Hindawi
Complexity
Volume 2020, Article ID 9285686, 11 pages
https://doi.org/10.1155/2020/9285686

Research Article

Analysis of Coupled System of Implicit Fractional Differential Equations Involving Katugampola–Caputo Fractional Derivative

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Received 9 December 2019; Accepted 13 March 2020; Published 21 April 2020

Academic Editor: Yan-Ling Wei

In this paper, we study the existence and uniqueness of solutions to implicit the coupled fractional differential system with the Katugampola–Caputo fractional derivative. Different fixed-point theorems are used to acquire the required results. Moreover, we derive some sufficient conditions to guarantee that the solutions to our considered system are Hyers–Ulam stable. We also provided an example that explains our results.

1. Introduction

From the last few years, fractional differential equations ($\mathbb{FDES}$) theory has gained significant attraction and importance. It arises naturally in various models in areas such as control theory, biology, nonlinear waves of earthquake, mechanics, signal processing, modeling the seepage flow in porous media, and in fluid dynamics, memory mechanism and hereditary properties of materials. Some recent existence and uniqueness ($\mathbb{EU}$) results of solutions for $\mathbb{FDES}$ with initial as well as boundary conditions can be found in [1–8]. In fact, $\mathbb{FDES}$ are the effective tools in real-world problems that motivate many researchers to work in this field, see [6, 9–20] and references cited therein.

Another important aspect in the qualitative theory of differential equations ($\mathbb{DES}$), which is exclusively studied for integer-order $\mathbb{DES}$, is Hyers–Ulam ($\mathbb{HU}$) stability and its various types. This stability concept was originated in 1940 from the question of Ulam [21], which was answered by Hyers [22]. Many researchers extended and generalized Hyers's results in which the work of Rassias [23] is considered to be the first notable contribution. Many researchers studied $\mathbb{HU}$ and $\mathbb{HU}$–Rassias stability of various functional equations, see [24–44] and references cited therein. This field got notorious attention when mathematicians started studying the $\mathbb{HU}$ stability for the solution of differential equations, initiated by Obloza [45, 46]. Motivated by the work of Obloza, various classes of integer-order ordinary differential equations were investigated [39, 47]. The idea was then extended for nonintegral-order differential equations; for some recent work, we refer to [48, 49]. As far as we know, only few researchers studied the different kinds of Ulam's type stabilities for the coupled system of $\mathbb{FDES}$. For details, see [38, 50–52].

Nowadays, both Riemann–Liouville-type ($\mathbb{RL}$) and Caputo-type derivatives are introduced generally, and the impact of applying it in mathematical physics and equations associated with probability is exposed. The fractional integral that generalizes both $\mathbb{RL}$- and Hadamard-type integrals into a single form was initiated by Katugampola [53]. Later on, in [54], new fractional derivative that generalizes the two derivatives was introduced by Katugampola.

Motivated by the work [54, 55], in this paper, we study the $\mathbb{EU}$ and $\mathbb{HU}$ stability of the following implicit switched coupled system of $\mathbb{FDE}$ involving the Katugampola–Caputo ($\mathbb{KC}$) fractional derivative:

$$\begin{cases} {}_c^{\sigma}D^{\alpha}u(\sigma) - f(\varsigma, u(\sigma), {}_c^{\sigma}D^{\alpha}u(\sigma)) = \psi(\sigma, u(\sigma), \nu(\sigma)), \\ \sigma \in \mathbf{J} = [0, T], T > 0, \quad 0 < \alpha < 1, \\ {}_c^{\sigma}D^{\alpha}\nu(\sigma) - g(\varsigma, \nu(\sigma), {}_c^{\sigma}D^{\alpha}\nu(\sigma)) = \varphi(\sigma, u(\sigma), \nu(\sigma)), \\ u(0) + h(u) = u_0, \nu(0) + h(\nu) = \nu_0, \end{cases} \tag{1}$$

where σ is a positive real number and ${}_c^{\sigma}D^{\alpha}u(\sigma)$ and ${}_c^{\sigma}D^{\alpha}\nu(\sigma)$ are the Katugampola–Caputo fractional derivatives of $u(\sigma)$ and $\nu(\sigma)$, respectively. The functions f, g: $\mathbf{J} \times \mathbb{X} \times \mathbb{X} \times \longrightarrow \mathbb{X}$ are closed and bounded. Also, ψ and φ are nonlocal continuous functions.

2. Preliminaries

In this portion, we introduce some notions and preliminaries. Suppose $\mathbb{C}(\mathbf{J}, \mathbb{X})$ denotes the Banach space ($\mathbb{BS}$) of continuous functions from $\mathbf{J} = [0, T]$ into $\mathbb{X}$, defined by $\mathbb{C}(\mathbf{J}, \mathbb{X}) = \{u: \mathbf{J} \longrightarrow \mathbb{X}, \sigma \in \mathbf{J}\}$, endowed with norms as $\|u\| = \max_{\sigma\in\mathbf{J}}\|u(\sigma)\|$, $\|\nu\| = \max_{\sigma\in\mathbf{J}}\|\nu(\sigma)\|$; indeed, these are $\mathbb{BS}$s' under these norms, and hence, their product is also $\mathbb{BS}$ with $\|(u, \nu)\| = \|u\| + \|\nu\|$, where u and ν are in $\mathbb{C}(\mathbf{J}, \mathbb{X})$.

Definition 1 (see [5]). Let $\alpha > 0$, and the arbitrary order integral in the $\mathbb{RL}$ sense for a function p: $\mathbf{J} \longrightarrow \mathbb{R}$ is

$$\mathbb{I}^{\alpha}p(\sigma) = \frac{1}{\Gamma(\alpha)} \int_0^{\sigma} (\sigma - \upsilon)^{\alpha-1} p(\upsilon)\mathrm{d}\upsilon, \tag{2}$$

where the integral on the right-hand side (RHS) is pointwise defined on $\mathbb{R}^+$.

Definition 2 (see [54]). $\mathbb{KC}$ left-sided noninteger-order integral ${}^{\rho}I_{a^+}^{\alpha}f$ of the function f on a closed interval $[a, b]$ of order α is defined as

$$\left({}^{\rho}\mathrm{I}^{\alpha}f\right)(\sigma) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^{\sigma} \left(\sigma^{\rho} - \upsilon^{\rho}\right)^{\alpha-1} \upsilon^{\rho-1}\mathrm{d}\upsilon, \quad \text{where } \alpha \in C, \text{ with } \mathfrak{R}(\alpha) > 0. \tag{3}$$

The corresponding $\mathbb{KC}$ fractional derivative to the above integral is given by

$$\left({}_c^{\rho}D_{a^+}^{\alpha}f\right)(\sigma) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-1)}\left(\sigma^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^{\sigma} \left(\sigma^{\rho} - \upsilon^{\rho}\right)^{n-\alpha-1} f(\upsilon)\mathrm{d}\upsilon. \tag{4}$$

Definition 3 (see [5]). The noninteger-order derivative in the Caputo sense of p on closed interval $[a, b]$ is

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}\sigma^{\alpha}}p(\sigma) = \int_a^{\sigma} \frac{(\sigma - \upsilon)^{n-\alpha-1}}{\Gamma(n-\alpha)}\left(\frac{d^n}{d\upsilon^n}p(\upsilon)\right)\mathrm{d}\upsilon, \quad \alpha \in (n-1, n], \tag{5}$$

where $n - 1 = [\alpha]$. In particular,

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}\sigma^{\alpha}}p(\sigma) = \frac{1}{\Gamma(1-\alpha)} \int_a^{\sigma} \frac{p'(\upsilon)}{(\sigma - \upsilon)^{\alpha}}\mathrm{d}\upsilon, \quad \text{where } \varphi'(\upsilon) = \frac{\mathrm{d}\varphi(\upsilon)}{\mathrm{d}\upsilon}, \alpha \in (0, 1]. \tag{6}$$

Moreover, the integral on the RHS is pointwise defined on $\mathbb{R}^+$.

