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Research Article

Sufficient Conditions for Graphs to Be k-Connected, Maximally Connected, and Super-Connected

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Let G be a connected graph with minimum degree $\delta(G)$ and vertex-connectivity $\kappa(G)$. The graph G is k-connected if $\kappa(G) \geq k$, maximally connected if $\kappa(G) = \delta(G)$, and super-connected if every minimum vertex-cut isolates a vertex of minimum degree. In this paper, we present sufficient conditions for a graph with given minimum degree to be k-connected, maximally connected, or super-connected in terms of the number of edges, the spectral radius of the graph, and its complement, respectively. Analogous results for triangle-free graphs with given minimum degree to be k-connected, maximally connected, or super-connected are also presented.

1. Introduction

Let $G = (V, E)$ be a simple connected undirected graph, where $V = V(G)$ is the vertex-set of G and $E = E(G)$ is the edge-set of G. The *order* and *size* of G are defined by $n = |V(G)|$ and $m = |E(G)|$, respectively; $d_G(x)$ is the *degree* of a vertex x in G, that is, the number of edges incident with x in G; $\delta(G) = \min\{d_G(x): x \in V(G)\}$ is the *minimum degree* of G. For a subset $X \subset V(G)$, use $G[X]$ to denote the subgraph of G induced by X. For two subsets X and Y of $V(G)$, let $[X, Y]$ be the set of edges between X and Y. The *complement* of G is denoted by $\overline{G}$. Let $G_1 \cup G_2$ denote the disjoint union of graphs G_1 and G_2, and let $G_1 \vee G_2$ denote the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2. The graph G is called a *triangle-free graph* if G contains no triangle. Denote by $\rho(G)$ the largest eigenvalue or the spectral radius of the adjacency matrix of G and it is called the *spectral radius* of G. If G is connected, then, by Perron-Frobenius Theorem, $\rho(G)$ is simple and there exists a unique (up to a multiple) corresponding positive eigenvector.

A *vertex-cut* of a connected graph G is a set of vertices whose removal disconnects G. The *vertex-connectivity* or simply the *connectivity* $\kappa = \kappa(G)$ of a connected graph G is the minimum cardinality of a vertex-cut of G if G is not complete, and $\kappa(G) = n - 1$ if G is the complete graph K_n of order n. A vertex-cut S is a *minimum vertex-cut* or a *κ-cut* of G if $|S| = \kappa(G)$. Apparently, $\kappa(G) \leq \delta(G)$ for any graph G. The graph G is *k-connected* if $\kappa(G) \geq k$, *maximally connected* if $\kappa(G) = \delta(G)$, and super-connected (or *super-κ*) if every minimum vertex-cut isolates a vertex of minimum degree. Hence, every super-connected graph is also maximally connected. An *edge-cut* of a connected graph G is a set of edges whose removal disconnects G. The *edge connectivity* $\lambda = \lambda(G)$ of a connected graph G is defined as the minimum cardinality of an edge-cut over all edge-cuts of G. An edge-cut S is a *minimum edge-cut* if $|S| = \lambda(G)$. The inequality $\lambda(G) \leq \delta(G)$ is obvious. The graph G is *maximally edge-connected* if $\lambda(G) = \delta(G)$, and it is *super-edge-connected* if every minimum edge-cut consists of edges incident with a vertex of minimum degree. Therefore, every super-edge-connected graph is also maximally edge-connected. For graph-theoretical terminology and notation not defined here, one can refer to [1, 2].

Sufficient conditions for graphs to be maximally (edge-) connected or super-(edge-) connected were given by several authors, depending on the order, the maximum and minimum degree, the diameter, the girth, the degree sequence, the clique number and so on. The paper in [3] by Hellwig and Volkmann gives a survey on this topic. Recently, Volkmann and Hong [4] proved that a connected graph or a connected triangle-free graph is maximally edge-connected or super-edge-connected if the number of edges is large enough, and the results corresponding to triangle-free graphs were generalized to connected graphs with given clique number by Volkmann [5].

On the other hand, the relationship between graph properties and eigenvalues has attracted much attention. Fiedler [6] initiated the research on the relationship between graph connectivity and graph eigenvalues, and Fiedler and Nikiforov [7] initiated the investigation on the spectral conditions for graphs to be Hamiltonian or traceable. Cioabă [8] investigated the relationship between edge-connectivity and adjacency eigenvalues of regular graphs. From then on, the edge-connectivity problem has been intensively studied by many researchers, such as Duan et al. [9], Gu et al. [10], Liu et al. [11], Liu et al. [12], and Suil O [13]. For vertex-connectivity, Li [14] presented sufficient conditions for a graph to be k-connected using spectral radius and signless Laplacian spectral radius; Feng et al. [15] demonstrated sufficient conditions based on spectral radius for a graph to be k-connected and k-edge-connected; Feng et al. [16] obtained a tight sufficient condition for a connected graph with fixed minimum degree to be k-connected based on its spectral radius, for sufficiently large order. Vertex-connectivity and the second largest adjacency eigenvalue of regular graphs were studied by Abiad et al. [17], Cioabă and Gu [18], O [19], and Zhang [20]. The relationship between vertex-connectivity and adjacency eigenvalues or Laplacian eigenvalues of graphs has been investigated by Hong et al. [21–23] and Liu et al. [24].

Motivated by the researches mentioned above, this paper presents sufficient conditions for a graph with given minimum degree to be k-connected, maximally connected, or super-connected in terms of the number of edges, the spectral radius of the graph, or its complement, respectively. In addition, we also give sufficient conditions for a triangle-free graph with given minimum degree to be k-connected, maximally connected, or super-connected in terms of the number of edges or its spectral radius, respectively. The results on k-connected graph in this paper improve the result in [16] by Feng et al. to some extent.

The rest of this paper is organized as follows. In Section 2, we present sufficient conditions for a graph with given minimum degree to be k-connected in terms of the number of edges, the spectral radius of the graph, and its complement, respectively. In terms of the same parameters as in Section 2, by setting $k = \delta$, we get sufficient conditions for a graph with given minimum degree to be maximally connected in Section 3, and we obtain sufficient conditions for a graph with given minimum degree to be super-connected in Section 4. In Section 5, sufficient conditions for a triangle-free graph to be k-connected, maximally connected, or super-connected are acquired in terms of the number of edges and the spectral radius of the graph, respectively.

2. k-Connected Graphs

Let G be a connected graph of order n, minimum degree δ, and vertex-connectivity κ. If $n \le 4$ or $\delta = 1$, then $\kappa = \delta$. If $\delta = n-1$, then $G = K_n$ and $\kappa = \delta$. If $\delta = n-2$, then when u and v are nonadjacent, the other $n-2$ vertices are all common neighbors of u and v. It is necessary to delete all common neighbors of some pair of vertices to separate the graph, so $\kappa \ge n-2 = \delta$. Therefore, we only need to consider $n \ge 5$ and $2 \le \delta \le n-3$ in the following.

Theorem 1. *Let G be a connected graph of order $n \ge 5$, size m, and minimum degree $\delta \ge k \ge 2$.*

(a) If

$$m \ge \frac{1}{2}n(n-1) - (\delta - k + 2)(n - \delta - 1), \tag{1}$$

then G is k-connected, unless $G = K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$.

(b) If $n \ge (1/2)(k+1)(\delta - k + 2) + (\delta + 2)$ and

$$m \ge \frac{1}{2}n(n-1) - \frac{1}{2}(\delta - k + 2)(2n - 2\delta + k - 3), \tag{2}$$

then G is k-connected, unless G is a subgraph of $K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$.

