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Research Article

Classical Solutions to the Initial-Boundary Value Problems for Nonautonomous Fractional Diffusion Equations

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In this paper, we investigate a class of nonautonomous fractional diffusion equations (NFDEs). Firstly, under the condition of weighted Hölder continuity, the existence and two estimates of classical solutions are obtained by virtue of the properties of the probability density function and the evolution operator family. Secondly, it focuses on the continuity and an estimate of classical solutions in the sense of fractional power norm. The results generalize some existing results on classical solutions and provide theoretical support for the application of NFDE.

1. Introduction

Due to the nonlocal kernel of fractional differential operators, the time fractional diffusion equations (FDEs) of order 0 to 1 can describe irregular diffusion phenomena with long tails. In real life, the regular diffusion phenomenon (integer order case) only occurs in a few special cases. Therefore, FDE has attracted the attention of many scholars.

Qualitative analysis on FDE is the premise of practical application. At present, the research in this area mainly includes the existence, regularity, and stability of solutions. El-Sayed and Herzallah [1] discussed the maximal regularity of strong solutions of the autonomous fractional nonhomogeneous evolution equations under the condition of Hölder continuity. The existence and uniqueness of the mild solution of the autonomous fractional evolution equations (AFEE) involving almost sectorial operators and the existence of classical solutions under the condition of Hölder continuity were researched by Wang et al. [2]. Other studies on the maximal regularity of classical solutions in the autonomous case in the function space of Hölder continuous functions can refer to [3–5]. It is known that Hölder continuity is a special case of weighted Hölder continuity. Mu et al. [6] studied the existence, maximum regularity, and spatial regularity of classical solutions to the autonomous fractional diffusion equations (AFDE) under the condition of weighted Hölder continuity and extended some results in existing research. Later, the Mittag-Leffler function and eigenfunction expansion were employed by Zhou et al. [7] to study the existence, uniqueness, and regularity of mild solutions to the backward problem of the AFDE in the function space of weighted Hölder continuous functions. For other relevant results, please refer to [8–15].

The FDE are often nonautonomous in practical problems, which makes it necessary to research the NFDE. The diffusion coefficient of the NFDE is related not only to the spatial variable but also to the time variable, which brings great difficulties to the research. For example, the diffusion term generates a continuous semigroup in the autonomous case, rather than a two-parameter family of evolution operators in the nonautonomous case. Nevertheless, El-Borai [16] obtained the existence of classical solutions of non-autonomous fractional evolution equations (NFEE) under the condition of Hölder continuity. A new resolvent family concept and a fixed point theorem were used by Debbouche and Baleanu [17] to establish some control results for nonlocal impulsive quasilinear

delay integrodifferential systems. Chalishajar et al. [18] used Sadovskii's fixed point theorem and Banach's fixed point theorem to study the existence of mild solutions to nonlocal problems of NFEE. In [19], the fractional resolvent family and the fixed point theorem are applied to investigate the global existence of mild solutions to NFEE. Chen et al. [20] applied noncompactness measure and Sadovskii's fixed point theorem to study the local existence and blow up of mild solutions to the Volterra-type NFEE. For other studies, see [21, 22]. On the basis of the above analysis, it can be found that the regularity of solutions to the NFDE and NFEE need to further study.

In this paper, we consider NFDE

$$\begin{cases} \partial_t^\alpha u(x,t) + \sum_{|p|\le 2m} b_p(x,t) D^p u(x,t) = f(x,t), & (x,t) \in \Omega \times I, \\ D^p u(x,t) = 0, \quad |p| \le m, & (x,t) \in \partial\Omega \times I, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where ∂_t^α is the Caputo fractional partial derivative with respect to t, $\Omega \subset R^n$ is a bounded open domain, whose boundary $\partial\Omega$ is sufficiently smooth, $I = (0,T)\ T > 0$, u_0 is the initial data for u. $A(x,t) = \sum_{|p|\le 2m} b_p(x,t) D^p u(x,t)$, m is a positive integer, multi-index $x = (x_1, x_2, \ldots, x_n)$, $p = (p_1, p_2, \ldots, p_n)$, $|p| = \sum_{i=1}^n p_i$, $D^p = ((\partial^{p_1}/\partial x_1^{p_1})(\partial^{p_2}/\partial x_1^{p_2}) \cdots (\partial^{p_n}/\partial x_1^{p_n}))$. Additionally, the following hypotheses are satisfied:

(H_0)The operators $A(x,t)$ are uniformly elliptic operator in $\overline{\Omega} \times \overline{I}$. That is, there exists a constant C such that

$$C|\xi|^{2m} \le (-1)^m \mathrm{Re} \sum_{|p|=2m} b_p(x,t)\xi^p, \quad (x,t) \in \overline{\Omega} \times \overline{I}, \tag{2}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$, $|\xi|^2 = \sum_{i=1}^n \xi_i^2$;

