Research Article

High-Order Breather Solutions, Lump Solutions, and Hybrid Solutions of a Reduced Generalized (3 + 1)-Dimensional Shallow Water Wave Equation

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We investigate a reduced generalized (3 + 1)-dimensional shallow water wave equation, which can be used to describe the nonlinear dynamic behavior in physics. By employing Bell’s polynomials, the bilinear form of the equation is derived in a very natural way. Based on Hirota’s bilinear method, the expression of $N$-soliton wave solutions is derived. By using the resulting $N$-soliton expression and reasonable constraining parameters, we concisely construct the high-order breather solutions, which have periodicity in $(x, y)$-plane. By taking a long-wave limit of the breather solutions, we have obtained the high-order lump solutions and derived the moving path of lumps. Moreover, we provide the hybrid solutions which mean different types of combinations in lump(s) and line wave. In order to better understand these solutions, the dynamic phenomena of the above breather solutions, lump solutions, and hybrid solutions are demonstrated by some figures.

1. Introduction

The study to exact solutions of nonlinear equation is one of the hot topics in nonlinear science [1–3]. It is known that all integrable equations possess soliton solutions exponentially localized in certain directions. In the past few decades, a variety of methods have been developed by scientist, such as the Darboux transformation method [4], the inverse scattering method [1, 5], Hirota bilinear method [6, 7], Lie group method [8], Bäcklund transformation [9, 10], and variable separation approaches method [11, 12].

Different from the stable solitons, breather waves and lumps are a special kind of rational solutions and localized structures with the unpredictability and instability. The rogue wave first appeared in studies of oceanography [13, 14] and gradually spread to other fields of physics such as Bose–Einstein condensates [15, 16], optical system [17], superfluid, and plasma. Recently, Ma et al. proposed the positive quadratic function to obtain the lump solutions, and some special examples of lump solutions have been found, such as the KdV equation [18, 19], the KP equation [20, 21], the BKP equation [22], the SK equation [23], the JM equation [24], the shallow water wave equation [25], the coupled Boussinesq equations [26], and nonlinear evolution equation [27, 28]. More recently, high-order rogue waves in a variety of soliton equations have been studied, including the generalized Kadomtsev–Petviashvili equation [29], nonlinear Schrödinger equation [30–33], the Boussinesq equation [34], the breaking soliton equation [35], the Sasa–Satsuma equation [36], the Davey–Stewartson equations [37], the complex short pulse equation [38], and many other equations. More importantly, collision will happen among different solitons. There are two kinds of collision, either elastic or inelastic. It is reported that lump solutions will keep their shapes, velocities, and amplitudes after the collision with soliton solutions, which means that the collision is completely elastic. On the basis of different conditions, the collision will change essentially. The main purpose of this article is to study the high-order breathers solutions, lump solutions and hybrid solutions of the generalized (3 + 1)-dimensional shallow water wave equation, which is usually written as.
This equation has been used in weather simulations, tidal waves, river and irrigation, and tsunami prediction and researched in different ways. Tian and Gao obtained the soliton-type solutions of equation (1) by using the generalized tanh algorithm method [39]. Zayed got the traveling wave solutions of equation (1) by using the \((G'/G)\) expansion method [40]. Wang et al. obtained some interaction solutions of equation (1) by the Hirota bilinear form [25]. Tang et al. presented the Grammian and Pfaffian solutions of equation (1) by the Hirota bilinear form [41]. Multiple soliton solutions of equation (1) are discussed by Zeng et al. [42]. New periodic solitary wave solutions of equation (1) are obtained by Liu and He [43].

The main content of this paper is organized as follows. In Section 2, we obtain the bilinear form of equation (1) by Bell's polynomials. In Section 3, starting from Hirota's bilinear method and reasonable constraining parameters, the high-order breather of this equation have been discussed. By a long-wave limit of these obtained breather solution, high-order lump solutions are studied. In Sections 4–6, we further obtain the breather solutions and lump solutions of this equation under some conditions including the first-order breather and lump solutions, the second-order breather and lump solutions, and the third-order breather and lump solutions. At the same time, we have derived the moving path of lumps. In Section 7, the hybrid solutions are presented, including the first-order lump and a line wave solutions and the second-order lump and a line wave solutions. Moreover, the dynamic properties of these exact solutions are displayed vividly by some figures. Section 8 is a short summary.