Proof. Let $\kappa = \kappa(G)$. On the contrary, suppose that G is not k-connected; that is, $1 \le \kappa \le k-1$. Let S be an arbitrary minimum vertex-cut, and let $X_0, X_1, \ldots, X_{p-1}$ $(p \ge 2)$, denote the vertex sets of the components of $G - S$, where $|X_0| \le |X_1| \le \cdots \le |X_{p-1}|$. Each vertex in X_0 is adjacent to at most $|X_0| - 1$ vertices of X_0 and $\kappa = |S|$ vertices of S. Thus,

$$\delta|X_0| \le \sum_{x \in X_0} d(x) \le |X_0|\left(|X_0| + \kappa - 1\right), \tag{3}$$

and so $|X_0| \ge \delta - \kappa + 1$. Let $Y = \cup_{i=1}^{p-1} X_i$; then $|Y| = n - \kappa - |X_0|$. Therefore,

$$\delta - \kappa + 1 \le |X_0| \le |Y| \le n - \delta - 1. \tag{4}$$

Since $G - S$ is disconnected, there are no edges between X_0 and Y in G and

$$m \le \frac{1}{2}n(n-1) - |X_0| \cdot |Y|. \tag{5}$$

(a) Since we suppose that G is not k-connected, it suffices to prove $G = K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$. By (4) and $|X_0| + |Y| = n - \kappa$, and since $\kappa \le k-1$, we obtain

$$\begin{aligned} |X_0| \cdot |Y| &\ge (\delta - \kappa + 1)(n - \delta - 1) \\ &\ge (\delta - k + 2)(n - \delta - 1). \end{aligned} \tag{6}$$

Substituting (6) into (5), it follows that

$$m \le \frac{1}{2}n(n-1) - (\delta - k + 2)(n - \delta - 1). \tag{7}$$

Combining this with (1), we obtain $m = (1/2)n(n-1) - (\delta - k + 2)(n - \delta - 1)$. Hence, all the inequalities in (6) must be equalities and so $\kappa = k - 1$, $|X_0| = \delta - k + 2$, and $|Y| = n - \delta - 1$. Thus, G is obtained from K_n by deleting all the edges of the complete bipartite subgraph $K_{|X_0|,|Y|}$ of K_n. That is, $G[X_0] = K_{\delta-k+2}$, $G[S] = K_{k-1}$, $G[Y] = K_{n-\delta-1}$, and $G = K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$.

(b) To prove that G is a subgraph of $K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$, we first show that $|X_0| = \delta - k + 2$. Suppose that $|X_0| \ge \delta - k + 3$. Since $|X_0| \le |Y|$, $|X_0| + |Y| = n - \kappa$, and $\kappa \le k - 1$, we have

$$\begin{aligned} |X_0| \cdot |Y| &\ge (\delta - k + 3)(n - \kappa - (\delta - k + 3)) \\ &\ge (\delta - k + 3)(n - \delta - 2). \end{aligned} \tag{8}$$

Substituting (8) into (5), it follows that

$$m \le \frac{1}{2}n(n-1) - (\delta - k + 3)(n - \delta - 2). \tag{9}$$

Combining this with (2), it is easy to get $n \le (1/2)(k+1)(\delta - k + 2) + (\delta + 2)$. By the hypothesis, we have $n = (1/2)(k+1)(\delta - k + 2) + (\delta + 2)$. Hence, $m = (1/2)n(n-1) - (\delta - k + 3)(n - \delta - 2)$ and all the inequalities in (8) must be equalities. Thus, $\kappa = k - 1$, $|X_0| = \delta - k + 3$, $|Y| = n - \delta - 2$, and G is obtained from K_n by deleting all the edges of the complete bipartite subgraph $K_{|X_0|,|Y|}$ of K_n. That is, $G = K_{k-1} \vee (K_{\delta-k+3} \cup K_{n-\delta-2})$. However, $\delta(G) = \delta(K_{k-1} \vee (K_{\delta-k+3} \cup K_{n-\delta-2})) = \delta + 1 > \delta$, a contradiction. Thus, $|X_0| \le \delta - k + 2$. Combining this with $|X_0| \ge \delta - \kappa + 1 \ge \delta - k + 2$, we get $|X_0| = \delta - k + 2$. Since $|S| = \kappa \le k - 1$ and $d_G(x) \ge \delta$ for each $x \in X_0$, we have that each vertex of X_0 is adjacent to each vertex of S and $|S| = k - 1$. Therefore, $G[X_0 \cup S] = K_{\delta+1}$ and G is a subgraph of $K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$. □

Theorem 2. *Let G be a connected graph of order n and minimum degree $\delta \ge k \ge 2$. If*

$$\rho(G) \ge \rho(K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})), \tag{10}$$

then G is k-connected, unless $G = K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$, where $\rho(K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1}))$ is the largest root of the equation

$$\begin{aligned} &\lambda^3 - (n-3)\lambda^2 + ((\delta - k + 2)(n - \delta - 1) - 2n + 3)\lambda \\ &\quad + (\delta - k + 2)(n - \delta - 1)k - n + 1 = 0. \end{aligned} \tag{11}$$

Proof. Let $\kappa = \kappa(G)$. Assume that (10) holds but $1 \le \kappa \le k - 1$. Let S be an arbitrary minimum vertex-cut of G, and let $X_0, X_1, \ldots, X_{p-1}$ $(p \ge 2)$, denote the vertex-sets of the components of $G - S$, where $|X_0| \le |X_1| \le \cdots \le |X_{p-1}|$. Each vertex in X_i is adjacent to at most $|X_i| - 1$ vertices of X_i and $\kappa = |S|$ vertices of S. Thus,

$$\delta|X_i| \le \sum_{x \in X_i} d(x) \le |X_i|(|X_i| - 1 + \kappa), \tag{12}$$

and so $|X_i| \ge \delta - \kappa + 1$ for each $i = 0, 1, \ldots, p - 1$. Let $Y = \cup_{i=1}^{p-1} X_i$. Then, $\delta - \kappa + 1 \le |X_0| \le |Y| \le n - \delta - 1$ and $|X_0| + |Y| = n - \kappa$. Since there are no edges between X_0 and Y in G, G is a subgraph of $K_\kappa \vee (K_{|X_0|} \cup K_{|Y|})$ and $\rho(G) \le \rho(K_\kappa \vee (K_{|X_0|} \cup K_{|Y|}))$.