(H_1)For $t \in \overline{I}$, the coefficients $b_q(x,t)$ are smooth functions with respect to$x \in \overline{\Omega}$, and there exists $0 < \sigma \le 1$ such that

$$\left| b_p(x,t) - b_p(x,s) \right| \le C|t-s|^\sigma, \quad \text{for } x \in \overline{\Omega}, \text{ and } t, s \in \overline{I}. \tag{3}$$

Firstly, when the inhomogeneous term satisfied the weighted Hölder continuity f and the initial value u_0 belonged to $D(A(0))$, the recursive method is applied to determine the representation of the solution to (1). The existence of a unique classical solution to (1) is proved by virtue of the properties of the probability density function and the evolution operator family. In addition, some estimates of the classical solution, directly connected with the regularity of u_0 and f, are carried out. Finally, the continuity of the classical solution to (1) in some fractional power norm is proved and a reasonable estimate is obtained. Theorem 1 extends Theorem 2.2 in [16], where f is Höler continuous.

The structure of this paper is as follows. The second section expounds the basic knowledge used later. In the third section, the existence and uniqueness of the classical solution to (1) and its continuity in the sense of fractional power are described, and the corresponding estimates are presented.

2. Preliminaries

Throughout this paper, the notation C represents a constant in a particular situation, $\Gamma(\cdot)$ denotes the Gamma function, and $B(\cdot,\cdot)$ denotes the Beta function. Let X be a Banach space with the norm $\|\cdot\|_X$ and γ, β are two constants satisfying $0 < \gamma < \beta \le 1$. We define the space of weighted Hölder continuous functions, which is Hölder continuous with the exponent γ and the weight $s^{1-\beta+\gamma}$, by $C^{\beta,\gamma}(I,X) = \{h\colon \overline{I} \longrightarrow X \mid h$ is continuous on I for $\beta \in (0,1)$(or $\overline{I}$ for $\beta = 1$), $\lim_{t\longrightarrow 0} t^{1-\beta} f(t)$ exsits for $\beta \in (0,1)$,

$$\sup_{0\le s<t\le T} \frac{s^{1-\beta+\gamma}\|h(t)-h(s)\|_X}{(t-s)^\gamma} < \infty, \qquad \lim_{t\longrightarrow 0} \sup_{0\le s<t} \frac{s^{1-\beta+\gamma}\|h(t)-h(s)\|_X}{(t-s)^\gamma} \longrightarrow \Big\}. \tag{4}$$

The norm is

$$\|h\|_{C^{\beta,\gamma}} = \sup_{0\le t\le T} t^{1-\beta}\|h(t)\|_X + \sup_{0\le s<t\le T} \frac{s^{1-\beta+\gamma}\|h(t)-h(s)\|_X}{(t-s)^\gamma}. \tag{5}$$

Remark 1. If h is Hölder continuous with exponent $\gamma(\gamma \in 0,1])$ on $\overline{I}$, then $h \in C^{1,\gamma}\ (I,X)$ [23]. Because of this, our results could generalize some existing conclusions which need Hölder continuity.

Set $u(t)(x) = u(x,t)$, $f(t)(x) = f(x,t)$. Define $A(t)\colon D \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ by

$$A(t)u(t)(x) = A(x,t)u(x,t), \qquad D(A(t)) = D = H^{2m}(\Omega) \cap H_0^m(\Omega). \tag{6}$$

Then we turn (1) into the abstract fractional equations

$$\begin{cases} D_t^\alpha u(t) + A(t)u(t) = f(t), & t \in I, \\ u(0) = u_0, \end{cases} \tag{7}$$

in the Hilbert space $L^2(\Omega)$, where D_t^α is the Caputo fractional derivative. We say that $u\colon \overline{I} \longrightarrow L^2(\Omega)$ is a classical solution of (7), if $u(t)$ is continuous on $\overline{I}$, $A(t)u(t)$ and $D_t^\alpha u$ exist and are continuous on I, and (7) is satisfied on $\overline{I}$.

It is well known that each $-A(s)(s \in \overline{I})$ generates an analytic evolution family $\{T(t,s)\}_{t\ge 0}$. Under the assumptions (H_0) and (H_1), there exists a constant $k \ge 0$ such that $B(t) := A(t) + kI$ satisfies the following properties [24]:

(H_2) For all λ satisfying $\mathrm{Re}\lambda \le 0$, the resolvent $R(\lambda; B(t))$ of $B(t)$ exists and

$$\|R(\lambda; B(t))\|_{L^2(\Omega)} \le \frac{C}{|\lambda|+1}, \tag{8}$$

for each $t \in \overline{I}$.