2. Bilinear Form

We start from a potential field \(q\) to construct the bilinear form of the \((3+1)\)-dimensional shallow water wave equation, which is defined by

\[
\begin{align*}
u_{\mu t} - u_{\mu xy} - 3u_{\mu x}u_y + 3u_{\mu x}u_y + u_{\mu x} + u_{\mu z} = 0. \quad (1) 
\end{align*}
\]

with the aid of following transformation

\[
q = 2\ln(f) \iff u = c(t)q_x = 2\ln(f)_{xx}. \quad (6)
\]

Take \(z = x\), and equation (5) becomes the following form:

\[
\begin{align*}
B(f \cdot f) = (D_x^2D_t - D_x^2D_y + D_x^2)(f \cdot f)
&= 2f_{xx}f - 2f^2_x - 2f_{xxyy}f^2 + 6f_{xxy}f_x - 6f_{xy}f_{xx} \\
&+ 2f_y f - 2f_{jx}f_x = 0. \quad (7)
\end{align*}
\]

3. High-Order Breather and Lump Solutions

3.1. High-Order Breather Solution. To obtain a higher-order breather solution of equation (1), we assume the auxiliary function \(f\) has much higher order expansions in terms of \(\varepsilon\):

\[
f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots + \varepsilon^n f_n + \cdots. \quad (8)
\]

Substitute equation (8) into bilinear equation (7) and then collect the coefficients of \(\varepsilon\), eliminating the coefficients of all power of \(\varepsilon\), and we can obtain overdetermined systems of ordinary differential equations (ODEs). By solving these ODEs with symbolic computation, we will arrive at the following results:

\[
f = \sum_{\mu=0,1} \exp \left( \sum_{j<s}^N \mu_j A_{j\mu} + \sum_{j=1}^N \mu_j \eta_j \right), \quad (9)
\]

where

\[
\eta_j = \kappa_j x + p_j y + \omega_j t + \phi_j,
\]

\[
\varepsilon^{\delta_{sn}} = \frac{(\kappa_j - \kappa_j^*)^3(p_j - p_j) - (\kappa_j^* - \kappa_j)^3(\omega_j - \omega_j)(p_j - p_j)}{(\kappa_j - \kappa_j^*)^3(p_j - p_j) - (\kappa_j^* + \kappa_j)^3(\omega_j + \omega_j)(p_j + p_j)},
\]

which need to satisfy

\[
\kappa_j^3 p_j - \kappa_j^* p_j \omega_j = 0, \quad (11)
\]

where \(\kappa_j, p_j, \omega_j, \) and \(\phi_j\) are analytic complex constants. The notation \(\sum_{\mu=0}^0\omega_{\mu}\) indicates summation over all possible combinations of \(\mu_1 = 0, 1, \mu_2 = 0, 1\), and \(\mu_{N} = 0, 1\); the \(\sum_{j<s}^N\) summation is over all possible combinations of the \(N\) elements with the specific condition \(j<s\). The \(s\)-th order breather solutions can be generated from \(2n\)-soliton solutions by taking parameter conjugations in equation (9).

As discussed in earlier works [37] in the literature, by suitable constraints of the parameters \(\kappa_j, p_j, \omega_j, \) and \(\phi_j\) in equation (9),

\[
N = 2n,
\]

\[
\kappa_{n+j}^* = \kappa_j,
\]

\[
p_{n+j} = p_j,
\]

\[
\omega_{n+j}^* = \omega_j,
\]

\[
\phi_{n+j}^* = \phi_j.
\]

2 Complexity

--polynomials for equation (3) as follows:

\[
E(q) = c(t)q_{\mu t} - c(t)q_{\mu x y} - 3c(t)^2q_{\mu x}q_y
- 3c(t)^2q_{\mu y}q_x + c(t)q_{\mu z} = 0, \quad (3)
\]

where \(E\) is a polynomial of \(q\). Taking \(c(t) = 1\) and referring to the results presented in [44, 45], we can get the form of the \(P\)-polynomials for equation (3) as follows:

\[
E(P) = P_{\mu t} - P_{\mu x y} + P_{\mu z} = 0. \quad (4)
\]