Lemma 1 (see [56]). *Let $\alpha \in [n-1, n)$, for $p \in \mathbb{C}([a, b])$, and the only one solution of $(d^{\alpha}/d\sigma^{\alpha})p(\sigma) = 0$ has the formula $p(\sigma) = \sum_{k=0}^{[\alpha]} c_k\sigma^k$, where $c_k \in \mathbb{R}$, $k = 1, 2, \ldots, [\alpha]$, $[\alpha] = n - 1$.*

Lemma 2 (see [56]). *Let $\alpha \in [n-1, n)$, for $p \in \mathbb{C}([a, b])$ and $\mathbb{I}^{\alpha}(d^{\alpha}/d\sigma^{\alpha})p(\sigma) = p(\sigma) + \sum_{k=0}^{[\alpha]} a_k\sigma^k$, for some $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, [\alpha]$, $[\alpha] = n - 1$.*

Definition 4 (see [57]). Consider $\mathbb{BS}\,\mathbb{X}$. Let Ω_1, Ω_2: $\mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}$ be two operators, and the operator system

$$\begin{cases} p(\sigma) = \Omega_1(p, q)(\sigma) \\ q(\sigma) = \Omega_2(p, q)(\sigma) \end{cases} \tag{7}$$

is called $\mathbb{HU}$ stable if we can find constants $C_i\,(i = 1, 2, 3, 4) > 0$ such that, for each $\varrho_j\,(j = 1, 2) > 0$ and for each solution $(\widehat{p}, \widehat{q}) \in \mathbb{X} \times \mathbb{X}$ of the inequalities

$$\begin{cases} \|\widehat{p} - \psi(\widehat{p}, \widehat{q})\| \le \varrho_1 \\ \|\widehat{q} - \varphi(\widehat{p}, \widehat{q})\| \le \varrho_2 \end{cases} \tag{8}$$

hold, then there exists a solution $(\widetilde{p}, \widetilde{q}) \in \mathbb{X}$ of system (7), which satisfies the inequalities

$$\begin{cases} \|\widehat{p} - \widetilde{p}\| \le C_1\varrho_1 + C_2\varrho_2 \\ \|\widehat{q} - \widetilde{q}\| \le C_3\varrho_1 + C_4\varrho_2. \end{cases} \tag{9}$$

Lemma 3 (see [57], Theorem 4). *Consider $\mathbb{BS}\,\mathbb{X}$ with operators Ω_1, Ω_2: $\mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}$ such that*

$$\begin{cases} \left\|\Omega_1(p, q) - \Omega_1(\widehat{p}, \widehat{q})\right\| \le \Lambda_1\|p - \widehat{p}\| + \Lambda_2\|q - \widehat{q}\|, \\ \left\|\Omega_2(p, q) - \Omega_2(\widehat{p}, \widehat{q})\right\| \le \Lambda_3\|p - \widehat{p}\| + \Lambda_4\|q - \widehat{q}\|. \end{cases} \tag{10}$$

If the spectral radius of

$$\mathbb{H} = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{pmatrix}, \tag{11}$$

is less than one, then the fixed points corresponding to operational system (10) are $\mathbb{HU}$ stable.

3. Existence and Uniqueness of the Solution

In this section, we prove $\mathbb{EU}$ of system (1). We consider the following assumptions:

($\mathbb{H}_1$): suppose ψ, φ: $\mathbf{J} \times \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}$ are continuous, and for all $(\mu, \nu), (\overline{\mu}, \overline{\nu}) \in \mathbb{X} \times \mathbb{X}$ with $\sigma \in \mathbf{J}$, there exist $\mathbb{M}_{\psi}, \mathbb{M}_{\varphi}, \mathbb{M}_{\psi}', \mathbb{M}_{\varphi}' > 0$ such that

$$\|\psi(\sigma,\mu(\sigma),\nu(\sigma))-\psi(\sigma,\overline{\mu}(\sigma),\overline{\nu}(\sigma))\|\leq \mathbb{M}_{\psi}\|\mu(\sigma)-\overline{\mu}(\sigma)\| + \mathbb{M}'_{\psi}\|\nu(\sigma)-\overline{\nu}(\sigma)\|,$$
$$\|\varphi(\sigma,\mu(\sigma),\nu(\sigma))-\varphi(\sigma,\overline{\mu}(\sigma),\overline{\nu}(\sigma))\|\leq \mathbb{M}_{\varphi}\|\mu(\sigma)-\overline{\mu}(\sigma)\| + \mathbb{M}'_{\varphi}\|\nu(\sigma)-\overline{\nu}(\sigma)\|. \tag{12}$$

($\mathbb{H}_2$): there exist $\mathbb{L}_f,\mathbb{L}_g,\mathbb{L}'_f,\mathbb{L}'_g>0$ such that

$$\|f(\sigma,\mu(\sigma),\nu(\sigma))-f(\sigma,\overline{\mu}(\sigma),\overline{\nu}(\sigma))\|\leq \mathbb{L}_f\|\mu-\overline{\mu}\| + \mathbb{L}'_f\|\nu-\overline{\nu}\|,$$
$$\|g(\sigma,\mu(\sigma),\nu(\sigma))-g(\sigma,\overline{\mu}(\sigma),\overline{\nu}(\sigma))\|\leq \mathbb{L}_g\|\mu-\overline{\mu}\| + \mathbb{L}'_g\|\nu-\overline{\nu}\|, \tag{13}$$

for all $\sigma\in\mathbf{J}$, (μ,ν), $(\overline{\mu},\overline{\nu})\in\mathbb{X}\times\mathbb{X}$.

($\mathbb{H}_3$): suppose $h\colon C(\mathbf{J},\mathbb{X})\longrightarrow\mathbb{R}$ is continuous, and for all $\mu,\nu,\overline{\mu},\overline{\nu}\in\mathbb{X}$, there exist $a,b>0$ such that

$$\|h(\mu)-h(\overline{\mu})\|\leq a\|\mu-\overline{\mu}\| \text{ and } \|h(\nu)-h(\overline{\nu})\|\leq b\|\nu-\overline{\nu}\|. \tag{14}$$

($\mathbb{H}_4$): let $l_f,l_g,p,q,p^*,q^*\in C(\mathbf{J},\mathbb{X})$ with $l_{gh}=\sup\{|l_f(\sigma),l_g(\sigma)|\}<1$ such that

$$\|f(\sigma,\mu(\sigma),\nu(\sigma))\|\leq l_f(\sigma)+p(\sigma)\|\mu\|_{PC}+q(\sigma)\|\nu\|,$$
$$\|g(\sigma,\mu(\sigma),\nu(\sigma))\|\leq l_g(\sigma)+p^*(\sigma)\|\mu\|_{PC}+q^*(\sigma)\|\nu\|, \tag{15}$$

where $\sigma\in\mathbf{J}$.

