightarrow H^-(x_2, 0)$, the periodic wave solutions $z_{\text{peri}}^p(\xi)$ and $z_{\text{peri}}^q(\xi)$ evolve into the bell-shaped solitary wave solutions $z_{\text{sol}}^q(\xi)$ and $z_{\text{sol}}^q(\xi)$, respectively, where $z_{\text{peri}}^p(\xi)$ and $z_{\text{peri}}^q(\xi)$ correspond to the periodic trajectories contained in the homoclinic trajectories.

Next, we consider the relationships between the periodic wave solutions $z_{\text{peri}}^p(\xi)$ and $z_{\text{sol}}^q(\xi)$, which correspond to the periodic trajectories surrounding the asymmetric homoclinic trajectories.

From Figure 6, we can know that the periodic trajectory corresponding to the $Y = -F_h(x)$ function curve in the case of $h > H(x_2, 0)$ surrounds the homoclinic trajectory corresponding to the $-F_h(x)$ function curve in the case of $h = H(x_2, 0)$. Set the abscissas of the intersections of the periodic trajectory and the $x$-axis as $s_1$ and $s_2$ when $h > H(x_2, 0)$, and set them as $e_i (i = 1, 2, 3)$ when $h = H(x_2, 0)$. $s_1$, $s_2$, and $e_i (i = 1, 2, 3)$ have the following relationship:

$$
s_1 < e_1 < e_2 = e_3 = x_2 < e_4 < s_2.
$$

When $h \longrightarrow H^+(x_2, 0)$, we have $s_1 \longrightarrow -x_0 - \sqrt{-2p - 2x_0^2}$, $s_2 \longrightarrow -x_0 + \sqrt{-2p - 2x_0^2}$, $\alpha = -(1/2)(s_1 + s_2) \longrightarrow x_2$, $\beta \longrightarrow 0$, $m \longrightarrow 1$, $\lambda_1 \longrightarrow 0$, $\lambda_2 \longrightarrow -((p + 3x_2^2)/(p + x_2^2))$, and $\sigma \longrightarrow -2p - 6x_2^2$, so we can obtain
that is, the periodic trajectory surrounding the homoclinic trajectory contracts into the homoclinic trajectory. So we can deduce that the periodic wave solution $z_{P_4}^\pm (\xi)$ of the form (2) for equation (1) evolves into the bell-shaped solitary wave solutions $z_{S_1}^\pm (\xi)$.

**Theorem 7.** Assume $\Delta < 0$, $q < 0$, when Hamilton energy $h \rightarrow H^+ (x_2, 0)$, the periodic wave solutions $z_{P_3} (\xi)$ and $z_{P_4} (\xi)$ of the form (2) corresponding to the periodic trajectories contained in the homoclinic trajectory evolve into bell-shaped solutions $z_{S_1}^\pm (\xi)$ and $z_{S_2}^\pm (\xi)$, respectively. When Hamilton energy $h \rightarrow H^- (x_2, 0)$, the periodic wave solution $z_{P_4}^\pm (\xi)$ ($m^2 > 1/2$) of the form (2) corresponding to the periodic trajectory in the homoclinic trajectory evolves into the bell-shaped solutions $z_{S_1}^\pm (\xi)$.

5.2.3. The Relationships between the Traveling Wave Solutions inside or outside a Single Homoclinic Orbit and the Limit of Hamilton Energy. Now we take the case of $\Delta = 0$ and $q > 0$ as an example. In this case, as shown in Figure 5 and according to the above solving process of the solitary wave solutions and the periodic wave solutions, we know that when $h = H (x_2, 0)$, homoclinic trajectory with $P_{2,3}$ as a cusp and $P_1$ as a center in system (15) corresponds to the bell-shape solution $z_{S_2}^\pm (\xi)$. When $h < H (x_2, 0)$ ($h > H (x_2, 0)$), there exists the periodic trajectory contained in(surrounding) the homoclinic trajectory in system (15), and it corresponds to the periodic wave solution $z_{P_4}^\pm (\xi)$ of the form (2) for equation (1), whose modulo satisfies $m^2 < 1/2$ ($m^2 > 1/2$).