Next, we will show

$$\begin{aligned} \rho\left(K_\kappa \vee \left(K_{|X_0|} \cup K_{|Y|}\right)\right) &\le \rho(K_\kappa \vee (K_{\delta-\kappa+1} \cup K_{n-\delta-1})) \\ &\le \rho(K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})). \end{aligned} \tag{13}$$

Denote $G(a, b, \kappa) = K_\kappa \vee (K_a \cup K_b)$ for short, where $b \ge a \ge \delta - \kappa + 1$ and $a + b + \kappa = n$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be the unique positive unit eigenvector corresponding to $\rho(G(a, b, \kappa))$. By symmetry, let $x := x_i$ for any $i \in K_a$; $y := x_j$ for any $j \in K_\kappa$; $z := x_\ell$ for any $\ell \in K_b$. According to $\lambda x_i = \sum_{ij \in E(G(a,b,\kappa))} x_j$ and the uniqueness of $\mathbf{x}$, we have that $\rho(G(a, b, \kappa))$ is the largest root of the following equations:

$$\begin{cases} \lambda x = (a-1)x + \kappa y, \\ \lambda y = ax + (\kappa - 1)y + bz, \\ \lambda z = \kappa y + (b-1)z. \end{cases} \tag{14}$$

Thus, $\rho(G(a, b, \kappa))$ is the largest root of the equation

$$\begin{aligned} f(\lambda; a, b, \kappa) := \lambda^3 &- (n-3)\lambda^2 + (ab - 2n + 3)\lambda \\ &+ ab(\kappa + 1) - n + 1 = 0. \end{aligned} \tag{15}$$

Then, we have

$$\begin{aligned} &f(\lambda; a, b, \kappa) - f(\lambda; \delta - \kappa + 1, n - \delta - 1, \kappa) \\ &\quad = (\lambda + \kappa + 1)(ab - (\delta - \kappa + 1)(n - \delta - 1)) \ge 0, \end{aligned} \tag{16}$$

for any $\lambda > 0$ and $b \ge a \ge \delta - \kappa + 1$. Therefore, $\rho(G(a, b, \kappa)) \le \rho(G(\delta - \kappa + 1, n - \delta - 1, \kappa))$ for any $b \ge a \ge \delta - \kappa + 1$, which means that

$$\rho\left(K_\kappa \vee \left(K_{|X_0|} \cup K_{|Y|}\right)\right) \le \rho(K_\kappa \vee (K_{\delta-\kappa+1} \cup K_{n-\delta-1})). \tag{17}$$

Since $K_\kappa \vee (K_{\delta-\kappa+1} \cup K_{n-\delta-1})$ is a subgraph of $K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$ for any $\kappa \le k - 1$,

$$\rho(K_\kappa \vee (K_{\delta-\kappa+1} \cup K_{n-\delta-1})) \le \rho(K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})). \tag{18}$$

Hence, from the discussion above, we have

$$\begin{aligned} \rho(G) &\le \rho\left(K_\kappa \vee \left(K_{|X_0|} \cup K_{|Y|}\right)\right) \\ &\le \rho(K_\kappa \vee (K_{\delta-\kappa+1} \cup K_{n-\delta-1})) \\ &\le \rho(K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})). \end{aligned} \tag{19}$$

By (10), the above inequalities must be equalities. Thus, $|X_0| = \delta - k + 2$, $\kappa = k - 1$, $|Y| = n - \delta - 1$, and so $G = K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$. The result follows from (15). □

Remark 1. In Corollary 3.5 in [16], the authors showed that if G is a connected graph of minimum degree $\delta(G) \geq \delta \geq k \geq 3$ and order $n \geq (\delta - k + 2)(k^2 - 2k + 4) + 3$, and $\rho(G) \geq \rho(K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1}))$, then G is k-connected, unless $G = \rho(K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1}))$. Apparently, without restriction on the order of graph, Theorem 2 improves Corollary 3.5 in [16].

Theorem 3. *Let G be a connected graph of order n and minimum degree $\delta \geq k \geq 3$. If G is a subgraph of $K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$ and $n \geq (1/2)(\delta - k + 2)(k^2 - 2k + 7)$, then*

$$\rho(G) < n - \delta + k - 3, \tag{20}$$

unless $G = K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$.

Proof. Denote $H = K_{k-1} \vee (K_{\delta-k+2} \cup K_{n-\delta-1})$ for short. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be the unique positive unit eigenvector corresponding to $\rho(G)$. Recall that Rayleigh's principle implies that

$$\rho(G) = \mathbf{x}^T A(G)\mathbf{x} = 2 \sum_{ij \in E(G)} x_i x_j. \tag{21}$$

Assume that G is a proper subgraph of H. Clearly, we could assume that G is obtained by omitting just one edge uv of H. Let X, Y, Z be the set of vertices of H of degree $\delta, n-1, n-\delta+k-3$, respectively, where $|X| = \delta - k + 2$, $|Y| = k - 1$, and $|Z| = n - \delta - 1$. Since $\delta(G) = \delta$, G must contain all the edges between X and Y. Therefore, $\{u, v\} \subset Y \cup Z$, with three possible cases: (a) $\{u, v\} \subset Y$; (b) $u \in Y, v \in Z$; and (c) $\{u, v\} \subset Z$. We shall show that case (c) yields a graph whose spectral radius is not smaller than the spectral radius of the graph in case (b) and that case (b) yields a graph whose spectral radius is not smaller than the spectral radius of the graph in case (a).

Firstly, suppose that case (a) occurs; that is, $\{u, v\} \subset Y$. Choose a vertex $w \in Z$. If $x_u \geq x_w$, then by removing the edge vw and adding the edge uv we obtain a new graph G_1 which is covered by case (b). By the Rayleigh principle,

$$\rho(G_1) - \rho(G) \geq \mathbf{x}^T A(G_1)\mathbf{x} - \mathbf{x}^T A(G)\mathbf{x} = 2x_v(x_u - x_w) \geq 0. \tag{22}$$

If $x_w > x_u$, then by removing all the edges between X and $\{u\}$ and adding all the edges between X and $\{w\}$ we obtain a new graph G_1' which is also covered by case (b). By the Rayleigh principle,

$$\begin{aligned} \rho(G_1') - \rho(G) &\geq \mathbf{x}^T A(G_1')\mathbf{x} - \mathbf{x}^T A(G)\mathbf{x} \\ &= 2(x_w - x_u) \sum_{i \in X} x_i > 0. \end{aligned} \tag{23}$$

Secondly, suppose that case (b) occurs; that is, $u \in Y$, $v \in Z$. Choose a vertex $w \in Z$ and $w \neq v$. If $x_u \geq x_w$, then by removing the edge vw and adding the edge uv we obtain a new graph G_2 which is covered by case (c). By the Rayleigh principle,

$$\rho(G_2) - \rho(G) \geq \mathbf{x}^T A(G_2)\mathbf{x} - \mathbf{x}^T A(G)\mathbf{x} = 2x_v(x_u - x_w) \geq 0. \tag{24}$$

If $x_w > x_u$, then by removing all the edges between X and $\{u\}$ and adding all the edges between X and $\{w\}$ we obtain a new graph G_2' which is also covered by case (c). By the Rayleigh principle,

$$\begin{aligned} \rho(G_2') - \rho(G) &\geq \mathbf{x}^T A(G_2')\mathbf{x} - \mathbf{x}^T A(G)\mathbf{x} \\ &= 2(x_w - x_u) \sum_{i \in X} x_i > 0. \end{aligned} \tag{25}$$

Therefore, we could assume that $\{u, v\} \in Z$. By symmetry, let $x := x_i$ for any $i \in X$; $y := x_j$ for any $j \in Y$; $z := x_\ell$ for any $\ell \in Z$, $\{u, v\}$; and $t := x_u = x_v$. According to $\lambda x_i = \sum_{ij \in E(G)} x_j$ and the uniqueness of $\mathbf{x}$, we have that ρ is the largest root of following equations:

$$\begin{cases} \lambda x = (\delta - k + 1)x + (k - 1)y, \\ \lambda y = (\delta - k + 2)x + (k - 2)y + (n - \delta - 3)z + 2t, \\ \lambda z = (k - 1)y + (n - \delta - 4)z + 2t, \\ \lambda t = (k - 1)y + (n - \delta - 3)z. \end{cases} \tag{26}$$