The above expression leads to the following bilinear equation:

\[
(D_t D_x - D_x^3D_y + D_x^2D_y)(f \cdot f) = 0, \quad (5)
\]

\[
\begin{align*}
\text{Substituting the transformation equation (2) into equation (1) and integrating equation (1) with respect to } x \text{ twice, then one can obtain the following equation:}
\end{align*}
\]

\[
\begin{align*}
E(q) = c(t)q_{\mu t} - c(t)q_{\mu x y} - 3c(t)^2q_{\mu x}q_y
- 3c(t)^2q_{\mu y}q_x + c(t)q_{\mu z} = 0, \quad (3)
\end{align*}
\]

\[
\begin{align*}
\text{where } E \text{ is a polynomial of } q. \text{ Taking } c(t) = 1 \text{ and referring to the results presented in [44, 45], we can get the form of the } P\text{-polynomials for equation (3) as follows:}
\end{align*}
\]

\[
\begin{align*}
E(P) = P_{\mu t} - P_{\mu x y} + P_{\mu z} = 0. \quad (4)
\end{align*}
\]

\[
\begin{align*}
\text{The above expression leads to the following bilinear equation:}
\end{align*}
\]

\[
\begin{align*}
(D_t D_x - D_x^3D_y + D_x^2D_y)(f \cdot f) = 0, \quad (5)
\end{align*}
\]
3.2. High-Order Lump Solution. Besides, the nth-order lump solution can also be generated from equation (9), and we take a long-wave limit with the provision
\[ \exp(\phi_j) = -1, \]
and setting
\[ \kappa_j = \delta \kappa_j, \]
\[ \rho_j = \delta \rho_j, \]
\[ \omega_j = \delta \omega. \]

Then, taking \( \delta \to 0 \) in the expansions of \( f \) in equation (9), the high-order lump solution could be obtained as follows:
\[ u = 2 (\ln f)_{xx}, \]
where
\[ f_N = \prod_{j=1}^{N} \theta_j + \frac{1}{2} \sum_{j \neq k \neq s} a_j a_k \prod_{m \neq j \neq k \neq s} \theta_m \]
\[ + \frac{1}{M!2^M} \times \sum_{j \neq \ldots \neq \ldots \neq m} a_j a_k \ldots a_m \prod_{p \neq j \neq \ldots \neq \ldots \neq m} \theta_p + \ldots, \]
(16)
with
\[ \theta_j = \frac{-\kappa_j \rho_j x - \rho_j^2 y + \kappa_j^2 t}{\rho_j}, \]
\[ a_{js} = \frac{6 \kappa_j \kappa_s \rho_j \rho_s (\kappa_j \rho_s + \kappa_s \rho_j)}{(\kappa_j \rho_s - \kappa_s \rho_j)^2}. \]
(17)
where the two positive integers \( j \) and \( s \) are not larger than \( N \), \( \kappa_{rs} = \kappa_r^* \) and \( \rho_{rs} = \rho_r^* \) (\( j = 1, 2, \ldots, n \)) are complex constants with \( N = 2n. \)

4. First-Order Breather and Lump Solutions

4.1. First-Order Breather Solution. To seek first-order breather solutions of equation (1), we assume \( n = 1 \) and \( N = 2 \), and the function \( f \) in equation (8) is the following form:
\[ f = 1 + e^{f_1} + \epsilon^2 f_2, \]
(18)
where
\[ f_1 = e^{\phi_1} + e^{\eta_1}, \]
\[ f_2 = e^{\kappa x \epsilon \rho y + \omega t + \phi_2}, \]
\[ \eta_s = \kappa_s x + \rho_s y + \omega_s t + \phi_2, \quad s = 1, 2, \]
\[ e^{\alpha_2} = \frac{1}{ \kappa_1 \kappa_2} \left( (\kappa_1 - \kappa_2)^3 (p_1 - p_2) - (\kappa_1 - \kappa_2)^2 (p_1 - p_2) (p_1 + p_2) \right) \]
\[ + \frac{1}{\kappa_1 \kappa_2} \left( (\kappa_1 + \kappa_2)^3 (p_1 + p_2) - (\kappa_1 + \kappa_2)^2 (p_1 + p_2) (p_1 + p_2) \right) \]
\[ \kappa_1^3 p - \kappa_2^3 p - \mu \omega_s = 0, \]
(19)
(20)
where the coefficients \( \kappa_s, \omega_s, \rho_s, \) and \( \phi_s \) are freely complex parameters, and we further take parameter constraints:
\[ \kappa_2 = \kappa_1^*, \quad p_2 = p_1^*, \quad \omega_2 = \omega_1^*, \quad \phi_1 = \phi_2 = 0. \]
(21)