($\mathbb{H}_5$): let $\psi,\varphi\colon\mathbf{J}\times\mathbb{X}\times\mathbb{X}\longrightarrow\mathbb{X}$ be continuous, and for all $(\mu,\nu)\in\mathbb{X}$ with $\varsigma\in\mathbf{J}$, there exist $l_\psi,l^1_\psi,l^2_\psi,l_\varphi,l^1_\varphi,l^2_\varphi\in C(\mathbf{J},\mathbb{X})$ such that

$$\|\psi(\sigma,\mu(\sigma),\nu(\sigma))\|\leq l_\psi(\sigma)+l^1_\psi(\sigma)\|\mu(\sigma)\|+l^2_\psi\|\nu(\sigma)\|,$$
$$\|\varphi(\sigma,\mu(\sigma),\nu(\sigma))\|\leq l_\varphi(\sigma)+l^1_\varphi(\sigma)\|\mu(\sigma)\|+l^2_\varphi\|\nu(\sigma)\|. \tag{16}$$

Theorem 1. *Let $\theta_1,\theta_2\in C(\mathbf{J},\mathbb{X})$ and $f,g\colon J\times\mathbb{X}\times\mathbb{X}\longrightarrow\mathbb{X}$ be continuous, and the solution of*

$$\begin{cases} {}^{\sigma}_{c}D^{\alpha}\mu(\sigma)-f(\varsigma,\mu(\sigma),{}^{\sigma}_{c}D^{\alpha}\mu(\sigma))=\theta_1(\sigma),\\ {}^{\sigma}_{c}D^{\alpha}\nu(\sigma)-f(\varsigma,\nu(\sigma),{}^{\sigma}_{c}D^{\alpha}\nu(\sigma))=\theta_2(\sigma),\\ \mu(0)+h(\mu)=\mu_0,\nu(0)+h(\nu)=\nu_0, \end{cases} \tag{17}$$

is equivalent to

$$\begin{cases} \mu(\sigma)=\mu_0-h(\mu)+\dfrac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_0^\sigma(\sigma^\sigma-\upsilon^\sigma)^{\alpha-1}\upsilon^{\sigma-1}f(\upsilon,\mu(\upsilon),{}^{\sigma}_{c}D^{\alpha}\mu(\upsilon))d\upsilon+\dfrac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_0^\sigma(\sigma^\sigma-\upsilon^\sigma)^{\alpha-1}\upsilon^{\sigma-1}\theta_1(\upsilon)d\upsilon,\\ \nu(\sigma)=\nu_0-h(\nu)+\dfrac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_0^\sigma(\sigma^\sigma-\upsilon^\sigma)^{\alpha-1}\upsilon^{\sigma-1}g(\upsilon,\nu(\upsilon),{}^{\sigma}_{c}D^{\alpha}\nu(\upsilon))d\upsilon+\dfrac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_0^\sigma(\sigma^\sigma-\upsilon^\sigma)^{\alpha-1}\upsilon^{\sigma-1}\theta_2(\upsilon)d\upsilon. \end{cases} \tag{18}$$

Proof. Consider

$${}^{\sigma}_{c}D^{\alpha}\mu(\sigma)-f(\sigma,\mu(\sigma),{}^{\sigma}_{c}D^{\alpha}\mu(\sigma))=\theta_1(\sigma),\quad 0<\alpha<1,\ \sigma\in\mathbf{J}. \tag{19}$$

Applying integral ${}^{\rho}I^{\alpha}$, we get

$${}^{\sigma}I^{\alpha}({}^{\sigma}_{c}D^{\alpha}\mu(\sigma))-{}^{\rho}I^{\alpha}(f(\sigma,\mu(\sigma),{}^{\sigma}_{c}D^{\alpha}\mu(\sigma)))={}^{\rho}\mathrm{I}^{\alpha}(\theta_1(\sigma))-c_0$$
$$\mu(\sigma)=\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^\sigma(\sigma^\sigma-\upsilon^\sigma)^{\alpha-1}\upsilon^{\sigma-1}f(\upsilon,\mu(\upsilon),{}^{\sigma}_{c}D^{\alpha}\mu(\upsilon))d\upsilon+\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^\sigma(\sigma^\sigma-\upsilon^\sigma)^{\alpha-1}\upsilon^{\sigma-1}\theta_1(\upsilon)d\upsilon-c_0. \tag{20}$$

Using $\mu(0)+h(\mu)=\mu_0$, we have

$$\mu(\sigma)=\mu_0-h(\mu)+\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^\sigma(\sigma^\sigma-\upsilon^\sigma)^{\alpha-1}\upsilon^{\sigma-1}\cdot f(\upsilon,\mu(\upsilon),{}^{\sigma}_{c}D^{\alpha}\mu(\upsilon))d\upsilon+\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^\sigma(\sigma^\sigma-\upsilon^\sigma)^{\alpha-1}\upsilon^{\sigma-1}\theta_1(\upsilon)d\upsilon. \tag{21}$$

Similarly,

$$\nu(\sigma)=\nu_0-h(\nu)+\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^\sigma(\sigma^\sigma-\upsilon^\sigma)^{\alpha-1}\upsilon^{\sigma-1}\cdot g(\upsilon,\nu(\upsilon),{}^{\sigma}_{c}D^{\alpha}\nu(\upsilon))d\upsilon+\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^\sigma(\sigma^\sigma-\upsilon^\sigma)^{\alpha-1}\upsilon^{\sigma-1}\theta_2(\upsilon)d\upsilon. \tag{22}$$

For the upcoming result, here, we define an operator.

($\mathbb{H}_6$): choose $\mathbb{B}_r=\{(\mu,\nu)\in\mathbb{X}\times\mathbb{X}\colon \|(\mu,\nu)\|\leq r, \|u\|\leq(r/2), \|\nu\|\leq(r/2)\}\subset\mathbb{X}$, with $r\geq((\mu_1+\|\mu_0\|+\|\nu_0\|+H+G)/((1-\mu_2))$, $p=\sup\{p(\sigma)\colon\sigma\in\mathbf{J}\}$, $q=\sup\{q(\sigma)\colon\sigma\in\mathbf{J}\}$, $H=\max\{\|h(\mu)\|,\mu\in\mathbb{X}\}$, $G=\max\{\|h(\nu)\|,\nu\in\mathbb{X}\}$, $l_\psi=\sup\{l_\psi(\sigma)\colon\sigma\in\mathbf{J}\}$, $l^1_\psi=\sup\{l^1_\psi(\sigma)\colon\sigma\in\mathbf{J}\}$, and $l^2_\psi=\sup\{l^2_\psi(\sigma)\colon\sigma\in\mathbf{J}\}$.