Thus, $\rho(G)$ is the largest root of the equation

$$\begin{aligned} f(\lambda) := {} & \lambda^4 - (n-5)\lambda^3 + ((n - \delta - 1)(\delta - k - 2) - 4\delta + 7)\lambda^2 \\ & + \left[(\delta k + 2\delta + 2)(n - \delta + k - 3) - \left(k^2 + 3\right)(n - 1) + 6\right]\lambda \\ & + 2((\delta - k + 1)(k(n - \delta - 2) - 1) \\ & + (k - 1)(n - \delta - 3)) = 0. \end{aligned} \tag{27}$$

By some basic calculations, we have

$$\begin{aligned} f(n - \delta + k - 3) = {} & 2n^2 - (\delta - k + 2)\left(k^2 - 2k + 7\right)n \\ & + (\delta - k + 2)((\delta - k + 1) \\ & \cdot \left(k^2 - 2k + 5\right) - 2(k - 3)). \end{aligned} \tag{28}$$

Set $g(x) = 2x^2 - (\delta - k + 2)(k^2 - 2k + 7)x + (\delta - k + 2)((\delta - k + 1)(k^2 - 2k + 5) - 2(k - 3))$. It is easy to see that the function $g(x)$ is strictly increasing when $x > (1/2)(\delta - k + 2)(k^2 - 2k + 7)$. Since $n \geq (1/2)(\delta - k + 2)(k^2 - 2k + 7) > (1/4)(\delta - k + 2)(k^2 - 2k + 7)$, we get

$$\begin{aligned} f(n - \delta + k - 3) = g(n) &\geq g\left(\frac{1}{2}(\delta - k + 2)\left(k^2 - 2k + 7\right)\right) \\ &= (\delta - k + 2)((\delta - k + 1) \\ &\quad \cdot \left(k^2 - 2k + 5\right) - 2(k - 3)) \\ &\geq (\delta - k + 2)\left(k^2 - 4k + 11\right) > 0. \end{aligned} \tag{29}$$

By $3 \leq k \leq \delta \leq n - 3$ and $n \geq (1/2)(\delta - k + 2)(k^2 - 2k + 7)$, we have $n \geq 2(\delta - k + 3)$ and then

$$\begin{aligned} f(n-\delta+k-4) &= -n^3+4(\delta-k+3)n^2 \\ &\quad -\left(\left(k^2-8k+5\delta+23\right)(\delta-k+2)-5\right)n \\ &\quad +\left(\left(k^2-5k+2\delta+15\right)\right. \\ &\quad \left.\cdot(\delta-k+2)-(2\delta-5)\right)(\delta-k+2)+2 \\ &= -n(n-2(\delta-k+3))^2 \\ &\quad -\left(\left(k^2-4k+\delta+7\right)(\delta-k+2)+1\right) \\ &\quad \cdot(n-2(\delta-k+3)) \\ &\quad -(\delta-k+2)\left(\left(k^2-3k+3\right)\right. \\ &\quad \left.\cdot(\delta-k+3)+k(k-1)\right) \\ &\le -2(3\cdot 3+k(k-1))<0, \end{aligned} \tag{30}$$

$f(0)=2((\delta-k+1)(k(n-\delta-2)-1)+(k-1)(n-\delta-3)) \ge 2(k-1)>0$, $f(-2)=-2(k-2)(\delta-k+2)+2<0$, and $f(-\infty)>0$. Therefore, it is easy to find that the largest root of $f(x)=0$ is in the interval $(n-\delta+k-4, n-\delta+k-3)$, which yields $\rho(G)<n-\delta+k-3$. □

The following lemma gives a sharp upper bound of the spectral radius of connected graphs with given number of edges and minimum degree.

Lemma 1 (see [25]). *Let G be a connected graph with n vertices and m edges. Let $\delta=\delta(G)$ be the minimum degree of G and let $\rho(G)$ be the spectral radius of the adjacency matrix of G. Then,*

$$\rho(G)\le\frac{\delta-1}{2}+\sqrt{2m-\delta n+\frac{(\delta+1)^2}{4}}. \tag{31}$$

Equality holds if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$.

Theorem 4. *Let G be a connected graph of order n and minimum degree $\delta\ge k\ge 3$. If $n\ge(1/2)(\delta-k+2)(k^2-2k+7)$ and*

$$\rho(G)\ge n-\delta+k-3, \tag{32}$$

then G is k-connected, unless $G=K_{k-1}\vee(K_{\delta-k+2}\cup K_{n-\delta-1})$.

Proof. On the contrary, suppose that $\kappa(G)<k$. Since G is connected and $\rho(G)\ge n-\delta+k-3$, by Lemma 1, we have

$$n-\delta+k-3\le\rho(G)\le\frac{\delta-1}{2}+\sqrt{2|E(G)|-\delta n+\frac{(\delta+1)^2}{4}}, \tag{33}$$

which yields

$$|E(G)|\ge\frac{1}{2}n(n-1)-\frac{1}{2}(\delta-k+2)(2n-2\delta+k-3). \tag{34}$$

Since $n\ge(1/2)(\delta-k+2)(k^2-2k+7)$, we obtain $n\ge(1/2)(k+1)(\delta-k+2)+(\delta+2)$. By Theorem 1 (b), G is a subgraph of $K_{k-1}\vee(K_{\delta-k+2}\cup K_{n-\delta-1})$. Since $\rho(G)\ge n-\delta+k-3$, by Theorem 3, $G=K_{k-1}\vee(K_{\delta-k+2}\cup K_{n-\delta-1})$. The proof is completed. □

Remark 2. In Theorem 3.4 in [16], the authors proved that if G is a connected graph of minimum degree $\delta(G)\ge\delta\ge k\ge 3$ and order $n\ge(\delta-k+2)(k^2-2k+4)+3$, and $\rho(G)\ge n-\delta+k-3$, then G is k-connected unless $G=\rho(K_{k-1}\vee(K_{\delta-k+2}\cup K_{n-\delta-1}))$. Obviously, Theorem 4 improves Theorem 3.4 in [16] from the perspective of the restriction on the order of graph.

Another sufficient condition for graphs to be k-connected can be obtained by using the spectral radius of the complement of a graph.

Theorem 5. *Let G be a connected graph of order $n\ge 5$ and minimum degree $\delta\ge k\ge 2$. If*

$$\rho(\overline{G})\le\sqrt{(\delta-k+2)(n-\delta-1)}, \tag{35}$$

then G is k-connected, unless $G=K_{k-1}\vee(K_{\delta-k+2}\cup K_{n-\delta-1})$.