For simplicity, taking parameters
\[ \kappa_1 = \kappa_{11} + i \kappa_{12}, \quad p_1 = p_{11} + i p_{12}, \quad \omega_1 = \omega_{11} + i \omega_{12}, \]
(22)
the function \( f \) in equation (18) can be rewritten as
\[ f = \sqrt{M} \cos \theta_1 (\Theta_1) + \cos \theta_2 (\Theta_2), \]
(23)
with
\[ \Theta_1 = \kappa_{11} x + p_{11} y + \omega_{11} t + \Theta_0, \]
\[ \Theta_2 = \kappa_{12} x + p_{12} y + \omega_{12} t, \]
\[ \Theta_0 = \sqrt{M} e^{\phi_1}, \]
\[ M = \frac{4 \kappa_{12}^3 p_{12} + \kappa_{12}^2 + p_{12} \omega_{12}}{4 \kappa_{11}^3 p_{11} - \kappa_{11}^2 - p_{11} \omega_{11}} \]
(24)

Below, we focus on the asymptotic behaviors of the periodic solutions generated by equation (23). From the quadratic dispersion relation in equation (20). We can know that the angular frequency \( \omega \) to the solution
\[ \omega_s = \frac{\kappa_s^2 (\rho_s p_s - 1)}{\rho_s}, \]
(25)
When \( \omega_s > 0 \), then
\[ f \to e^{2\phi_s}, \quad t \to -\infty, \]
(26)
namely,
\[ u \to 0, \quad t \to -\infty, \]
(27)
\[ f \to Me^{((2\kappa (\kappa p - 1))/\rho)^2 + 2\phi_s}, \quad t \to +\infty, \]
which also results in
\[ u \to 0, \quad t \to +\infty. \]
(28)
Hence, the asymptotic behavior of this breather solution is
\( u \to 0, \ t \to \infty. \) \hfill (29)

Set \( \kappa = i, \) \( \kappa = -i, \) \( p_1 = 1 + i, \)
\( p_2 = 1 - i, \) \( \phi_0 = 0. \) \hfill (30)

The dynamic graphs of first-order breather solutions are shown in Figure 1. The breather is periodic in the \( x \) direction and localized in the \( y \) axis. As times goes on, the breather keeps moving in the \((x, y)\)-plane along the positive \( y \)-axis to the negative \( y \)-axis.

4.2. First-Order Lump Solution. For the first-order lump solution, we take \( n = 1 \) and \( N = 2, \) and equation (16) can be rewritten as
\[
\begin{align*}
\theta & = -\frac{\kappa_p x - p_y y + \kappa^2 t}{p}, \\
a_{12} & = -\frac{6\kappa_1 \kappa_2 p_1 p_2 (\kappa_1 p_2 + \kappa_2 p_1)}{\kappa_1 p_2 - \kappa_2 p_1},
\end{align*}
\]
where \( p_1, \kappa_1, (s = 1, 2) \) are complex constant, and the lump keeps moving by the line
\[
y = -\frac{\kappa_1 \kappa_2 x}{\kappa_1 p_2 + \kappa_2 p_1}. \quad (33)
\]

For instance, taking parameters as
\( \kappa_1 = 1 + i, \)
\( \kappa_2 = 1 - i, \)
\( p_1 = 1, \)
\( p_2 = 1, \) \hfill (34)
the function \( f \) in equation (31) can be rewritten as
\[
f = (2t - x)^2 + (x + y)^2 + 6. \quad \hfill (35)
\]

In this case, the corresponding solution is first-order lump solution, see Figure 2. As times goes on, the lump keeps moving in the \((x, y)\)-plane by the blue line \( y = -x. \)