$$\mu_1 = \frac{T^{\sigma\alpha} l_{gh}\left(2-(q+q^*)+(1-q)(1-q^*)\left(l_\psi + l_\varphi\right)\right)}{\sigma^\alpha (1-q)(1-q^*)\Gamma(\alpha+1)},$$

$$\mu_2 = \frac{T^{\sigma\alpha}\left((1-q^*)p+(1-q)p^*+(1-q)\left(1-q^*\left(l_\psi^1+l_\varphi^1+l_\psi^2+l_\varphi^2\right)\right)\right)}{2\sigma^\alpha (1-q)(1-q^*)\Gamma(\alpha+1)}, \quad (23)$$

$$\max\{\mu_1,\mu_2\}<1.$$

Construct $\mathbb{F}=(\mathbb{F}_1,\mathbb{F}_2)$ and $\mathbb{G}=(\mathbb{G}_1,\mathbb{G}_2)$ on $\mathbb{B}_r$ as

$$\begin{cases} (\mathbb{F}_1\mu)(\sigma) = \mu_0 - h(\mu) + \dfrac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_0^\sigma (\sigma^\sigma - v^\sigma)^{\alpha-1} v^{\sigma-1} f(v,\mu(v), {}_c^\sigma D^\alpha \mu(v))\mathrm{d}v, \\ (\mathbb{F}_2\nu)(\sigma) = \nu_0 - h(\nu) + \dfrac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_0^\sigma (\sigma^\sigma - v^\sigma)^{\alpha-1} v^{\sigma-1} g(v,\mu(v), {}_c^\sigma D^\alpha \mu(v))\mathrm{d}v, \end{cases}$$

$$\begin{cases} (\mathbb{G}_1\mu)(\sigma) = \dfrac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_0^\sigma (\sigma^\sigma - v^\sigma)^{\alpha-1} v^{\sigma-1} \psi(v,\mu(v),\nu(v))\mathrm{d}v, \\ (\mathbb{G}_2\nu)(\sigma) = \dfrac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\displaystyle\int_0^\sigma (\sigma^\sigma - v^\sigma)^{\alpha-1} v^{\sigma-1} \varphi(v,\mu(v),\nu(v))\mathrm{d}v. \end{cases} \quad (24)$$

∎

Theorem 2. *Let the conditions from* $(\mathbb{H}_1)$ *to* $(\mathbb{H}_6)$ *hold; then, system (1) has only one solution.*

Proof. For any $(\mu,\nu)\in\mathbb{B}_r$,

$$\begin{aligned} \|\mathbb{F}(\mu,\nu)+\mathbb{G}(\mu,\nu)\| &\le \|\mathbb{F}(\mu,\nu)\| + \|\mathbb{G}(\mu,\nu)\| \\ &= \left\|\mathbb{F}_1\mu\right\| + \left\|\mathbb{F}_2\nu\right\| + \left\|\mathbb{G}_1(\mu,\nu)\right\| + \left\|\mathbb{G}_2(\mu,\nu)\right\|. \end{aligned} \quad (25)$$

From (24), we have

$$\begin{aligned} \left\|\mathbb{F}_1\mu(\sigma)\right\| &= \left|\mu_0 - h(\mu) + \frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^\sigma (\sigma^\sigma - v^\sigma)^{\alpha-1} v^{\sigma-1} f(v,\mu(v), {}_c^\sigma D^\alpha\mu(v))\mathrm{d}v\right| \\ &\le \left\|\mu_0 - h(\mu)\right\| + \frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^\sigma (\sigma^\sigma - v^\sigma)^{\alpha-1} v^{\sigma-1} \|f(v,\mu(v), {}_c^\sigma D^\alpha\mu(v))\|\mathrm{d}v \\ &\le \left\|\mu_0\right\| + \|h(\mu)\| + \frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^\varsigma (\sigma^\sigma - v^\sigma)^{\alpha-1} v^{\sigma-1}\left\|K_\mu(v)\right\|\mathrm{d}v, \text{ where} \\ \left\|K_\mu(\sigma)\right\| &= \|f(\varsigma,\mu(\sigma), {}_c^\sigma D^\alpha\mu(\sigma))\| = \left\|f\left(\sigma,\mu_0 - h(\mu) + {}_c^\sigma I^\alpha K_\mu(\sigma), K_\mu(\sigma)\right)\right\| \\ &\le l_{gh} + p(\sigma)\|\mu\| + q(\sigma)\left\|K_\mu(\sigma)\right\| \\ &\le l_{gh} + p\|\mu\| + q\left\|K_\mu(\sigma)\right\|, \text{ or} \\ \left\|K_\mu(\sigma)\right\| &\le \frac{l_{gh}+p\|\mu\|}{1-q}. \end{aligned} \quad (26)$$

Therefore,

$$\|\mathbb{F}_1\mu\| \le \|\mu_0\| + H + \frac{(2l_{gh} + rp)T^{\sigma\alpha}}{2(1-q)\sigma^\alpha\Gamma(\alpha+1)},$$
$$\|\mathbb{F}_2\mu\| \le \|\nu_0\| + G + \frac{(2l_{gh} + rp^*)T^{\sigma\alpha}}{2(1-q^*)\sigma^\alpha\Gamma(\alpha+1)}. \tag{27}$$

Furthermore, (24) gives

$$\begin{aligned}\|(\mathbb{G}_1(\mu,\nu))(\sigma)\| &= \left\|\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^\sigma (\sigma^\sigma - v^\sigma)^{\alpha-1} v^{\sigma-1}\psi(v,\mu(v),\nu(v))\mathrm{d}v\right\| \\ &\le \frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^\sigma (\sigma^\sigma - v^\sigma)^{\alpha-1} v^{\sigma-1}\|\psi(v,\mu(v),\nu(v))\|\mathrm{d}v \\ &\le \frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^\sigma (\sigma^\sigma - v^\sigma)^{\alpha-1} v^{\sigma-1} l_\psi(\sigma) + l_\psi^1(\sigma)\|\mu(\sigma)\| + l_\psi^2(\sigma)\|\nu(\sigma)\|\mathrm{d}v \\ &\le \frac{T^{\sigma\alpha}\left(l_\psi(\sigma) + l_\psi^1(\sigma)\|\mu(\sigma)\| + l_\psi^2(\sigma)\|\nu(\sigma)\|\right)}{\sigma^\alpha\Gamma(\alpha+1)} \\ &\le \frac{T^{\sigma\alpha}\left(l_\psi + l_\psi^1\|\mu\| + l_\psi^2\|\nu\|\right)}{\sigma^\alpha\Gamma(\alpha+1)} \\ &\le \frac{T^{\sigma\alpha}\left(2l_\psi + r\left(l_\psi^1 + l_\psi^2\right)\right)}{2\sigma^\alpha\Gamma(\alpha+1)}.\end{aligned} \tag{28}$$

In a similar way, we get

$$\|(\mathbb{G}_2(\mu,\nu))\| \le \frac{T^{\sigma\alpha}\left(2l_\varphi + r\left(l_\psi^1 + l_\psi^2\right)\right)}{2\sigma^\alpha\Gamma(\alpha+1)}. \tag{29}$$

Therefore, (25) gives

$$\|\mathbb{F}(\mu,\nu) + \mathbb{G}(\mu,\nu)\| \le r. \tag{30}$$

So, $\mathbb{F}(\mu,\nu) + \mathbb{G}(\mu,\nu) \in \mathbb{B}_r$.