(H_3)

$$\left\|(B(t)-B(s))B^{-1}(\tau)\right\|_{L^2(\Omega)} \le C|t-s|^\sigma, \tag{9}$$

for all $t, s, \tau \in \overline{I}$.

Without losing generality, we suppose $A(t)$ satisfies (H_2) and (H_3) in the following sections.

Set $U(t,s)=\alpha\int_0^\infty \theta\zeta_\alpha(\theta)T(t^\alpha\theta,s)\mathrm{d}\theta$, $\varphi_1(t,s)=(t-s)^{\alpha-1}[A(t)-A(s)]U(t-s,s)$, $\varphi_{n+1}(t,s)=\int_s^t\varphi_n(t,\tau)\varphi_1(\tau,s)\mathrm{d}\tau$, $n\in\mathbb{Z}^+$, $\varphi(t,s)=\sum_{n=1}^\infty\varphi_n(t,s)$, $V(t)=-A(t)A^{-1}(0)-\int_0^t\varphi(t,s)A(s)A^{-1}(0)\mathrm{d}s$, where $t,s\in\overline{I}$, ζ_α is probability density function defined on $(0,\infty)$:

$$\zeta_\alpha(\theta)=\frac{1}{\alpha}\theta^{-1-(1/\alpha)}\rho_\alpha\left(\theta^{(-1/\alpha)}\right),$$

$$\rho_\alpha(\theta)=\frac{1}{\pi}\sum_{n=0}^\infty(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\quad \theta\in(0,\infty), \tag{10}$$

and $\int_0^\infty\theta^\lambda\zeta_\alpha(\theta)\mathrm{d}\theta=(\Gamma(\lambda+1)/\Gamma(\alpha\lambda+1))$ for $\lambda\in(-1,\infty)$ [25]. In order to obtain the main results, we need to recall the fractional powers of $A(t)$ [24]. Let us denote by

$$A^{-q}(t)=\frac{1}{\Gamma(q)}\int_0^\infty s^{q-1}T(s,t)\mathrm{d}s,\quad \text{for } q>0,t\geq 0. \tag{11}$$

Then we define the fractional power of $A(t)$ by $A^q(t)=(A^{-q}(t))^{-1}$ for $q>0$, and $A^0(t)=I$. The following conclusions follow from some results of [16, 23, 24, 26].

Lemma 1

(i) $(t-s)^{\alpha-1}U(t-s,s)$ *and* $(t-s)^{\alpha-1}A(t)U(t-s,s)$ *are uniformly continuous, where* $t,s\in\overline{I}$, $t-s\geq\varepsilon$, *and* ϵ *is an arbitrary positive number;*

(ii) $\|A(t)(U(t-s,s)-U(t-s,t))\|_{L^2(\Omega)}\leq C(t-s)^{\sigma-\alpha}$ *for* $0\leq s<t\leq T$;

(iii) $\|A^q(s)U(t,s)\|_{L^2(\Omega)}\leq(C/t^{\alpha q})$ *for* $q\geq 0,t>0,s\in\overline{I}$;

(iv) $\|A^q(s)T(t,s)\|_{L^2(\Omega)}\leq(C/t^q)$ *for* $q\geq 0,t>0,s\in\overline{I}$;

(v) $\|\varphi_n(t,s)\|_{L^2(\Omega)}\leq(C^n(t-s)^{\sigma n-1}/\Gamma(\sigma n))$, $\|\varphi(t,s)\|_{L^2(\Omega)}\leq C(t-s)^{\sigma-1}$, $\|V(t)\|_{L^2(\Omega)}\leq C+Ct^\sigma$, $\varphi(t,s)$ *and* $V(t)$ *are uniformly continuous in* t,s, *and*

$$\begin{aligned}\|\varphi(t,\tau)-\varphi(s,\tau)\|_{L^2(\Omega)}&\leq C(t-s)^{\sigma-\nu}(s-\tau)^{\nu-1},\quad \text{for } 0\leq\tau<s<t\leq T,\\ \|V(t)-V(s)\|_{L^2(\Omega)}&\leq C(t-s)^\sigma+C(t-s)^{\sigma-\nu}s^\nu,\quad \text{for } 0\leq s<t\leq T,\end{aligned} \tag{12}$$

where $\nu\in(0,\sigma)$,

(vi) $\|A^{q_1}(t)A^{-q_2}(s)\|\leq C$, *where* $t,s\in\overline{I},0\leq q_1<q_2$;

(vii) $\|A(t)A^{-1}(s)\|\leq C$, *where* $t,s\in\overline{I}$;

(viii) $A^q(t)(t\geq 0)$ *is a closed operator, whose domain* $D(A^q(t))$ *is a Banach space;*

(ix) $A^{q_1+q_2}(t)=A^{q_1}(t)A^{q_2}(t)$, *for* $q_1,q_2\in\mathbb{R}$;

(x) $D(A^{q_1}(t))\subset D(A^{q_2}(t))$, *for* $0<q_2\leq q_1$.