5. Second-Order Breather and Lump Solutions

5.1. Second-Order Breather Solution. The second-order breather solutions can be derived by a similar procedure as the first-order breather. We assume that the auxiliary function \( f \) has higher-order expansions in terms of \( \varepsilon: \)
\[
f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \varepsilon^4 f_4, \hfill (36)
\]
where
\[
f_1 = \sum e^{\eta_j}, \quad \hfill (37)
\]
\[
f_2 = \sum e^{\eta_j + \eta_s + \kappa_p j}, \quad \hfill (37)
\]
\[
f_3 = \sum e^{\eta_j + \eta_s + \kappa_p j + \kappa_s j}, \quad \hfill (37)
\]
\[
f_4 = e^{\eta_j + \eta_s + \kappa_p j + \kappa_s j}, \quad \hfill (37)
\]
where
\[
\eta_j = \kappa_j x + p_j y + \omega_j t + \phi_j, \quad \hfill (38)
\]
\[
e^{A_{ij}} = \frac{(k_j - k_s)^2 (p_j - p_s) - (k_j - k_s)^2 - (\omega_j - \omega_s) (p_j - p_s)}{(k_j - k_s)^2 (p_j - p_s) - (k_j + k_s)^2 - (\omega_j + \omega_s) (p_j + p_s)}, \quad \hfill (39)
\]
\[
e^{A_{il}} = e^{A_{il}^j} e^{A_{il}^k}, \quad \hfill (39)
\]
\[
k_j^2 p_j - k_s^2 - p_j \omega_j = 0, \quad \hfill (39)
\]
where \( j = 1, 2, 3, 4, s = 1, 2, 3, 4, \) and \( l = 3, 4. \) For simplicity, we take parameter choices in equation (36):
\[
k_1 = 1 + i, \quad \hfill (39)
\]
\[
k_2 = 1 - i, \quad \hfill (39)
\]
\[
k_3 = 3i, \quad \hfill (39)
\]
\[
k_4 = -3i, \quad \hfill (39)
\]
\[
p_1 = 1 + 3i, \quad \hfill (39)
\]
\[
p_2 = 1 - 3i, \quad \hfill (39)
\]
\[
p_3 = 2 + 5i, \quad \hfill (39)
\]
\[
p_4 = 2 - 5i. \quad \hfill (39)
\]

Similarly, we can also give the dynamic graphs of second-order breather solution in Figure 3. The two breathers keep a regular movement in \((x, y)\)-plane at all times, and they are always tangled.

5.2. Second-Order Lump Solution. For the second-order lump solution, we take \( n = 2 \) and \( N = 4 \) to equation (16) which can be rewritten as
\[
f = \theta_1 \theta_2 \theta_3 \theta_4 + a_{12} \theta_1 \theta_4 + a_{14} \theta_1 \theta_3 + a_{14} \theta_1 \theta_3 + a_{23} \theta_1 \theta_4
\]
\[
+ a_{23} \theta_1 \theta_3 + a_{34} \theta_1 \theta_2 + a_{12} a_{34} + a_{13} a_{24} + a_{14} a_{23}, \quad \hfill (40)
\]

where
\[
\theta_j = \frac{-\kappa_j p_j x - p_j y + \kappa_j^2 t}{p_j}, \quad j = 1, 2, 3, 4, \quad \hfill (41)
\]
\[
a_{js} = \frac{6 \kappa_j p_j p_s (\kappa_j p_s + \kappa_j p_s)}{(\kappa_j p_s - \kappa_s p_s)^2}, \quad 1 \leq j < s \leq 4. \quad \hfill (41)
Figure 1: Evolution graphs of the first-order breather solution: (a), (b), and (c) three-dimensional plot at \( t = -20, t = 0, \) and \( t = 20 \) and (d) density plot \( (t = 0) \).

Figure 2: Continued.
Figure 2: Evolution graphs of the first-order lump solution: (a) three-dimensional plot ($t = 0$), (b) density plot ($t = 0$), and (c) the contour plot at $t = -10, 0, 10$ about the moving path described by the blue line.

Figure 3: Evolution graphs of the second-order breather solution: (a), (b), and (c) are the three-dimensional plots at $t = -3, t = 0$, and $t = 3$, and (d), (e), and (f) are the corresponding density plots at $t = -3, t = 0$, and $t = 3$. 

Complexity
Taking the following parameters into equation (40),

\[
\begin{align*}
k_1 &= 1 + i, \\
k_2 &= 1 - i, \\
k_3 &= \frac{1}{2} + i, \\
k_4 &= \frac{1}{2} - i,
\end{align*}
\]

we can obtain the expression of \( f \), which is

\[
f = \frac{7}{2} x^3 y - \frac{427}{3} t y + \frac{1825}{24} x^2 - \frac{233}{8} x y - 4 \sqrt{2} y t + \frac{17}{2} t^2 x y
\]

\[
- \frac{1}{2} t x y^2 + 6 t^3 y - 35 t^3 x - 35 t^3 y + x y^3 + \frac{3295}{12} t^2
\]

\[
- \frac{5105}{12} t x - \frac{1387}{48} y^2 + \frac{11}{4} x^3 y^2 - 10 x^3 t + \frac{55}{2} t^2 x^2 + \frac{29}{4} t^2 y^2
\]

\[
+ \frac{3}{2} t y^3 + 25 t^4 + \frac{1}{4} y^4 + \frac{5}{2} x^4 + \frac{85045}{72}
\]

\[
(42)
\]