Now, for $(\mu,\nu), (\overline{\mu},\overline{\nu}) \in \mathbb{B}_r$ and $\varsigma \in \mathbf{J}$, we have

$$\begin{aligned}\|\mathbb{F}(\mu,\nu) - \mathbb{F}(\overline{\mu},\overline{\nu})\| &\le \|\mathbb{F}_1(\mu) - \mathbb{F}_1(\overline{\mu})\| + \|\mathbb{F}_2(\nu) - \mathbb{F}_1(\overline{\nu})\| \\ &\le \|h(\mu) - h(\overline{\mu})\| + \frac{\sigma^{\alpha-1}}{\Gamma(\alpha)}\int_0^\sigma (\sigma^\sigma - v^\sigma)^{\alpha-1} v^{\sigma-1}\left\|K_\mu(v) - K_{\overline{\mu}}(v)\right\|\mathrm{d}v \\ &\quad + \|h(\nu) - h(\overline{\nu})\| + \frac{\sigma^{\alpha-1}}{\Gamma(\alpha)}\int_0^\sigma (\sigma^\sigma - v^\sigma)^{\alpha-1} v^{\sigma-1}\left\|K_\mu(v) - K_{\overline{\mu}}(v)\right\|\mathrm{d}v,\end{aligned} \tag{31}$$

where

$$\begin{aligned}K_\mu(v) &= f(v,\mu(v),{}_c^\sigma D^\alpha \mu(v)), \\ K_{\overline{\mu}}(v) &= f(v,\overline{\mu}(v),{}_c^\sigma D^\alpha \overline{\mu}(v)), \\ K_\nu(v) &= g(v,\mu(v),{}_c^\sigma D^\alpha \mu(v)), \\ K_{\overline{\nu}}(v) &= g(v,\overline{\mu}(v),{}_c^\sigma D^\alpha \mu(v)).\end{aligned} \tag{32}$$

Since $K_\mu(v) = f(v,\mu(v),{}_c^\sigma D^\alpha \mu(v))$, therefore,

$$\begin{aligned}\left\|K_\mu(\sigma) - K_{\overline{\mu}}(\sigma)\right\| &\le \left\|f\left(v,\mu(\sigma),K_\mu(\sigma)\right) - f\left(v,\overline{\mu}(\sigma),K_{\overline{\mu}}(\sigma)\right)\right\| \\ &\le \mathbb{L}_f\|\mu(\sigma) - \overline{\mu}(\sigma)\| + \mathbb{L}_f'\left\|K_\mu(\sigma) - K_{\overline{\mu}}(\sigma)\right\|, \\ \left\|K_\mu(\sigma) - K_{\overline{\mu}}(\sigma)\right\| &\le \frac{\mathbb{L}_f\|\mu - \overline{\mu}\|}{1 - \mathbb{L}_f'}\end{aligned} \tag{33}$$

Similarly, we get

$$\left\|K_\nu(\sigma) - K_{\overline{\nu}}(\sigma)\right\| \le \frac{\mathbb{L}_g\|\nu - \overline{\nu}\|}{1 - \mathbb{L}_g'}. \tag{34}$$

Using (31), we get

$$\begin{aligned}\|\mathbb{F}(\mu,\nu)-\mathbb{F}(\overline{\mu},\overline{\nu})\| &\leq a\|\mu-\overline{\mu}\|+\frac{\mathbb{L}_f T^{\sigma\alpha}(\|\mu-\overline{\mu}\|)}{\sigma^{\alpha}\left(1-\mathbb{L}_f'\right)\Gamma(\alpha+1)} \\ &+b\|\nu-\overline{\nu}\|+\frac{T^{\sigma\alpha}\mathbb{L}_g\|\nu-\overline{\nu}\|}{\sigma^{\alpha}\left(1-\mathbb{L}_g'\right)\Gamma(\alpha+1)} \\ &\leq\left(a+\frac{\mathbb{L}_f T^{\sigma\alpha}}{\sigma^{\alpha}\left(1-\mathbb{L}_f'\right)\Gamma(\alpha+1)}\right)\|\mu-\overline{\mu}\| \\ &+\left(b+\frac{T^{\sigma\alpha}\mathbb{L}_g}{\sigma^{\alpha}\left(1-\mathbb{L}_g'\right)\Gamma(\alpha+1)}\right)\|\nu-\overline{\nu}\| \\ &\leq\gamma_1\|\mu-\overline{\mu}\|+\gamma_2\|\nu-\overline{\nu}\| \\ &\leq\gamma(\|\mu-\overline{\mu}\|+\|\nu-\overline{\nu}\|).\end{aligned}\tag{35}$$

Here, $\gamma=\max\{\gamma_1,\gamma_2\}$, with

$$\begin{aligned}\gamma_1&=a+\left(\frac{\mathbb{L}_f T^{\sigma\alpha}}{\left(\sigma^{\alpha}\left(1-\mathbb{L}_f'\right)\Gamma(\alpha+1)\right)}\right),\\ \gamma_2&=b+\left(\frac{T^{\sigma\alpha}\mathbb{L}_g}{\left(\sigma^{\alpha}\left(1-\mathbb{L}_g'\right)\Gamma(\alpha+1)\right)}\right).\end{aligned}\tag{36}$$

Thus, $\mathbb{F}$ is a contraction. For the continuity and compactness of $\mathbb{G}$, take a sequence $\{\mathbb{S}_n=(\mu_n,\nu_n)\}$ in $\mathbb{B}_r$ with (μ_n,ν_n) approaching to (μ,ν) as n approaches to ∞ in $\mathbb{B}_r$. So,

$$\begin{aligned}\left\|\mathbb{G}(\mu_n,\nu_n)(\sigma)-\mathbb{G}(\mu,\nu)(\sigma)\right\| &=\left\|\mathbb{G}_1(\mu_n,\nu_n)(\sigma)-\mathbb{G}_1(\mu,\nu)(\sigma)\right\|+\left\|\mathbb{G}_2(\mu_n,\nu_n)(\sigma)-\mathbb{G}_2(\mu,\nu)(\sigma)\right\| \\ &\leq\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\sigma}\left(\sigma^{\sigma}-v^{\sigma}\right)^{\alpha-1}v^{\sigma-1}\left\|\psi(v,\mu_n(v),\nu_n(v))-\psi(v,\mu(v),\nu(v))\right\|\mathrm{d}v \\ &+\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\sigma}\left(\sigma^{\sigma}-v^{\sigma}\right)^{\alpha-1}v^{\sigma-1}\left\|\varphi(v,\mu_n(v),\nu_n(v))-\varphi(v,\mu(v),\nu(v))\right\|\mathrm{d}v \\ &\leq\frac{T^{\sigma\alpha}\left(\mathbb{M}_{\psi}\left\|\mu_n-\mu\right\|+\mathbb{M}_{\psi}'\left\|\nu_n-\nu\right\|\right)}{\sigma^{\alpha}\Gamma(\alpha+1)}+\frac{T^{\sigma\alpha}\left(\mathbb{M}_{\varphi}\left\|\mu_n-\mu\right\|+\mathbb{M}_{\varphi}'\left\|\nu_n-\nu\right\|\right)}{\sigma^{\alpha}\Gamma(\alpha+1)}\end{aligned}\tag{37}$$