3. Classical Solutions

In the following, we state the main results.

Theorem 1. *If* $f\in C^{\beta,\gamma}(I,L^2(\Omega))$, $u_0\in D(A(0))$, $0<\gamma<\beta\leq 1$, *and* $\alpha+\beta>1$, *then (7) has a unique classical solution:*

$$\begin{aligned}u(t)=u_0&+\int_0^t(t-s)^{\alpha-1}U(t-s,s)V(s)A(0)u_0\mathrm{d}s\\&+\int_0^t(t-s)^{\alpha-1}U(t-s,s)f(s)\mathrm{d}s\\&+\int_0^t\int_0^\tau(t-s)^{\alpha-1}U(t-s,s)\varphi(s,\tau)f(\tau)\mathrm{d}\tau\mathrm{d}s,\quad \text{for } t\in\overline{I},\end{aligned} \tag{13}$$

$$\begin{aligned}\|u(t)\|_{L^2(\Omega)}&\leq C\max\{1,t^{\alpha+\sigma}\}\|u_0\|_{L^2(\Omega)}\\&+C\max\{t^{\alpha+\beta-1},t^{\alpha+\sigma+\beta-1}\}\|f\|_{C^{\beta,\gamma}},\quad \text{for } t\in\overline{I},\end{aligned} \tag{14}$$

$$\begin{aligned}&\|A(t)u(t)\|_{L^2(\Omega)}+\|D_t^\alpha u(t)\|_{L^2(\Omega)}\\&\leq C\|A(0)u_0\|_{L^2(\Omega)}+C\max\{t^{\beta-1},t^{2\sigma+\beta-1}\}\|f\|_{C^{\beta,\gamma}},\quad \text{for } t\in I.\end{aligned} \tag{15}$$

Proof. We set

$$\begin{aligned}u(t)=u_0&+\int_0^t(t-s)^{\alpha-1}U(t-s,s)V(s)A(0)u_0\mathrm{d}s\\&+\int_0^t(t-s)^{\alpha-1}U(t-s,s)w(s)\mathrm{d}s.\end{aligned} \tag{16}$$

Substituting it into (7), by Theorem 2.2 in [16], we get

$$\begin{aligned}&D^\alpha\left(u_0+\int_0^t(t-s)^{\alpha-1}U(t-s,s)V(s)A(0)u_0\mathrm{d}s\right)\\&+A(t)\left(u_0+\int_0^t(t-s)^{\alpha-1}U(t-s,s)V(s)A(0)u_0\mathrm{d}s\right)=0.\end{aligned} \tag{17}$$

Then formally using Lemma 1 of [16], we obtain that

$$\begin{aligned} f(t) &= D^{\alpha}\int_0^t (t-s)^{\alpha-1}U(t-s,s)w(s)\mathrm{d}s \\ &\quad + A(t)\int_0^t (t-s)^{\alpha-1}U(t-s,s)w(s)\mathrm{d}s \\ &= w(t) - \int_0^t (t-s)^{\alpha-1}A(s)U(t-s,s)w(s)\mathrm{d}s \\ &\quad + A(t)\int_0^t (t-s)^{\alpha-1}U(t-s,s)w(s)\mathrm{d}s \\ &= w(t) + \int_0^t \varphi_1(t-s,s)w(s)\mathrm{d}s. \end{aligned} \tag{18}$$

Write $w_0(t) = f(t)$, $w_{n+1} = -\int_0^t \varphi_1(t,s)w_n(s)\mathrm{d}s$, and $w(t) = \sum_{n=0}^{\infty} w_n(t)$. According to the character of φ_n and Fubini's theorem, we have

$$w_n(t) = \int_0^t \varphi_n(t,s)f(s)\mathrm{d}s. \tag{19}$$

In view of Lemma 1, we obtain that

$$\begin{aligned} \left\|\sum_{n=0}^{\infty} w_n(t)\right\|_{L^2(\Omega)} &\le \sum_{n=0}^{\infty}\int_0^t \frac{C^n(t-s)^{\sigma n-1}}{\Gamma(\sigma n)}s^{\beta-1}\mathrm{d}s\|f\|_{C^{\beta,\gamma}} \\ &\le \|f\|_{C^{\beta,\gamma}}\sum_{n=0}^{\infty}\frac{C^n B(\sigma n,\beta)}{\Gamma(\sigma n)}T^{n\sigma+\beta-1} \\ &\le \|f\|_{C^{\beta,\gamma}}T^{\beta-1}\Gamma(\beta)\sum_{n=0}^{\infty}\frac{(CT^{\sigma})^n}{\Gamma(\sigma n+\beta)} \\ &= \|f\|_{C^{\beta,\gamma}}T^{\beta-1}\Gamma(\beta)E_{\sigma,\beta}(CT^{\sigma}), \end{aligned} \tag{20}$$