When $h = H (x_2, 0)$, according to the Hamilton energy function, we have the following:

1. The periodic trajectory contained in the homoclinic trajectory has two intersections $s_1$ and $s_2$ with the $x$-axis, where $s_1$ and $s_2$ satisfy

$$-3\sqrt{\frac{p}{3}} < s_1 < x_1 = -2\sqrt{\frac{p}{3}} < s_2 < x_{2,3} = \sqrt{-\frac{p}{3}}$$

(95)

2. The periodic trajectory surrounding the same trajectory has two intersections $s_1$ and $s_3$ with the $x$-axis, where $s_1$ and $s_2$ satisfy

$$s_1 < -3\sqrt{\frac{p}{3}} < x_1 = -2\sqrt{\frac{p}{3}} < x_{2,3} = \sqrt{-\frac{p}{3}} < s_2.$$  

(96)

For the periodic trajectory contained in the homoclinic trajectory, as shown in Figure 5, when Hamilton energy $h \rightarrow H^+ (x_2, 0)$, we have $s_1 \rightarrow -3\sqrt{-p/3}$, $s_2 \rightarrow \sqrt{-p/3}$, $\alpha \rightarrow -\sqrt{-p/3}$, and $\beta \rightarrow 0$, and $m \rightarrow 0$, so

$$\lim_{h \rightarrow H^+} (x_2, 0) u_{P_4}^\pm (\xi) = \lim_{m \rightarrow 0} u_{P_4}^\pm (\xi) = u_{S_1}^\pm (\xi).$$

(97)

At this time, the periodic wave solution $z_{P_4}^\pm (\xi)$ ($m^2 < 1/2$) of the form (2) for equation (1) evolves into the bell-shaped solitary solution $z_{S_1}^\pm (\xi)$.

For the periodic trajectory surrounding the homoclinic trajectory, it is known from Figure 5 that when the Hamilton energy $h \rightarrow H^- (x_2, 0)$, we have $s_1 \rightarrow -3\sqrt{-p/3}$ and $s_2 \rightarrow \sqrt{-p/3}$, then $\alpha \rightarrow -\sqrt{-p/3}$, $\beta \rightarrow 0$, and $m \rightarrow 1$. By calculating, we can obtain

$$\lim_{h \rightarrow H^-} (x_2, 0) u_{P_4}^\pm (\xi) = \lim_{m \rightarrow 1} u_{P_4}^\pm (\xi) = u_{S_2}^\pm (\xi).$$

(98)

In Figure 5, it can be seen that the periodic trajectory surrounding the homoclinic trajectory contracts into a homoclinic trajectory, which corresponds to the evolutionary process from periodic wave solution $z_{P_4}^\pm (\xi)$ ($m^2 > 1/2$) to the bell-shaped solitary wave solution $z_{S_2}^\pm (\xi)$. Therefore, we can obtain the following theorem.

**Theorem 8.** When $\Delta = 0$, $q > 0$, and Hamilton energy $h \rightarrow H^- (x_2, 0)$, the periodic wave solution $z_{P_4}^\pm (\xi)$ ($m^2 > 1/2$) of equation (1) corresponding to the periodic trajectory contained in the homoclinic trajectory in system (15) will evolve into the bell-shaped solution $z_{S_2}^\pm (\xi)$. When the Hamilton energy $h \rightarrow H^- (x_2, 0)$, the periodic wave solution $z_{P_4}^\pm (\xi)$ ($m^2 > 1/2$) of equation (1), which corresponds to the periodic trajectory surrounding the homoclinic trajectory, will evolve into the bell-shaped solution $z_{S_2}^\pm (\xi)$.

5.3. Diagrams of the Relationships between Hamilton Energy and Traveling Wave Solutions. In the last section, we have discussed the evolutionary relationships between the periodic wave solutions and the solitary wave solutions of equation (1), when the Hamilton energy of system (15) approaches a certain limit. In this section, we will give examples of 3D diagrams to intuitively demonstrate the relationships between these two solutions: When the Hamilton energy changes, the amplitudes $u_{P_3} (\xi)$ ($i = 1, 2, 3, 4$) of the periodic wave solutions for equation (1) will evolve into the amplitudes $u_{S_3} (\xi)$ ($i = 1, 2$) of the bell-shaped solutions for equation (1).