implies $\|\mathbb{G}(\mu_n,\nu_n)(\sigma)-\mathbb{G}(\mu,\nu)(\sigma)\|\longrightarrow 0$, as $n\longrightarrow\infty$. That is why $\mathbb{G}$ is continuous on $\mathbb{B}_r$. Now, for the uniform boundedness of $\mathbb{G}$ on $\mathbb{B}_r$, consider $\|\mathbb{G}(\mu,\nu)\|$, and by using (31), we have

$$\begin{aligned}\|\mathbb{G}(\mu,\nu)\| &\leq\left\|G_1(\mu,\nu)\right\|+\left\|\mathbb{G}_2(\mu,\nu)\right\| \\ &\leq\frac{T^{\sigma\alpha}\left(2l_{\psi}+r\left(l_{\psi}^1+l_{\psi}^2\right)\right)}{2\sigma^{\alpha}\Gamma(\alpha+1)}+\frac{T^{\sigma\alpha}\left(2l_{\varphi}+r\left(l_{\varphi}^1+l_{\varphi}^2\right)\right)}{2\sigma^{\alpha}\Gamma(\alpha+1)} \\ &=\left[\frac{T^{\sigma\alpha}\left(l_{\psi}+l_{\varphi}\right)}{r\sigma^{\alpha}\Gamma(\alpha+1)}+\frac{T^{\sigma\alpha}\left(l_{\psi}^1+l_{\psi}^2+l_{\varphi}^1+l_{\varphi}^2\right)}{2\sigma^{\alpha}\Gamma(\alpha+1)}\right]r.\end{aligned}\tag{38}$$

Thus, $\mathbb{G}$ is uniformly bounded on $\mathbb{B}_r$. Now, for the equicontinuity of the operator $\mathbb{G}$, take σ_1,σ_2 from $\mathbf{J}$ with $\sigma_1\geq\sigma_2$ and $(\mu,\nu)\in\mathbb{B}_r$. Since $\psi(\sigma,\mu(\sigma),\nu(\sigma))$ is bounded on $\mathbb{B}_r$, we can take $\sup\{\psi(\sigma,\mu(\sigma),\nu(\sigma))\}=M_0$. Thus,

$$
\begin{aligned}
\left\|\mathbb{G}_1(\mu,\nu)(\sigma_1)-\mathbb{G}_1(\mu,\nu)(\sigma_2)\right\| &\le \left\|\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\sigma_1}(\sigma_1^\sigma-v^\sigma)^{\alpha-1}v^{\sigma-1}\psi(v,\mu(v),\nu(v))\right.\\
&\quad\left.-\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\sigma_2}(\sigma_2^\sigma-v^\sigma)^{\alpha-1}v^{\sigma-1}\psi(v,\mu(v),\nu(v))\mathrm{d}v\right\|\\
&\le \frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\sigma_1}\left((\sigma_1^\sigma-v^\sigma)^{\alpha-1}-(\sigma_2^\sigma-v^\sigma)^{\alpha-1}\right)v^{\sigma-1}\|\psi(v,\mu(v),\nu(v))\|\mathrm{d}v\\
&\quad+\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\sigma_1}^{\sigma_2}(\sigma_2^\sigma-v^\sigma)^{\alpha-1}v^{\sigma-1}\|\psi(v,\mu(v),\nu(v))\|\mathrm{d}v\\
&\le M_0\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\left[\int_0^{\sigma_1}\left((\sigma_1^\sigma-v^\sigma)^{\alpha-1}-(\sigma_2^\sigma-v^\sigma)^{\alpha-1}\right)v^{\sigma-1}\mathrm{d}v\right.\\
&\quad\left.+\int_{\sigma_1}^{\sigma_2}(\sigma_2^\sigma-v^\sigma)^{\alpha-1}v^{\sigma-1}\mathrm{d}v\right]\\
&= M_0\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\left[\frac{(\sigma_2^\sigma-\sigma_1^\sigma)^\alpha}{\sigma\alpha}+\frac{(\sigma_2^\sigma-\sigma_1^\sigma)^\alpha}{\sigma\alpha}\right]\\
&= \frac{2M_0}{\sigma^\alpha\Gamma(\alpha+1)}(\sigma_2^\sigma-\sigma_1^\sigma)^\alpha, \text{ gives}
\end{aligned}
\tag{39}
$$

$\|\mathbb{G}_1(\mu,\nu)(\sigma_1)-\mathbb{G}_1(\mu,\nu)(\sigma_2)\|\longrightarrow 0$ as $\sigma_2\longrightarrow\sigma_1$. Using the same approach with $\sup\{\varphi(\sigma,\mu(\sigma),\nu(\sigma))\}=M_0'$, we have

$$
\begin{aligned}
\left\|\mathbb{G}_2(\mu,\nu)(\sigma_1)-\mathbb{G}_2(\mu,\nu)(\sigma_2)\right\| &= \frac{2M_0'}{\sigma^\alpha\Gamma(\alpha+1)}(\sigma_2^\sigma-\sigma_1^\sigma)^\alpha\\
&\longrightarrow 0, \sigma_1\longrightarrow\sigma_2.
\end{aligned}
\tag{40}
$$

Combining these inequalities, we get $\|\mathbb{G}(\mu,\nu)(\sigma_1)-\mathbb{G}(\mu,\nu)(\sigma_2)\|\longrightarrow 0$ as $\varsigma_2\longrightarrow\varsigma_1$; hence, $\mathbb{G}$ is relatively compact on $\mathbb{B}_r$. Thus, by Arzela–Ascolli ($\mathbb{AA}$) theorem, $\mathbb{G}$ is compact and continuous, so (1) has a unique solution. □

Theorem 3. *Under conditions* $(\mathbb{H}_1)$ *to* $(\mathbb{H}_5)$ *with* $\zeta_{fg}<1$, *system (1) has a unique solution.*

Proof. We define the operator $\Psi=(\Psi_1,\Psi_2)$: $\mathbb{X}\times\mathbb{X}\longrightarrow\mathbb{X}$ by

$$
\Psi=\left(\Psi_1(\mu,\nu)(\sigma),\Psi_2(\mu,\nu)(\sigma)\right),\quad \sigma\in\mathbf{J}, \tag{41}
$$

where

$$
\begin{aligned}
\Psi_1(\mu,\nu)(\sigma) &= \mu_0-h(\mu)+\frac{\sigma^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\sigma}(\sigma^\sigma-v^\sigma)^{\alpha-1}v^{\sigma-1}f(v,\mu(v),{}_c^\sigma D^\alpha\mu(v))\mathrm{d}v\\
&\quad+\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\sigma}(\sigma^\sigma-v^\sigma)^{\alpha-1}v^{\sigma-1}\psi(v,\mu(v),\nu(v))\mathrm{d}v,\\
\Psi_2(\mu,\nu)(\sigma) &= \nu_0-h(\nu)+\frac{\sigma^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\sigma}(\sigma^\sigma-v^\sigma)^{\alpha-1}v^{\sigma-1}g(v,\nu(v),{}_c^\sigma D^\alpha\nu(v))\mathrm{d}v\\
&\quad+\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\sigma}(\sigma^\sigma-v^\sigma)^{\alpha-1}v^{\sigma-1}\psi(v,\mu(v),\nu(v))\mathrm{d}v.
\end{aligned}
\tag{42}
$$