which implies that $\sum_{n=0}^{\infty} w_n(t)$ uniformly converges on $\overline{I}$ and

$$w(t) = f(t) + \int_0^t \varphi(t,s)f(s)\mathrm{d}s. \tag{21}$$

Besides,

$$\begin{aligned} \|w(t)-w(s)\|_{L^2(\Omega)} &\le \|f(t)-f(s)\|_{L^2(\Omega)} \\ &\quad + \int_0^s \|\varphi(t,\tau)-\varphi(s,\tau)\|_{L^2(\Omega)}\|f(\tau)\|_{L^2(\Omega)}\mathrm{d}\tau \\ &\quad + \int_s^t \|\varphi(t,\tau)f(\tau)\|_{L^2(\Omega)}\mathrm{d}\tau \\ &\le s^{-1+\beta-\gamma}(t-s)^{\gamma}\|f\|_{C^{\beta,\gamma}} + C\|f\|_{C^{\beta,\gamma}}\int_0^s (t-s)^{\sigma-\nu}(s-\tau)^{\nu-1}\tau^{\beta-1}\mathrm{d}\tau \\ &\quad + C\|f\|_{C^{\beta,\gamma}}\int_s^t (t-\tau)^{\sigma-1}\tau^{\beta-1}\mathrm{d}\tau \\ &\le s^{-1+\beta-\gamma}(t-s)^{\gamma}\|f\|_{C^{\beta,\gamma}} + CB(\nu,\beta)(t-s)^{\sigma-\nu}s^{\nu+\beta-1}\|f\|_{C^{\beta,\gamma}} \\ &\quad + \frac{C}{\sigma}s^{\beta-1}(t-s)^{\sigma}\|f\|_{C^{\beta,\gamma}}, \end{aligned} \tag{22}$$

provided $0 < s < t \le T$. Using Lemma 1, we deduce that u is continuous on I. In addition, we conclude from Lemma 1 that

$$\begin{aligned} \|u(t)-u(0)\|_{L^2(\Omega)} &\le C\int_0^t (t-s)^{\alpha-1}(1+s^{\sigma})\mathrm{d}s\|u_0\|_{L^2(\Omega)} + \left(C\int_0^t (t-s)^{\alpha-1}s^{\beta-1}\mathrm{d}s\right. \\ &\quad \left. + \int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{\sigma-1}\tau^{\beta-1}\mathrm{d}\tau\mathrm{d}s\right)\|f\|_{C^{\beta,\gamma}} \\ &\le C\left(\frac{t^{\alpha}}{\alpha} + CB(\alpha,\sigma+1)t^{\alpha+\sigma}\right)\|u_0\|_{L^2(\Omega)} \\ &\quad + C\left(B(\alpha,\beta)t^{\alpha+\beta-1} + B(\sigma,\beta)B(\alpha,\sigma+\beta)t^{\alpha+\sigma+\beta-1}\right)\|f\|_{C^{\beta,\gamma}}, \end{aligned} \tag{23}$$

implies that u is continuous at $t = 0$. Next, we show that $u(t) \in D$ for $t \in I$. In view of

$$\begin{aligned}
&\int_0^t (t-s)^{\alpha-1} A(t) U(t-s,s) V(s) A(0) u_0 \mathrm{d}s \\
&= \int_0^t (t-s)^{\alpha-1} A(t) U(t-s,t)(V(s)-V(t)) A(0) u_0 \mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha-1} A(t)(U(t-s,s)-U(t-s,t)) V(s) A(0) u_0 \mathrm{d}s \\
&\quad - \int_0^\infty \zeta_\alpha(\theta) T(t^\alpha \theta, t) V(t) A(0) u_0 d\theta + V(t) A(0) u_0,
\end{aligned} \tag{24}$$

we conclude from Lemma 1 and (22) that

$$\begin{aligned}
&\left\| A(t)\left(u_0 + \int_0^t (t-s)^{\alpha-1} U(t-s,s) V(s) A(0) u_0 \mathrm{d}s \right) \right\|_{L^2(\Omega)} \\
&\le \left\| A(t) A^{-1}(0) A(0) u_0 \right\|_{L^2(\Omega)} + C \int_0^t (t-s)^{-1} \left((t-s)^\sigma + (t-s)^{\sigma-\nu} s^\nu \right) \mathrm{d}s \left\| A(0) u_0 \right\|_{L^2(\Omega)} \\
&\quad + C \int_0^t (t-s)^{\sigma-1} (1+s^\sigma) \mathrm{d}s \left\| A(0) u_0 \right\|_{L^2(\Omega)} + C(1+t^\sigma) \left\| A(0) u_0 \right\|_{L^2(\Omega)} \\
&\le C \left\| A(0) u_0 \right\|_{L^2(\Omega)} + C t^\sigma \left\| A(0) u_0 \right\|_{L^2(\Omega)} + C t^{2\sigma} \left\| A(0) u_0 \right\|_{L^2(\Omega)}.
\end{aligned} \tag{25}$$