The moving path of the two lumps has the same expression as in equation (33):

\[
\begin{align*}
y &= -x, \\
y &= -\frac{5}{2} x.
\end{align*}
\]

\[
(44)
\]

Figure 5 shows that the third-order breather solution. In the following, we mainly consider the breather profile of solution in \((x, y)\)-space for fixed time. As time \(t\) goes on, the three breathers are always tangled with each other.

6. Third-Order Breather and Lump Solutions

6.1. Third-Order Breather Solution. We select the following parameters into equation (9):

\[
\begin{align*}
N &= 6, \\
k_1 &= 1 + 2i, \\
k_2 &= 1 - 2i, \\
k_3 &= -1 + 2i, \\
k_4 &= -1 - 2i, \\
k_5 &= -1 + 3i, \\
k_6 &= -1 - 3i, \\
p_1 &= 1 - 2i, \\
p_2 &= 1 + 2i, \\
p_3 &= -2 - 5i, \\
p_4 &= -2 + 5i, \\
p_5 &= 1 + 7i, \\
p_6 &= 1 - 7i.
\end{align*}
\]

\[
(45)
\]

Again, a long-wave limit is now taken to generate rational solutions. Indeed, take the limit as \( k_j, p_j \to 0, (1 \leq j \leq 6) \) with the provision
Figure 4: Evolution graphs of second-order lump solution: (a), (b), (c) three-dimensional plot at $t = -5, t = 0, \text{ and } t = 5$, (d) density plot at $t = 0$, and (e) the contour plot at $t = -14$ (black), $t = -7$ (green), $t = 0$ (blue), $t = 7$ (pink), and $t = 14$ (red) about the moving path described by the blue line $y = -x$ and red line $y = -(5/2)x$.

Figure 5: Evolution graphs of the third-order breather solution: (a), (b), and (c) are the three-dimensional plots at $t = -(1/2), t = 0, \text{ and } t = (1/2)$, and (d), (e), and (f) are the corresponding density plots at $t = -(1/2), t = 0, \text{ and } t = (1/2)$. 

Complexity
\[ \exp(\phi_j) = -1, \quad 1 \leq j \leq 6. \quad (47) \]

We take \( f \) as follows:

\[
\begin{align*}
f &= -\frac{7696909166}{1584375}tx^2y - \frac{38851750819}{9506250}tx^2y^2 + \frac{5885264754}{528125}t^3xy - \frac{2032}{15}t^4xy - \frac{524}{3}t^3x^2y - \frac{292}{3}t^2x^3y - \frac{443}{15}tx^4y + \frac{201380994224}{14259375}t^3x - \frac{3584}{45}t^3x^3 + \frac{776}{15}t^3x^3y^3 + \frac{512}{45}tx^5 + \frac{10296019829}{73125}x^2 + \frac{22977924016}{316875}xy + \frac{14014379}{260}y^2 - \frac{2955269230234}{14259375}txy, \\
&+ \frac{82066555632}{528125}tx^2y + \frac{298948123}{105625}x^3y + \frac{7007575069}{1267500}x^2y^2 + \frac{13163337777}{422500}x^3y^3 + \frac{1024}{45}t^6 - \frac{152349045356}{14259375}t^2x^2 + \frac{35892347297}{2851875}t^2y^2 - \frac{922}{15}tx^5 + \frac{54}{5}t^2x^6 + \frac{45}{4}t^6y^2 - \frac{303129614}{316875}x^4 + \frac{238646391}{169000}y^4 + \frac{25802142748}{4753125}tx^3 + \frac{2105630148}{528125}t^3y^3 + \frac{15192760076582}{42778125}t^2 + \frac{56x^5y + \frac{93}{2}t^2x^3y}{2} + \frac{110x^4y^2 + \frac{130x^3y^3 + \frac{203}{2}t^2x^2y^4 - \frac{3136}{45}t^3x^2y}{2}}{9} + \frac{800}{27}t^2y^2 - \frac{64}{9}tx^4y + \frac{7856}{45}t^4x^2y + \frac{9476}{45}t^3x^2y^2 - \frac{3872}{45}t^4xy^2 - \frac{64552308256}{14259375}t^4 + \frac{237011861}{1625},
&+ \frac{800}{27}t^2y^2 - \frac{64}{9}tx^4y + \frac{7856}{45}t^4x^2y + \frac{9476}{45}t^3x^2y^2 - \frac{3872}{45}t^4xy^2 - \frac{64552308256}{14259375}t^4 + \frac{237011861}{1625}.
\end{align*}
\]