For $(\mu,\nu), (\overline{\mu},\overline{\nu}) \in \mathbb{X}\times\mathbb{X}$ and $\varsigma \in \mathbf{J}$, we get

$$
\begin{aligned}
\|\Psi(\mu,\nu)(\sigma)-\Psi(\overline{\mu},\overline{\nu})(\sigma)\| &\leq a\|\mu-\overline{\mu}\| + \frac{T^{\sigma\alpha}\mathbb{L}_f\|\mu-\overline{\mu}\|}{\sigma^{\alpha}\left(1-\mathbb{L}'_f\right)\Gamma(\alpha+1)} \\
&\quad + \frac{T^{\sigma\alpha}\left(\mathbb{M}_{\psi}\|\mu-\overline{\mu}\| + \mathbb{M}'_{\psi}\|\nu-\overline{\nu}\|\right)}{\sigma^{\alpha}\Gamma(\alpha+1)} \\
&\quad + b\|\nu-\overline{\nu}\| + \frac{T^{\sigma\alpha}\mathbb{L}_g\|\nu-\overline{\nu}\|}{\sigma^{\alpha}\left(1-\mathbb{L}'_g\right)\Gamma(\alpha+1)} \\
&\quad + \frac{T^{\sigma\alpha}\left(\mathbb{M}_{\varphi}\|\mu-\overline{\mu}\| + \mathbb{M}'_{\psi}\|\nu-\overline{\nu}\|\right)}{\sigma^{\alpha}\Gamma(\alpha+1)} \\
&= \left[a + \frac{T^{\sigma\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\left[\frac{\mathbb{L}_f}{1-\mathbb{L}'_f} + \mathbb{M}_{\psi} + \mathbb{M}_{\varphi}\right]\right]\|\mu-\overline{\mu}\| \\
&\quad + \left[b + \frac{T^{\sigma\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\left[\frac{\mathbb{L}_g}{1-\mathbb{L}'_g} + \mathbb{M}'_{\psi} + \mathbb{M}'_{\psi}\right]\right]\|\nu-\overline{\nu}\| \\
&\leq \zeta_{fg}\left(\|\mu-\overline{\mu}\|, \|\nu-\overline{\nu}\|\right), \text{ with} \\
\zeta_f &= a + \frac{T^{\sigma\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\left[\frac{\mathbb{L}_f}{1-\mathbb{L}'_f} + \mathbb{M}_{\psi} + \mathbb{M}_{\varphi}\right], \\
\zeta_g &= b + \frac{T^{\sigma\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\left[\frac{\mathbb{L}_g}{1-\mathbb{L}'_g} + \mathbb{M}'_{\psi} + \mathbb{M}'_{\psi}\right], \\
\zeta_{fg} &= \max\left\{\zeta_f, \zeta_g\right\}.
\end{aligned}
\tag{43}
$$

Hence, Ψ is a contraction, and by the assumption that $\zeta_{fg} < 1$, (1) has only one solution. □

4. HU Stability

Now, we are analyzing different kinds of stabilities such as $\mathbb{HU}$ stability of the proposed system given in (1).

Theorem 4. *Let the hypothesis from* $(\mathbb{H}_1)$ *to* $(\mathbb{H}_5)$ *hold true with the conditions* $\zeta_{fg} < 1$, *and if the matrix* $\mathbb{Q} \longrightarrow 0$, *then (1) is* $\mathbb{HU}$ *stable.*

Proof. Proceeding from Theorem 3, for any $(\mu,\nu), (\mu^*,\nu^*) \in \mathbb{X}$ and $\varsigma \in \mathbf{J}$, we have

$$
\begin{aligned}
\left\|\Psi_1(\mu,\nu)(\sigma)-\Psi_1(\mu^*,\nu^*)(\sigma)\right\| &\leq a\left\|\mu-\mu^*\right\| + \frac{T^{\sigma\alpha}\mathbb{L}_f\left\|\mu-\mu^*\right\|}{\sigma^{\alpha}\left(1-\mathbb{L}'_f\right)\Gamma(\alpha+1)} \\
&\quad + \frac{T^{\sigma\alpha}\left(\mathbb{M}_{\psi}\left\|\mu-\mu^*\right\| + \mathbb{M}'_{\psi}\|\nu-\overline{\nu}\|\right)}{\sigma^{\alpha}\Gamma(\alpha+1)} \\
&\leq \left[a + \frac{T^{\sigma\alpha}\mathbb{L}_f}{\sigma^{\alpha}\left(1-\mathbb{L}'_f\right)\Gamma(\alpha+1)} + \frac{\mathbb{M}_{\psi}T^{\sigma\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)}\right]\left\|\mu-\mu^*\right\| \\
&\quad + \frac{T^{\sigma\alpha}\mathbb{M}'_{\psi}}{\sigma^{\alpha}\Gamma(\alpha+1)}\|\nu-\overline{\nu}\| \\
&= \mathbb{C}_1\left\|\mu-\mu^*\right\| + \mathbb{C}_2\left\|\nu-\nu^*\right\|.
\end{aligned}
\tag{44}
$$

Similarly,

$$\left\|\Psi_2(\mu,\nu)(\sigma)-\Psi_2(\mu^*,\nu^*)(\sigma)\right\| \leq \frac{\mathbb{M}_\varphi T^{\sigma\alpha}}{\sigma^\alpha\Gamma(\alpha+1)}\left\|\mu-\mu^*\right\| + \left[b+\frac{T^{\sigma\alpha}\mathbb{L}_g}{\sigma^\alpha\left(1-\mathbb{L}_g'\right)\Gamma(\alpha+1)}+\frac{\mathbb{M}_\varphi T^{\sigma\alpha}}{\sigma^\alpha\Gamma(\alpha+1)}\right]\left\|\nu-\nu^*\right\| = \mathbb{C}_3\left\|\mu-\mu^*\right\|+\mathbb{C}_4\left\|\nu-\nu^*\right\|, \tag{45}$$

where

$$\begin{aligned}\mathbb{C}_1 &= a+\frac{T^{\sigma\alpha}\mathbb{L}_f}{\sigma^\alpha\left(1-\mathbb{L}_f'\right)\Gamma(\alpha+1)}+\frac{\mathbb{M}_\psi T^{\sigma\alpha}}{\sigma^\alpha\Gamma(\alpha+1)},\\ \mathbb{C}_2 &= \frac{T^{\sigma\alpha}\mathbb{M}_\psi'}{\sigma^\alpha\Gamma(\alpha+1)},\\ \mathbb{C}_3 &= \frac{\mathbb{M}_\varphi T^{\sigma\alpha}}{\sigma^\alpha\Gamma(\alpha+1)},\\ \mathbb{C}_4 &= b+\frac{T^{\sigma\alpha}\mathbb{L}_g}{\sigma^\alpha\left(1-\mathbb{L}_g'\right)\Gamma(\alpha+1)}+\frac{\mathbb{M}_\varphi T^{\sigma\alpha}}{\sigma^\alpha\Gamma(\alpha+1)}.\end{aligned} \tag{46}$$