5.3.1. The Evolution from Periodic Solutions to Solitary Wave Solutions Caused by the Variation of Hamiltonian Energy in
The case of $\Delta < 0$ and $q < 0$. The case of $\Delta < 0$ and $q < 0$ is shown in the global phase portrait Figure 1(b). When $h = H(x_2, 0)$, equation (1) has bell-shaped solitary wave solutions $z_{S1}^d(\xi)$ of the form (2). When $H(x_2, 0) < h < H(x_2, 0)$, equation (1) has periodic wave solutions $z_{p2}^d(\xi)$ and $z_{p3}(\xi)$, which satisfy $\lim_{\xi \rightarrow H^-}(x_2, 0)z_{p2}(\xi) = z_{S1}^d(\xi)$ and $\lim_{\xi \rightarrow H^+}(x_2, 0)z_{p3}(\xi) = z_{S1}^d(\xi)$. When $h > H(x_2, 0)$, equation (1) has a periodic wave solution $z_{p3}(\xi) (m^2 > 1/2)$ of the form (2), which satisfies $\lim_{\xi \rightarrow H^+}(x_2, 0)z_{p3}(\xi) = z_{S1}^d(\xi)$. Due to the space limitation, we only give the diagrams of the evolution from the amplitude $u_{p3}(\xi)$ of the periodic wave solution $z_{p3}(\xi)$ to the amplitude $u_{S1}^d(\xi)$ of the solitary wave solution $z_{S1}^d(\xi)$.

Here, we take $p = -7, q = -6, c = 25/12$, and $x_2 = -1$, and then, we have $H(x_2, 0) = 11$. When Hamilton energy $h \rightarrow H^-(x_2, 0)$, the amplitude of the periodic wave solution for equation (1) satisfies $u_{p3}(\xi) \rightarrow u_{S1}^d(\xi)$. The corresponding diagrams are shown in Figures 7(a)–7(d).

5.3.2. The Evolution from Periodic Solutions to Solitary Wave Solutions Caused by the Variation of Hamilton Energy in the Case of $\Delta = 0$ and $q > 0$. The case of $p < 0, \Delta = 0, q > 0$ corresponds to the global phase portrait Figure 1(e). When $h = H(x_2, 0)$, equation (1) has a bell-shape solution $z_{S1}^d(\xi)$ in the form of (2). When $H(x_2, 0) < h < H(x_2, 0)$, equation (1) has a periodic wave solution $z_{p4}(\xi) (m^2 > 1/2)$ of the form (2), which satisfies $\lim_{\xi \rightarrow H^-}(x_2, 0)z_{p4}(\xi) = z_{S1}^d(\xi)$. When $h > H(x_2, 0)$, equation (1) has a periodic wave solution $z_{p4}(\xi) (m^2 > 1/2)$ of the form (2), which satisfies $\lim_{\xi \rightarrow H^+}(x_2, 0)z_{p4}(\xi) = z_{S1}^d(\xi)$. Due to the space limitation, here we only give the diagrams of the evolution from the amplitude $u_{p4}(\xi)$ of the periodic wave solution $z_{p4}(\xi)$ (m$^2 > 1/2$) to the amplitude $u_{S1}^d(\xi)$ of solitary wave solution $z_{S1}^d(\xi)$.

Take $p = -3, q = 2, x_2 = 2$, and $c = (4/\sqrt{3})$, we have $H(x_2, 0) = 3$. When Hamilton energy $h \rightarrow H^+(x_2, 0)$, the diagrams of the amplitude $u_{p4}(\xi) (m^2 > 1/2) \rightarrow u_{S1}^d(\xi)$ of
the traveling wave solution for equation (1) are shown in Figures 8(a)–8(d) (for the sake of beauty, we adjusted the positive and negative axes of the vertical direction of the graphic).

It can be seen that by the qualitative analysis and the first integral method, we not only obtain all the unimodal bell-shaped solitary wave solutions and the periodic wave solutions of the S-S equation (1), but also show the mathematical meaning of the obtained solutions, i.e., the relationships between solutions and bounded trajectories. Moreover, we reveal the relationships between the solitary wave solutions, the periodic wave solutions of the S-S equation, and the energy $h$ of the Hamilton system.

For scholars who want to know more about the application of Hamilton energy in the research of solitary wave solution, refer to literatures [35–37].

Data Availability
The data used to support the findings of this study are included within the article.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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References
Complexity