Proof. Let $\kappa=\kappa(G)$. Assume that (35) holds but $1\le\kappa\le k-1$. Let S be an arbitrary minimum vertex-cut of G, and let $X_0, X_1, \ldots, X_{p-1}$ $(p\ge 2)$, denote the vertex-sets of the components of $G-S$, where $|X_0|\le|X_1|\le\cdots\le|X_{p-1}|$. Each vertex in X_i is adjacent to at most $|X_i|-1$ vertices of X_i and $\kappa=|S|$ vertices of S. Thus,

$$\delta|X_i|\le\sum_{x\in X_i}d(x)\le|X_i|\left(|X_i|-1+\kappa\right), \tag{36}$$

and so $|X_i|\ge\delta-\kappa+1$ for each $i=0,1,\ldots,p-1$. Let $Y=\cup_{i=1}^{p-1}X_i$. Then, $\delta-\kappa+1\le|X_0|\le|Y|\le n-\delta-1$ and $|X_0|+|Y|=n-\kappa$. Since there are no edges between X_0 and Y in G, $K_{|X_0|,|Y|}$ is a subgraph of $\overline{G}$. Thus,

$$\begin{aligned} \rho(\overline{G}) &\ge \rho\left(K_{|X_0|,|Y|}\right)=\sqrt{|X_0|\cdot|Y|}=\sqrt{|X_0|\left(n-\kappa-|X_0|\right)} \\ &\ge\sqrt{(\delta-\kappa+1)(n-\delta-1)}\ge\sqrt{(\delta-k+2)(n-\delta-1)}. \end{aligned} \tag{37}$$

By (35), the above inequalities must be equalities. Thus, $|X_0|=\delta-k+2$, $\kappa=k-1$ and $\overline{G}=K_{\delta-k+2,n-\delta-1}$, and so $G=K_{k-1}\vee(K_{\delta-k+2}\cup K_{n-\delta-1})$. □

3. Maximally Connected Graphs

If $\kappa(G)=\delta(G)$, then G is maximally connected. Therefore, by setting $k=\delta$ in Theorem 1, we obtain the following theorem.

Theorem 6. *Let G be a connected graph of order $n\ge 5$, size m, and minimum degree $\delta\ge 2$.*

(a) If $m\ge\binom{n-2}{2}+2\delta-1$, then G is maximally connected, unless $G=K_{\delta-1}\vee(K_2\cup K_{n-\delta-1})$.

(b) *If* $n \geq 2\delta + 3$ *and* $m \geq \binom{n-2}{2} + \delta$, *then* G *is maximally connected, unless* G *is a subgraph of* $K_{\delta-1} \vee (K_2 \cup K_{n-\delta-1})$.

Theorem 7. *Let* G *be a connected graph of order* $n \geq 5$ *and minimum degree* $\delta \geq 2$. *If*

$$\rho(G) \geq \frac{\delta - 1}{2} + \sqrt{(n-\delta-1)(n-4) + \frac{(\delta+1)^2}{4}}, \tag{38}$$

then G *is maximally connected, unless* $G = K_{n-4} \vee (K_2 \cup K_2)$.

Proof. On the contrary, suppose that $\kappa(G) < \delta$. Since G is connected, by (38) and Lemma 1, we have

$$\begin{aligned} &\frac{\delta - 1}{2} + \sqrt{(n-\delta-1)(n-4) + \frac{(\delta+1)^2}{4}} \\ &\quad \leq \rho(G) \leq \frac{\delta - 1}{2} + \sqrt{2|E(G)| - \delta n + \frac{(\delta+1)^2}{4}}, \end{aligned} \tag{39}$$

which yields

$$|E(G)| \geq \binom{n-2}{2} + 2\delta - 1. \tag{40}$$

By Theorem 6 (a), $G = K_{\delta-1} \vee (K_2 \cup K_{n-\delta-1})$. To complete the proof, we only need to show $\delta = n - 3$.

Since $|E(G)| = \binom{n-2}{2} + 2\delta - 1$, the equalities hold in (39). Thus, by Lemma 1, G is either a regular graph or a bidegreed graph in which each vertex is of degree δ or $n - 1$. However, the vertices of G have degrees from the set $\{\delta, n-3, n-1\}$. Therefore, $\delta = n - 3$ and the result follows. □

By setting $k = \delta$ in Theorem 2, we obtain the following result.

Theorem 8. *Let* G *be a connected graph of order* $n \geq 5$ *and minimum degree* $\delta \geq 2$. *If*

$$\rho(G) \geq \rho(K_{\delta-1} \vee (K_2 \cup K_{n-\delta-1})), \tag{41}$$

then G *is maximally connected, unless* $G = K_{\delta-1} \vee (K_2 \cup K_{n-\delta-1})$, *where* $\rho(K_{\delta-1} \vee (K_2 \cup K_{n-\delta-1}))$ *is the largest root of the equation*

$$\lambda^3 - (n-3)\lambda^2 - (2\delta - 1)\lambda + 2\delta(n-\delta-1) - n + 1 = 0. \tag{42}$$

Theorem 9. *Let* G *be a connected graph of order* n *and minimum degree* $\delta \geq 2$. *If* $n \geq \delta^2 - 2\delta + 7$ *and*

$$\rho(G) \geq n - 3, \tag{43}$$

then G *is maximally connected, unless* $G = K_{\delta-1} \vee (K_2 \cup K_{n-\delta-1})$.

Proof. Set $k = \delta$ in the proofs of Theorems 3 and 4. If $\delta \geq 3$, then the result follows from Theorem 4. If $\delta = 2$, then case (a) cannot occur in the proof of Theorems 3. In Theorem 3, by noting that $f(n-3) > 0$, $f(n-4) < 0$, $f(0) > 0$, $f(-\sqrt{3}) = 2\sqrt{3} - 4 < 0$, and $f(-\infty) > 0$, we have $\rho(G) < n - 3$ and so Theorem 3 holds for $\delta = k = 2$. Hence, Theorem 4 also holds for $\delta = k = 2$ and the result follows. □

Remark 3. In the proof of Theorem 3, if we take $k = \delta \geq 2$ and $n = \delta^2 - 2\delta + 6$, then $f(n-3) = g(n) = g(\delta^2 - 2\delta + 6) = 10 - 4\delta < 0$ when $\delta \geq 3$. Notice that $f(+\infty) = +\infty$. So, the largest root of $f(x) = 0$ is greater than $n - 3$ if $\delta \geq 3$, and it follows that $\rho(G) > n - 3$. That is to say, the requirement $n \geq \delta^2 - 2\delta + 7$ in Theorem 9 is best possible when $\delta \geq 3$.

By setting $k = \delta$ in Theorem 5, we have the following result.

Theorem 10. *Let* G *be a connected graph of order* $n \geq 5$ *and minimum degree* $\delta \geq 2$. *If*

$$\rho(\overline{G}) \leq \sqrt{2(n-\delta-1)}, \tag{44}$$

then G *is maximally connected, unless* $G = K_{\delta-1} \vee (K_2 \cup K_{n-\delta-1})$.

4. Super-Connected Graphs

For any connected graph G of order n, if $2 \leq n \leq 4$, then G is super-κ. Therefore, $n \geq 5$ is considered in this section.

Theorem 11. *Let* G *be a connected graph of order* $n \geq 5$, *size* m, *and minimum degree* δ. *If*

$$m \geq \binom{n-2}{2} + 2\delta, \tag{45}$$

then G *is super-*κ, *unless* $G = (K_\delta \vee (K_2 \cup K_{n-\delta-2})) - e$, *where* $e = xy$ *is an edge of* $K_\delta \vee (K_2 \cup K_{n-\delta-2})$ *with* $d(x) = \delta + 1$ *and* $d(y) = n - 1$.