Writing together the above inequalities, we have

$$\begin{aligned}\left\|\Psi_1(\mu,\nu)(\sigma)-\Psi_1(\mu^*,\nu^*)(\sigma)\right\| &\leq \mathbb{C}_1\left\|\mu-\mu^*\right\|+\mathbb{C}_2\left\|\nu-\nu^*\right\|,\\ \left\|\Psi_2(\mu,\nu)(\sigma)-\Psi_2(\mu^*,\nu^*)(\sigma)\right\| &\leq \mathbb{C}_3\left\|\mu-\mu^*\right\|+\mathbb{C}_4\left\|\nu-\nu^*\right\|.\end{aligned} \tag{47}$$

From this, we get

$$\left\|\Psi(\mu,\nu)-\Psi(\mu^*,\nu^*)\right\| \leq \mathbb{Q}\left\|\mu-\mu^*,\nu-\nu^*\right\|, \tag{48}$$

where $\mathbb{Q}=\begin{pmatrix}\mathbb{C}_1 & \mathbb{C}_2\\ \mathbb{C}_3 & \mathbb{C}_4\end{pmatrix}$. As given, $\mathbb{Q}$ converges to 0; therefore, (1) is $\mathbb{HU}$ stable. □

Example 1. Consider

$$\begin{aligned}{}^\sigma_c D^\alpha(\nu(\sigma)) &= \left(\frac{1}{4e^\sigma}+\frac{1}{5e^\sigma}\left(\|\mu(\sigma)\|+\|{}^\sigma_c D^\alpha(\nu(\sigma))\|\right)\right) + \frac{t+\sin|\mu(\sigma)|+\cos|\mu(\sigma)|}{50}, \quad \sigma\in\mathbf{J},\\ {}^\sigma_c D^\alpha(\nu(\sigma)) &= \left(\frac{1}{10e^{2t}}+\frac{1}{20e^\sigma}\left(\|\nu(\sigma)\|+\|{}^\sigma_c D^\alpha(\nu(\sigma))\|\right)\right) + \frac{\cos|\mu(\sigma)|+\|\nu(\sigma)\|}{t+100}.\\ \mu(0)&+\sum_{k=1}^{n}a_k\mu(\sigma_k)=0,\\ \nu(0)&+\sum_{k=1}^{n}b_k\nu(\sigma_k)=0,\end{aligned} \tag{49}$$

where $\sigma=0.4, \alpha\in(0,1), a_k, b_k>0, k=0,1,2,\ldots,n$. Set functions as

$$\begin{aligned}f(\sigma,\mu(\sigma),\nu(\sigma)) &= \frac{1}{4e^\sigma}+\frac{1}{5e^\sigma}\left(\|\mu(\sigma)\|+\|{}^\sigma_c D^\alpha(\nu(\sigma))\|\right),\\ g(\sigma,\mu(\sigma),\nu(\sigma)) &= \frac{1}{10e^{2t}}+\frac{1}{20e^\sigma}\left(\|\nu(\sigma)\|+\|{}^\sigma_c D^\alpha(\nu(\sigma))\|\right),\\ h(\mu) &= \sum_{k=1}^{n}b_k\nu(\sigma_k),\\ h(\nu) &= \sum_{k=1}^{n}b_k\nu(\sigma_k).\end{aligned} \tag{50}$$

Taking $(\mu,\nu), (\overline{\mu},\overline{\nu})$ from $\mathbb{X}$ and $\sigma\in[0,1]$, we have

$$\begin{aligned}\|f(\sigma,\mu(\sigma),\nu(\sigma))-f(\sigma,\overline{\mu}(\sigma),\overline{\nu}(\sigma))\| &\leq \frac{1}{5}\|\mu-\overline{\mu}\| + \frac{1}{5}\|\nu-\overline{\nu}\|,\\ \|g(\sigma,\mu(\sigma),\nu(\sigma))-g(\sigma,\overline{\mu}(\sigma),\overline{\nu}(\sigma))\| &\leq \frac{1}{20}\|\mu-\overline{\mu}\| + \frac{1}{20}\|\nu-\overline{\nu}\|,\\ \|\psi(\sigma,\mu(\sigma),\nu(\sigma))-\psi(\sigma,\overline{\mu}(\sigma),\overline{\nu}(\sigma))\| &\leq \frac{1}{50}\|\mu-\overline{\mu}\| + \frac{1}{50}\|\nu-\overline{\nu}\|,\\ \|\varphi(\sigma,\mu(\sigma),\nu(\sigma))-\varphi(\sigma,\overline{\mu}(\sigma),\overline{\nu}(\sigma))\| &\leq \frac{1}{100}\|\mu-\overline{\mu}\| + \frac{1}{100}\|\nu-\overline{\nu}\|,\\ \|h(\mu)-h(\overline{\mu})\| &\leq \sum_{k=1}^{m}a_k\left\|\mu(\sigma_k)-\overline{\mu}(\sigma_k)\right\|,\\ \|h(\nu)-h(\overline{\nu})\| &\leq \sum_{k=1}^{m}b_k\left\|\nu(\sigma_k)-\overline{\nu}(\sigma_k)\right\|.\end{aligned} \tag{51}$$

In particular, let we take $T=1, \alpha=1/2>0, \sigma=0.4$, and from the inequalities in (51), $\mathbb{L}_f=\mathbb{L}_f'=1/5, \mathbb{L}_g=\mathbb{L}_g'=1/20, \mathbb{M}_\psi=\mathbb{M}_\psi'=1/50, \mathbb{M}_\varphi=\mathbb{M}_\varphi'=1/100, \quad \sum a_k=\sum b_k=1/3$.

Thus, we have $\zeta_{fg}=0.8328<1$. Therefore, by Theorem 3, (5.1) has a unique solution. After calculations, we get

$$\mathbb{Q}=\begin{pmatrix}0.8151 & 0.0357\\ 0.0178 & 0.4450\end{pmatrix}. \tag{52}$$

Upon calculations, we obtained the eigenvalues 0.81675 and 0.4433, which show that $\mathbb{Q}$ is converging to zero. Using Theorem 4, the solution of (5.1) is $\mathbb{HU}$ stable.

5. Conclusion

In this manuscript, we used 𝔸𝔸 theorem and Banach contraction principle to achieve the sufficient conditions for 𝔼𝕌 of solutions to a nonlocal implicit switched system. With the help of assumptions, we proved the ℍ𝕌 stability result for the couple system given in (1).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the China Postdoctoral Science Foundation (Grant no. 2019M652348), the Technology Research Foundation of Chongqing Educational Committee (Grant no. KJQN201900539), and the Natural Science Foundation of Chongqing Normal University (Grant no. 16XYY24).

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