The moving path of the two lumps is given by
\[ y = -x, \quad (49) \]
and other one lump moves along a straight line:
\[ y = \frac{2}{3}x. \quad (50) \]

Figure 6 gives the dynamic graphs of the third-order lump. The shape of the three lumps keep covariant with \( t \) changing. Two of them keep moving in the \((x, y)\)-plane by the blue line \( y = -x \), and the other keeps moving by the red line \( y = -(2/3)x \). When time \( t = 0 \), the three lumps are tangling in the same line. As the time goes by, they keep moving in the \((x, y)\)-plane and one of the waves passes between the other two.

### 7. Hybrid solutions

To derive the hybrid solutions of high-order lump and a line wave, we take
\[ N = 2n + 1, \quad \exp(\phi_j) = -1, \quad 1 \leq n, 1 \leq j \leq 2n, \quad (51) \]
taking \( \kappa_i, p_i \rightarrow 0 \), \( 1 \leq i \leq 2n \), and the hybrid solutions of high-order lump and a line wave are derived.

#### 7.1. Hybrid Solutions of First-Order Lump and a Line Wave

Firstly, we take
\[ N = 3, \quad \exp(\phi_j) = -1, \quad 1 \leq j \leq 2, \quad (52) \]
setting \( \kappa_1, \kappa_2, p_1, p_2 \rightarrow 0 \), and we can obtain
\[ f = \theta_1\theta_2 + a_{12} + (\theta_1\theta_2 + a_{12}\theta_3 + a_{13}\theta_3 + a_{13}\theta_1 + a_1a_2)\phi_j, \quad (53) \]
where
\[ a_{ij} = \frac{\kappa_1^2(\kappa_3p_3 - 1) + p_3(\omega_1 - \omega_j)}{\kappa_3^2(\kappa_3p_3 - 1) - p_3(\omega_1 + \omega_j)}, \quad j = 1, 2. \]

Taking the following parameters into equation (53),
\[ \kappa_1 = -1 + 2i, \]
\[ \kappa_2 = -1 - 2i, \]
\[ \kappa_3 = 2, \]
\[ p_1 = 2 + i, \]
\[ p_2 = 2 - i, \]
\[ p_3 = -4, \]
\[ \phi_3 = 0. \]

The hybrid solutions of first-order lump and a line wave are displayed in Figure 7. In this case, these figures show the whole movement process between one lump and a line wave in different directions. As the time goes by, one dark-type lump keeps attracting the line wave step by step, until they collide then gradually separate. At certain time, the dark-type lump became bright-type lump.

#### 7.2. Hybrid Solutions of Second-Order Lump and a Line Wave

We take
\[ N = 5, \quad \exp(\phi_j) = -1, \quad 1 \leq j \leq 4. \quad (56) \]
Taking \( \kappa_1, \kappa_2, \kappa_3, p_1, p_2, p_3, p_4 \rightarrow 0 \), we can obtain
Figure 6: Evolution graphs of the third-order lump solution: (a), (b), and (c) three-dimensional plots at $t = -40$, $t = -25$, and $t = 0$, (d), (e), and (f) density plots at $t = 25$, $t = 40$, and $t = 0$, and (g) the contour plot at $t = -25$ (black), $t = 0$ (blue) and $t = 25$ (red) about moving path described by the blue line $y = -x$ and red line $y = -(2/3)x$.

Figure 7: Continued.
Figure 7: Evolution graphs of third-order lump solution given: (a), (b), and (c) three-dimensional plots at $t = -5, t = 0$, and $t = 5$ and (d), (e), and (f) density plots at $t = -5, t = 0$, and $t = 5$.