Proof. Since $m \geq \binom{n-2}{2} + 2\delta$, by Theorem 6 (a), $\kappa(G) = \delta$. On the contrary, suppose that G is not super-κ. Let S be an arbitrary minimum vertex-cut with δ vertices, and let $X_0, X_1, \ldots, X_{p-1}$ $(p \geq 2)$ denote the vertex-sets of the components of $G - S$, where $2 \leq |X_0| \leq |X_1| \leq \cdots \leq |X_{p-1}|$. Denote $Y = \cup_{i=1}^{p-1} X_i$. Since $G - S$ is disconnected, there are no edges between X_0 and Y in G, and

$$m \leq \frac{1}{2}n(n-1) - |X_0| \cdot |Y|. \tag{46}$$

Thus, by $|X_0| + |Y| = n - \delta$ and $2 \leq |X_0| \leq |Y| \leq n - \delta - 2$, we have

$$m \le \frac{1}{2}n(n-1) - |X_0| \cdot |Y| \le \frac{1}{2}n(n-1) - 2(n-\delta-2) = \binom{n-2}{2} + 2\delta + 1. \tag{47}$$

If $m = \binom{n-2}{2} + 2\delta + 1$, then all the inequalities in the above proof must be equalities. It is deduced that $p = 2$, $|X_0| = 2$, $|Y| = n-\delta-2$, and $d_G(s) = n-1$ for each $s \in S$, $d_{G[X_0]}(x) = 1$ for each $x \in X_0$, and $d_{G[Y]}(y) = n-\delta-3$ for each $y \in Y$. That is, $G[X_0] = K_2$, $G[S] = K_\delta$, $G[Y] = K_{n-\delta-2}$, and $G \cong K_\delta \vee (K_2 \cup K_{n-\delta-2})$. However, $\delta(G) = \delta + 1 > \delta$, which is a contradiction. Therefore, $m \le \binom{n-2}{2} + 2\delta$. By (45), $m = \binom{n-2}{2} + 2\delta$.

Next, we show that G is a proper subgraph of $K_\delta \vee (K_2 \cup K_{n-\delta-2})$. It suffices to prove that $|X_0| = 2$ and $p = 2$.

If $|X_0| \ge 3$, then $n \ge \delta + 6$. Combining (45) with (46), we obtain

$$\begin{aligned}\binom{n-2}{2} + 2\delta \le m &\le \frac{1}{2}\left(n^2 - n\right) - |X_0| \cdot |Y| \\ &\le \frac{1}{2}\left(n^2 - n\right) - 3(n-\delta-3) \\ &\le \binom{n-2}{2} + 2\delta.\end{aligned} \tag{48}$$

All the above inequalities must be equalities, and so $|X_0| = |Y| = 3$, $n = \delta + 6$, and $G \cong K_\delta \vee (K_3 \cup K_3)$. However, $\delta(G) = \delta + 2 > \delta$, which is a contradiction. Therefore, $|X_0| = 2$.

If $p \ge 3$, then $n \ge \delta + 6$. Let $Y_1 = \cup_{i=2}^{p-1} X_i$. Then $|X_1| + |Y_1| = n-\delta-2$ and $2 = |X_0| \le |X_1| \le |Y_1| \le n-\delta-4$. Since $G - S$ is disconnected, there are no edges among X_0, X_1, and Y_1 in G (i.e., $[X_0, X_1] = \varnothing$, $[X_0, Y_1] = \varnothing$, $[X_1, Y_1] = \varnothing$), and

$$\begin{aligned}m &\le \frac{1}{2}\left(n^2 - n\right) - |X_0| \cdot \left(|X_1| + |Y_1|\right) - |X_1||Y_1| \\ &\le \frac{1}{2}\left(n^2 - n\right) - 2(n-\delta-2) - 2 \cdot 2 \\ &= \binom{n-2}{2} + 2\delta - 3 < \binom{n-2}{2} + 2\delta,\end{aligned} \tag{49}$$

which is a contradiction. Therefore, $p = 2$.

Let $H = K_\delta \vee (K_2 \cup K_{n-\delta-2})$. Then $G \subset H$ and $|E(G)| = |E(H)| - 1$. Therefore, $G = H - e$. Since $\delta(H) = \delta + 1$ and $\delta(G) = \delta$, $e = xy$ is an edge of H with $d(x) = \delta + 1$ and $d(y) = n - 1$. □

Theorem 12. *Let G be a connected graph of order $n \ge 5$ and minimum degree δ. If*

$$\rho(G) \ge \frac{\delta - 1}{2} + \sqrt{2 + (n-\delta-1)(n-4) + \frac{(\delta+1)^2}{4}}, \tag{50}$$

then G is super-κ.

Proof. On the contrary, suppose that G is not super-κ. Since G is connected, by (50) and Lemma 1, we have

$$\begin{aligned}&\sqrt{2 + (n-\delta-1)(n-4) + \frac{(\delta+1)^2}{4}} \\ &\quad \le \rho(G) - \frac{\delta-1}{2} \le \sqrt{2|E(G)| - \delta n + \frac{(\delta+1)^2}{4}},\end{aligned} \tag{51}$$

which yields

$$|E(G)| \ge \binom{n-2}{2} + 2\delta. \tag{52}$$

By Theorem 11, $G = K_\delta \vee (K_2 \cup K_{n-\delta-2}) - e$, where $e = xy$ is an edge of $K_\delta \vee (K_2 \cup K_{n-\delta-2})$ with $d(x) = \delta + 1$ and $d(y) = n - 1$.

Since $|E(G)| = \binom{n-2}{2} + 2\delta$, the equalities hold in (51). Thus, by Lemma 1, G is either a regular graph or a bidegreed graph in which each vertex is of degree δ or $n - 1$. However, the vertices of G have degree from the set $\{\delta, \delta+1, n-3, n-2, n-1\}$. Thus, G cannot be a bidegreed graph, which yields a contradiction. Hence, G is super-κ. □

Theorem 13. *Let G be a connected graph of order $n \ge 5$ and minimum degree δ. If*

$$\rho(G) \ge \rho\left(K_\delta \vee \left(K_2 \cup K_{n-\delta-2}\right)\right), \tag{53}$$

then G is super-κ, where $\rho(K_\delta \vee (K_2 \cup K_{n-\delta-2}))$ is the largest root of the equation

$$\lambda^3 - (n-3)\lambda^2 - (2\delta+1)\lambda + 2(\delta+1)(n-\delta-2) - n + 1 = 0. \tag{54}$$

Proof. On the contrary, suppose that G is not super-κ. Let S be an arbitrary minimum vertex-cut with $\kappa(\le \delta)$ vertices, and let $X_0, X_1, \ldots, X_{p-1}$ $(p \ge 2)$ denote the vertex-sets of the components of $G - S$, where $2 \le |X_0| \le |X_1| \le \cdots \le |X_{p-1}|$. Denote $Y = \cup_{i=1}^{p-1} X_i$. Then $2 \le |X_0| \le |Y| \le n - \kappa - 2$ and $|X_0| + |Y| = n - \kappa$. Since there are no edges between X_0 and Y in G, G is a subgraph of $K_\kappa \vee (K_{|X_0|} \cup K_{|Y|})$ and $\rho(G) \le \rho(K_\kappa \vee (K_{|X_0|} \cup K_{|Y|}))$.

According to (15) in the proof of Theorem 2, $\rho(K_\kappa \vee (K_{|X_0|} \cup K_{|Y|}))$ is the largest root of the equation

$$f\left(\lambda;\left|X_0\right|,|Y|,\kappa\right) := \lambda^3-(n-3)\lambda^2 +\left(\left|X_0\right||Y|-2n+3\right)\lambda +\left|X_0\right||Y|(\kappa+1)-n+1=0. \tag{55}$$

Then, we have

$$f\left(\lambda;\left|X_0\right|,|Y|,\kappa\right)-f(\lambda;2,n-\kappa-2,\kappa) =(\lambda+\kappa+1)\left(\left|X_0\right||Y|-2(n-\kappa-2)\right)\geq 0, \tag{56}$$

for any $\lambda>0$ and $|Y|\geq|X_0|\geq 2$. Therefore, $\rho(K_\kappa\vee(K_{|X_0|}\cup K_{|Y|}))\leq\rho(K_\kappa\vee(K_2\cup K_{n-\kappa-2}))$.