That is, $u_0 + \int_0^t (t-s)^{\alpha-1} U(t-s,s) V(s) A(0) u_0 \mathrm{d}s \in D$ for $t \in I$. We also know that $A(t)(u_0 + \int_0^t (t-s)^{\alpha-1} U(t-s,s) V(s) A(0) u_0 ds)$ is continuous for $t \in I$ ([16]). Next, we write

$$\begin{aligned}
&\int_0^t (t-s)^{\alpha-1} A(t) U(t-s,s) w(s) \mathrm{d}s \\
&= \int_0^t (t-s)^{\alpha-1} A(t) U(t-s,t)(w(s)-w(t)) \mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha-1} A(t)(U(t-s,s)-U(t-s,t)) w(s) \mathrm{d}s \\
&\quad - \int_0^\infty \zeta_\alpha(\theta) T(t^\alpha \theta, t) w(t) d\theta + w(t) \\
&:= I_1(t) + I_2(t) + I_3(t) + I_4(t).
\end{aligned} \tag{26}$$

Thanks to (22) and

$$\begin{aligned}
\|w(t)\|_{L^2(\Omega)} &\le t^{\beta-1} \|f\|_{C^{\beta,\gamma}} + \int_0^t (t-s)^{\sigma-1} s^{\beta-1} \mathrm{d}s \|f\|_{C^{\beta,\gamma}} \\
&\le \left(t^{\beta-1} + B(\beta,\sigma) t^{\sigma+\beta-1} \right) \|f\|_{C^{\beta,\gamma}},
\end{aligned} \tag{27}$$

drawn from Lemma 1, we come to the conclusion that

$$\begin{aligned}
&\left\|\int_0^t (t-s)^{\alpha-1}A(t)U(t-s,s)w(s)\mathrm{d}s\right\|_{L^2(\Omega)}\\
&\le C\int_0^t (t-s)^{-1}\left(s^{-1+\beta-\gamma}(t-s)^{\gamma}+(t-s)^{\sigma-\nu}s^{\nu+\beta-1}+s^{\beta-1}(t-s)^{\sigma}\right)\mathrm{d}s\|f\|_{C^{\beta,\gamma}}\\
&\quad+C\int_0^t (t-s)^{\sigma-1}\left(s^{\beta-1}+s^{\sigma+\beta-1}\right)\mathrm{d}s\|f\|_{C^{\beta,\gamma}}+C\left(t^{\beta-1}+t^{\sigma+\beta-1}\right)\|f\|_{C^{\beta,\gamma}}\\
&\le C\Big(B(\gamma,\beta-\gamma)t^{\beta-1}+B(\sigma-\nu,\nu+\beta)t^{\sigma+\beta-1}+B(\sigma,\beta)t^{\sigma+\beta-1}\Big)\\
&\quad+t^{\beta-1}+t^{\sigma+\beta-1}+B(\sigma,\sigma+\beta)t^{2\sigma+\beta-1}\Big)\|f\|_{C^{\beta,\gamma}}.
\end{aligned}\tag{28}$$

That is $\int_0^t (t-s)^{\alpha-1}U(t-s,s)w(s)\mathrm{d}s \in D$ for $t\in[\varepsilon,T]$ $(\forall\varepsilon\in I)$. Next, we show that $A\int_0^t (t-s)^{\alpha-1}U(t-s,s)w(s)\mathrm{d}s \in C(I,L^2(\Omega))$. Moreover,

$$\begin{aligned}
I_1(t)-I_1(s)&=\int_s^t (t-\tau)^{\alpha-1}A(t)U(t-\tau,t)(w(\tau)-w(t))\mathrm{d}\tau\\
&\quad+\int_0^s\Big((t-\tau)^{\alpha-1}A(t)U(t-\tau,t)-(s-\tau)^{\alpha-1}A(s)U(s-\tau,s)\Big)\\
&\quad\cdot(w(\tau)-w(s))\mathrm{d}\tau+\int_0^s (t-\tau)^{\alpha-1}A(t)U(t-\tau,t)(w(s)-w(t))\mathrm{d}\tau\\
&:=J_1(t)+J_2(t)+J_3(t),
\end{aligned}\tag{29}$$