Figure 8: Evolution graphs of the third-order lump solution: (a), (b), and (c) three-dimensional plots at $t = -5, t = 0$, and $t = 5$ and (d), (e), and (f) density plots at $t = -5, t = 0$, and $t = 5$.

$$f = \theta_1 \theta_2 \theta_3 \theta_4 + a_{12} \theta_2 \theta_3 + a_{13} \theta_1 \theta_4 + a_{14} \theta_1 \theta_2 + a_{23} \theta_1 \theta_4 + a_{24} \theta_1 \theta_2 + a_{34} \theta_1 \theta_2 + a_{12} a_{14} + a_{13} a_{24} + a_{14} a_{23}$$

$$+ \left[ \theta_1 \theta_2 \theta_3 \theta_4 + a_{45} \theta_1 \theta_2 \theta_3 + a_{35} \theta_1 \theta_2 \theta_4 + a_{25} \theta_1 \theta_2 \theta_4 + a_{15} \theta_1 \theta_2 \theta_4 + (a_{35} a_{45} + a_{34} \theta_1 \theta_2 + (a_{25} a_{34} + a_{24}) \theta_1 \theta_3) \right]$$

$$+ (a_{25} a_{35} a_{45} + a_{23} a_{45} + a_{23} a_{34} + a_{24} a_{35}) \theta_1 + (a_{15} a_{25} a_{45} + a_{14} a_{35} + a_{13} a_{45} + a_{13} a_{35}) \theta_2$$

$$+ (a_{25} a_{35} a_{45} + a_{23} a_{45} + a_{23} a_{34} + a_{24} a_{35}) \theta_1 + (a_{15} a_{25} a_{45} + a_{14} a_{35} + a_{13} a_{45} + a_{13} a_{35}) \theta_2$$

$$+ (a_{15} a_{25} a_{45} + a_{14} a_{35} + a_{13} a_{45} + a_{13} a_{35}) \theta_2$$

$$+ (a_{15} a_{25} a_{45} + a_{14} a_{35} + a_{13} a_{45} + a_{13} a_{35}) \theta_2$$

$$+ (a_{12} a_{34} + a_{13} a_{24} + a_{14} a_{23} + a_{12} a_{35} a_{45} + a_{14} a_{25} a_{35} + a_{13} a_{24} a_{35} + a_{15} a_{25} a_{35} a_{45}) e^{\gamma t},$$

(57)
where

\[ a_{ij} = \frac{\kappa_j^2 (\kappa_j p_3 - 1) + p_3 (\omega_1 - \omega_j)}{\kappa_j^2 (\kappa_j p_3 - 1) - p_3 (\omega_1 + \omega_j)} \quad 1 \leq j \leq 4. \]  

(58)

We select the following parameters from equation (57):

\[ \kappa_1 = 1 + i, \]
\[ \kappa_2 = 1 - i, \]
\[ \kappa_3 = \frac{1}{2} + i, \]
\[ \kappa_4 = \frac{1}{2} - i, \]
\[ \kappa_5 = -2, \]
\[ p_1 = 1, \]
\[ p_2 = 1, \]
\[ p_3 = \frac{1}{2}, \]
\[ p_4 = \frac{1}{2}, \]
\[ p_5 = 4, \]
\[ \phi_5 = 0. \]

The hybrid solutions second-order lump and a line wave is displayed in Figure 8. In this case, the correspond solution features two lumps and a line wave. As the time goes by, two lumps attract each other until they collided with the line wave and then gradually separate.

8. Conclusion

In this work, we investigate some high-order breather solutions, lump solutions, and hybrid solutions of a reduced generalized (3 + 1)-dimensional shallow water (SWW) equation by Hirota’s bilinear method and long wave limit method. Based on the bilinear form and reasonable constraining parameters, the high-order breather solutions of the SWW equation have been obtained. When \( N = 2, 4, 6 \), we obtained first-order, second-order, and third-order breather solutions, which are exhibited by three-dimensional figures, see Figures 1, 3, and 5. The high-order lump solution has been constructed by taking the corresponding long wave limit method. At the same time, we have obtained the moving path of first-order, second-order, and third-order lump displayed by plot Figures 2, 4, and 6. When \( N = 2n + 1 \), the hybrid solutions can be obtained by the long wave limit method. The first-order lump and a line wave are shown in Figure 7 and second-order lump and a line wave are shown in Figure 8. From these figures, we observe that the shapes and heights of a line wave are changed with the lumps coming, and the visible signs can give us an indication to escaping the attack of extreme waves. In the near future, based on this method, we would like to discuss the move path of lump in hybrid solutions which means different types of combinations in lump wave, line wave, or other types wave.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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