Since $K_\kappa\vee(K_2\cup K_{n-\kappa-2})$ is a subgraph of $K_\delta\vee(K_2\cup K_{n-\delta-2})$ for any $\kappa\leq\delta$, we get

$$\rho\left(K_\kappa\vee\left(K_2\cup K_{n-\kappa-2}\right)\right)\leq\rho\left(K_\delta\vee\left(K_2\cup K_{n-\delta-2}\right)\right). \tag{57}$$

Hence, from the above discussion, we obtain

$$\rho(G)\leq\rho\left(K_\kappa\vee\left(K_{|X_0|}\cup K_{|Y|}\right)\right) \leq\rho\left(K_\kappa\vee\left(K_2\cup K_{n-\kappa-2}\right)\right)\leq\rho\left(K_\delta\vee\left(K_2\cup K_{n-\delta-2}\right)\right). \tag{58}$$

By (53), the above inequalities must be equalities. Thus, $|X_0|=2$, $\kappa=\delta$, $|Y|=n-\delta-2$, and $G=K_\delta\vee(K_2\cup K_{n-\delta-2})$. However, $\delta(G)=\delta+1>\delta$, which is a contradiction. The result follows. □

Theorem 14. *Let G be a connected graph of order $n\geq 5$ and minimum degree δ. If*

$$\rho(\overline{G})\leq\sqrt{2(n-\delta-2)}, \tag{59}$$

then G is super-κ.

Proof. Assume that (59) holds but G is not super-κ. Let S be an arbitrary minimum vertex-cut of G with $\kappa(\leq\delta)$ vertices, and let $X_0,X_1,\ldots,X_{p-1}$ $(p\geq 2)$, denote the vertex-sets of the components of $G-S$, where $2\leq|X_0|\leq|X_1|\leq\cdots\leq|X_{p-1}|$. Denote $Y=\cup_{i=1}^{p-1}X_i$. Then $2\leq|X_0|\leq|Y|\leq n-\kappa-2$ and $|X_0|+|Y|=n-\kappa$. Since there are no edges between X_0 and Y in G, $K_{|X_0|,|Y|}$ is a subgraph of $\overline{G}$. Thus,

$$\rho(\overline{G})\geq\rho\left(K_{|X_0|,|Y|}\right)=\sqrt{\left|X_0\right|\cdot|Y|}=\sqrt{\left|X_0\right|\left(n-\kappa-\left|X_0\right|\right)} \geq\sqrt{2(n-\kappa-2)}\geq\sqrt{2(n-\delta-2)}. \tag{60}$$

By (59), the above inequalities must be equalities. Thus, $|X_0|=2$, $\kappa=\delta$, and $\overline{G}=K_{2,n-\delta-2}$, and so $G=K_\delta\vee(K_2\cup K_{n-\delta-2})$. However, $\delta(G)=\delta(K_\delta\vee(K_2\cup K_{n-\delta-2}))=\delta+1>\delta$, a contradiction. This completes the proof. □

5. Sufficient Conditions for Triangle-Free Graphs

Let us extend an interesting result by applying the famous theorem of Mantel [26] and Turán [27].

Theorem 15 (see [26, 27]). *For any triangle-free graph G of order n, we have $|E(G)|\leq\lfloor(1/4)n^2\rfloor$, with equality if and only if $G=K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$.*

Theorem 16. *Let G be a connected triangle-free graph of order n, size m, and minimum degree $\delta\geq k\geq 2$. If*

$$m\geq\delta^2+\left\lfloor\frac{1}{4}(n-2\delta+k-1)^2\right\rfloor, \tag{61}$$

then G is k-connected, unless $V(G)=X\cup S\cup Y$ and S is a minimum vertex-cut of G with $G[S]=\overline{K_{k-1}}$, $G[X\cup S]=K_{\delta,\delta}$, and $G[Y\cup S]=K_{\lfloor(n-2\delta+k-1)/2\rfloor,\lceil(n-2\delta+k-1)/2\rceil}$.

Proof. Let $\kappa=\kappa(G)$. On the contrary, suppose that $\kappa\leq k-1$. Let S be a minimum vertex-cut of G, and let $X,Y_1,\ldots,Y_{p-1}$ $(p\geq 2)$ denote the vertex-sets of the components of $G-S$, where $|X|\leq|Y_1|\leq\cdots\leq|Y_{p-1}|$. Set $Y=\cup_{i=1}^{p-1}Y_i$. Then $|X|\leq|Y|$ and $|X|+|Y|=n-\kappa$. By Theorem 15, we deduce that

$$|E(G[X\cup S])|\leq\left\lfloor\frac{(|X|+|S|)^2}{4}\right\rfloor, \qquad |E(G[Y\cup S])|\leq\left\lfloor\frac{(|Y|+|S|)^2}{4}\right\rfloor, \tag{62}$$

with equalities if and only if

$$G[X\cup S]=K_{\lfloor(|X|+|S|)/2\rfloor,\lceil(|X|+|S|)/2\rceil}, \qquad G[Y\cup S]=K_{\lfloor(|Y|+|S|)/2\rfloor,\lceil(|Y|+|S|)/2\rceil}. \tag{63}$$

If $x\in X$, then $\delta\leq d_G(x)\leq|X|+|S|$. The assumption $\kappa\leq k-1\leq\delta-1$ implies that x has at least one neighbor $y\in X$. Since G is triangle-free, we deduce that $N_G(x)\cap N_G(y)=\varnothing$, where $N_G(x)$ is the neighbor set of x. As $N_G(x)\cup N_G(y)\subseteq X\cup S$, it follows that

$$|X|+|S|=|X\cup S|\geq\left|N_G(x)\cup N_G(y)\right| =\left|N_G(x)\right|+\left|N_G(y)\right|\geq 2\delta, \tag{64}$$

and thus $|X|\geq 2\delta-|S|=2\delta-\kappa$. Therefore, we arrive at

$$2\delta-\kappa\leq|X|\leq|Y|\leq n-2\delta. \tag{65}$$

Together with $|X|+|Y|=n-\kappa$ and (62), it leads to

$$\begin{aligned} m &= |E(G[X\cup S])| + |E(G[Y\cup S])| - |E(G[S])| \\ &\le \lfloor \frac{1}{4}(|X|+|S|)^2 \rfloor + \lfloor \frac{1}{4}(|Y|+|S|)^2 \rfloor - |E(G[S])| \ (\text{by } (62)) \\ &\le \lfloor \frac{1}{4}(|X|+|S|)^2 + \frac{1}{4}(|Y|+|S|)^2 \rfloor \\ &= \lfloor \frac{(|X|+|Y|+|S|)^2+|S|^2}{4} - \frac{|X|\cdot|Y|}{2} \rfloor \\ &\le \lfloor \frac{n^2+\kappa^2}{4} - \frac{(2\delta-\kappa)(n-2\delta)}{2} \rfloor \ (\text{by } (65)) \\ &= \delta^2 + \lfloor \frac{1}{4}(n-2\delta+\kappa)^2 \rfloor \\ &\le \delta^2 + \lfloor \frac{1}{4}(n-2\delta+k-1)^2 \rfloor, \quad (\text{by } \kappa \le k-1). \end{aligned} \tag{66}$$