and for $t>s$, (22) and Lemma 1 imply that

$$\begin{aligned}
\|J_1(t)\|_{L^2(\Omega)}&\le C\int_s^t\left(\tau^{-1+\beta-\gamma}(t-\tau)^{\gamma-1}+(t-\tau)^{\sigma-\nu-1}\tau^{\nu+\beta-1}+\tau^{\beta-1}(t-\tau)^{\sigma-1}\right)\mathrm{d}\tau\\
&\le C\left(\frac{s^{-1+\beta-\gamma}}{\gamma}(t-s)^{\gamma}+\frac{\max\left\{s^{\nu+\beta-1},t^{\nu+\beta-1}\right\}}{\sigma-\nu}(t-s)^{\sigma-\nu}+\frac{s^{\beta-1}}{\sigma}(t-s)^{\sigma}\right).
\end{aligned}\tag{30}$$

Then $\lim_{t\longrightarrow s^+}J_1(t)=0$ $(s\in[\varepsilon,T],\forall\varepsilon\in I)$ holds. Furthermore, using Lemma 1 we have

$$\begin{aligned}
\left\|(t-\tau)^{\alpha-1}A(t)U(t-\tau,t)(w(\tau)-w(s))\right\|_{L^2(\Omega)}&\le C(t-\tau)^{-1}\|w(\tau)-w(s)\|_{L^2(\Omega)}\\
&\le C\left(\tau^{-1+\beta-\gamma}(s-\tau)^{\gamma-1}+(s-\tau)^{\sigma-\nu-1}\tau^{\nu+\beta-1}+\tau^{\beta-1}(s-\tau)^{\sigma-1}\right)\in L^1(I,\mathbb{R}).
\end{aligned}\tag{31}$$

Thus $\lim_{t\longrightarrow s^+}J_2(t)=0$ $(s\in[\varepsilon,T],\forall\varepsilon\in I)$ could be immediately gotten by Lebesgue's dominated convergence theorem. Applying Lemma 1, we also find that

$$\begin{aligned}\|J_3(t)\|_{L^2(\Omega)} &\le \left\|\left(\int_0^\infty \zeta_\alpha(\theta)T\left((t-s)^\alpha\theta,t\right)\mathrm{d}\theta - \int_0^\infty \zeta_\alpha(\theta)T\left(t^\alpha\theta,t\right)\mathrm{d}\theta\right)(w(t)-w(s))\right\|_{L^2(\Omega)} \\ &\le C\|w(t)-w(s)\|_{L^2(\Omega)}.\end{aligned} \tag{32}$$

Then $\lim_{t\longrightarrow s^+} J_3(t) = 0$ $(s \in [\varepsilon, T])$ follows from (22). Next, we write

$$\begin{aligned}I_2(t) - I_2(s) &= \int_s^t (t-\tau)^{\alpha-1}A(t)(U(t-\tau,\tau)-U(t-\tau,t))w(\tau)\mathrm{d}\tau \\ &\quad + \int_0^s \left((t-\tau)^{\alpha-1}A(t)(U(t-\tau,\tau)-U(t-\tau,t)) - (s-\tau)^{\alpha-1}A(s)(U(s-\tau,\tau)-U(s-\tau,s))\right)w(\tau)\mathrm{d}\tau =: K_1 + K_2.\end{aligned} \tag{33}$$

Then

$$\begin{aligned}\|K_1(t)\|_{L^2(\Omega)} &\le C\int_s^t (t-\tau)^{\sigma-1}\left(\tau^{\beta-1}+\tau^{\sigma+\beta-1}\right)\mathrm{d}\tau \\ &\le \frac{C}{\sigma}\left(s^{\beta-1}+\max\left\{s^{\sigma+\beta-1},t^{\sigma+\beta-1}\right\}\right)(t-s)^\sigma,\end{aligned} \tag{34}$$

derived by Lemma 1 and (27). That is, $\lim_{t\longrightarrow s^+} K_1(t) = 0$ $(s \in [\varepsilon, T], \forall \varepsilon \in I)$. Owing to

$$\begin{aligned}&\left\|(s-\tau)^{\alpha-1}A(s)(U(s-\tau,\tau)-U(s-\tau,t))w(\tau)\right\|_{L^2(\Omega)} \\ &\quad \le C(s-\tau)^{\sigma-1}\left(\tau^{\beta-1}+\tau^{\sigma+\beta-1}\right) \in L^1(I,\mathbb{R}),\end{aligned} \tag{35}$$

$\lim_{t\longrightarrow s^+} K_2(t) = 0$ $(s \in [\varepsilon, T]$, ε is arbitrary and $\varepsilon \in I)$ could be obtained by Lebesgue's dominated convergence theorem. In addition, Lemma 1 and the formula (22) imply that $\lim_{t\longrightarrow s^+} I_3(t) = \lim_{t\longrightarrow s^+} I_4(t) = 0$ $(s \in [\varepsilon, T])$. When $t < s$, the above limits are similar. Then by (16) and the properties of $w(t)$, using arguments similar to the ones in Theorem 2.2 and Lemma 1 in [16] one can easily obtain that $D_t^\alpha u$ exists and is continuous on I, and u satisfies (7). Therefore, one obtains u is a classical solution of (7). It is also easily seen that (14) and (15) hold, by (23), (25), and (28). □

Remark 2. Theorem 1 extends Theorem 2.2 in [16], where f is Höler continuous.

Theorem 2. *If $f \in C^{\beta,\gamma}(I, L^2(\Omega))$, $u_0 \in D(A(0))$, $0 < \gamma < \beta \le 1$, $\alpha + \beta > 1$, and $0 < \beta_1 < 1 - (1 - \beta/\alpha)$, then the classical solution to (7) has the property: $A^{\beta_1}(t)u(t)$ is continuous on $\overline{I}$, and*

$$\left\|A_1^\beta(t)u(t)\right\|_{L^2(\Omega)} \le C\left\|D\left(A^\beta(0)\right)u_0\right\|_{L^2(\Omega)} + C\|f\|_{C^{\beta,\gamma}}. \tag{36}$$

Proof. The existence of the classical solution could be gotten immediately from Theorem 1. Since $D(A(0)) \subset D(A^\beta(0))$, $u_0 \in D(A^\beta(0))$. In view of $A^{\beta_1}(t) = A^{\beta_1-1}(t)A(t)$ and $A^{\beta_1-1}(t)$ is a bounded linear operator for $t \in \overline{I}$, using Theorem 1 we may find $A^{\beta_1}(t)u(t)$ is continuous on I. If $\alpha + \beta > 1$, then $0 < \beta_1 < 1 - (1 - \beta/\alpha) < \beta$. Next we show that $A^{\beta_1}(t)u(t)$ is continuous at $t = 0$. In fact, note that

$$\begin{aligned}A^{\beta_1}(t)u(t) - A^{\beta_1}(0)u_0 &= \left(A^{\beta_1}(t)A^{-\beta}(0) - A^{\beta_1-\beta}(0)\right)A^\beta(0)u_0 \\ &\quad + \int_0^t (t-s)^{\alpha-1}A^\mu(s)U(t-s,s)V(s)A^{-\mu}(t)A^{\beta_1}(t)A^{1-\beta}(0)A^\beta(0)u_0\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1}A^{\beta_1-1}(t)A(t)U(t-s,s)w(s)\mathrm{d}s \\ &=: P_1(t) + P_2(t) + P_3(t),\end{aligned} \tag{37}$$

where $1 - \beta + \beta_1 < \mu < 1$.

It is clear that (H_1) implies that $\lim_{t\longrightarrow 0} P_1(t) = 0$. We now estimate, using Lemma 1 and (27),

$$\begin{aligned}
\left\|P_2(t)\right\|_{L^2(\Omega)} &\le C\int_0^t (t-s)^{\alpha-\alpha\mu-1}\left(1+s^{\sigma}\right)\mathrm{d}s\left\|A^{\beta}(0)u_0\right\|_{L^2(\Omega)} \\
&= C\left(t^{\alpha(1-\mu)}+B(\alpha(1-\mu),\sigma+1)t^{\alpha(1-\mu)+\sigma}\right)\left\|A^{\beta}(0)u_0\right\|_{L^2(\Omega)}, \\
\left\|P_3(t)\right\|_{L^2(\Omega)} &\le \left\|\int_0^t (t-s)^{\alpha-1}A^{\beta_2}(s)A^{-\beta_2}(s)A^{\beta_1}(t)U(t-s,s)w(s)\mathrm{d}s\right\| \\
&\le C\int_0^t (t-s)^{\alpha-\alpha\beta_2-1}\left(s^{\beta-1}+B(\beta,\sigma)s^{\sigma+\beta-1}\right)\|f\|_{C^{\beta,\gamma}}\mathrm{d}s \\
&\le C\left(B(\alpha-\alpha\beta_2,\beta)t^{\alpha-\alpha\beta_2+\beta-1}+B(\beta,\sigma)B(\alpha-\alpha\beta_2,\beta+\sigma)t^{\alpha-\alpha\beta_2+\beta+\sigma-1}\right)\|f\|_{C^{\beta,\gamma}},
\end{aligned} \tag{38}$$

where $\beta_1<\beta_2<1-(1-\beta/\alpha)$. These prove $\lim_{t\longrightarrow 0}P_2(t)=\lim_{t\longrightarrow 0}P_3(t)=0$. From the above, we see that $A^{\beta_1}(t)u(t)$ is continuous on $\overline{I}$ and (36) holds. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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