Combining this with (61), we have $m = \delta^2 + \left\lfloor (1/4)(n-2\delta+k-1)^2 \right\rfloor$, and so $|S| = \kappa = k-1$, $|X| = 2\delta - k + 1$, $|Y| = n - 2\delta$, $|E(G[S])| = 0$, $|E(G[X\cup S])| = \delta^2$, and $|E(G[Y\cup S])| = \left\lfloor (1/4)(n-2\delta+k-1)^2 \right\rfloor$. Therefore, $G[S] = \overline{K_{k-1}}$, $G[X\cup S] = K_{\delta,\delta}$, and $G[Y\cup S] = K_{\lfloor (n-2\delta+k-1)/2 \rfloor, \lceil (n-2\delta+k-1)/2 \rceil}$. This completes the proof. □

Theorem 17. *Let G be a connected triangle-free graph of order n and minimum degree $\delta \ge k \ge 2$. If*

$$\rho(G) \ge \frac{\delta-1}{2} + \sqrt{2\lfloor \frac{1}{4}(n-2\delta+k-1)^2 \rfloor - \delta(n-2\delta) + \frac{(\delta+1)^2}{4}}, \tag{67}$$

then G is k-connected.

Proof. On the contrary, suppose that $\kappa(G) < k$. Since G is connected, by (67) and Lemma 1, we have

$$\begin{aligned} &\sqrt{2\lfloor \frac{1}{4}(n-2\delta+k-1)^2 \rfloor - \delta(n-2\delta) + \frac{(\delta+1)^2}{4}} \\ &\le \rho(G) - \frac{\delta-1}{2} \le \sqrt{2|E(G)| - \delta n + \frac{(\delta+1)^2}{4}}, \end{aligned} \tag{68}$$

which yields

$$|E(G)| \ge \delta^2 + \lfloor \frac{1}{4}(n-2\delta+k-1)^2 \rfloor. \tag{69}$$

By Theorem 16, $V(G) = X\cup S\cup Y$ and S is a minimum vertex-cut of G with $G[S] = \overline{K_{k-1}}$, $G[X\cup S] = K_{\delta,\delta}$, and $G[Y\cup S] = K_{\lfloor (n-2\delta+k-1)/2 \rfloor, \lceil (n-2\delta+k-1)/2 \rceil}$.

Since $|E(G)| = \delta^2 + \left\lfloor (1/4)(n-2\delta+k-1)^2 \right\rfloor$, the equalities hold in (68). Thus, by Lemma 1, G is either a regular graph or a bidegreed graph in which each vertex is of degree δ or $n-1$. However, G is neither a regular graph nor the complete bipartite graph $K_{1,n-1}$, which yields a contradiction. Hence, G is k-connected.

By setting $k = \delta$ in Theorems 16 and 17, we obtain the two following theorems. □

Theorem 18. *Let G be a connected triangle-free graph of order n, size m, and minimum degree $\delta \ge 2$. If*

$$m \ge \delta^2 + \lfloor \frac{1}{4}(n-\delta-1)^2 \rfloor, \tag{70}$$

then G is maximally connected, unless $V(G) = X\cup S\cup Y$, and S is a minimum vertex-cut of G with $G[S] = \overline{K_{\delta-1}}$, $G[X\cup S] = K_{\delta,\delta}$, and $G[Y\cup S] = K_{\lfloor (n-\delta-1)/2 \rfloor, \lceil (n-\delta-1)/2 \rceil}$.

Theorem 19. *Let G be a connected triangle-free graph of order n and minimum degree $\delta \ge 2$. If*

$$\rho(G) \ge \frac{\delta-1}{2} + \sqrt{2\lfloor \frac{1}{4}(n-\delta-1)^2 \rfloor - \delta(n-2\delta) + \frac{(\delta+1)^2}{4}}, \tag{71}$$

then G is maximally connected.

For super-connected graphs, we have the following results.

Theorem 20. *Let G be a connected triangle-free graph of order n, size m, and minimum degree $\delta \ge 2$. If*

$$m \ge \delta^2 + \lfloor \frac{1}{4}(n-\delta)^2 \rfloor, \tag{72}$$

then G is super-κ.

Proof. Let $\kappa = \kappa(G)$. On the contrary, suppose that G is not super-κ. Since $m \ge \delta^2 + \left\lfloor (1/4)(n-\delta)^2 \right\rfloor$, by Theorem 18, $\kappa = \delta$. Let S be a minimum vertex-cut of G with δ vertices, and let $X, Y_1, \ldots, Y_{p-1}$ $(p \ge 2)$ denote the vertex-sets of the components of $G - S$, where $2 \le |X| \le |Y_1| \le \cdots \le |Y_{p-1}|$. Set $Y = \cup_{i=1}^{p-1} Y_i$; then $|Y| \ge |X| \ge 2$. Therefore, with the same proceeding of the proof of Theorem 16 (from (62) to (65)), we arrive at

$$\delta \le |X| \le |Y| \le n - 2\delta. \tag{73}$$

Together with $|X| + |Y| = n - \delta$ and (62), it leads to

$$\begin{aligned} m &= |E(G[X\cup S])| + |E(G[Y\cup S])| - |E(G[S])| \\ &\le \lfloor \frac{1}{4}(|X|+|S|)^2 \rfloor + \lfloor \frac{1}{4}(|Y|+|S|)^2 \rfloor - |E(G[S])| \\ &\le \lfloor \frac{1}{4}(|X|+|S|)^2 + \frac{1}{4}(|Y|+|S|)^2 \rfloor \\ &= \lfloor \frac{(|X|+|Y|+|S|)^2+|S|^2}{4} - \frac{|X|\cdot|Y|}{2} \rfloor \\ &\le \lfloor \frac{n^2+\delta^2}{4} - \frac{\delta(n-2\delta)}{2} \rfloor \\ &= \delta^2 + \lfloor \frac{1}{4}(n-\delta)^2 \rfloor. \end{aligned} \tag{74}$$

Combining this with (72), we have $m = \delta^2 + \lfloor (1/4)(n-\delta)^2 \rfloor$, and so $|X| = |S| = \delta$, $|Y| = n - 2\delta$, $|E(G[S])| = 0$, $|E(G[X \cup S])| = \delta^2$, and $|E(G[Y \cup S])| = \lfloor (1/4)(n-\delta)^2 \rfloor$. Therefore, $G[S] = \overline{K_\delta}$, $G[X \cup S] = K_{\delta,\delta}$, and $G[Y \cup S] = K_{\lfloor (n-\delta)/2 \rfloor, \lceil (n-\delta)/2 \rceil}$. Thus, $G[X] = \overline{K_\delta}$, which contradicts the fact that $G[X]$ is a component of G with at least two vertices. The result follows. □

Theorem 21. *Let G be a connected triangle-free graph of order n and minimum degree $\delta \geq 2$. If*

$$\rho(G) \geq \frac{\delta - 1}{2} + \sqrt{2\lfloor \frac{1}{4}(n-\delta)^2 \rfloor - \delta(n - 2\delta) + \frac{(\delta+1)^2}{4}}, \tag{75}$$

then G is super-κ.

Proof. Since G is connected, by (75) and Lemma 1, we have

$$\begin{aligned} &\sqrt{2\lfloor \frac{1}{4}(n-\delta)^2 \rfloor - \delta(n - 2\delta) + \frac{(\delta+1)^2}{4}} \\ &\leq \rho(G) - \frac{\delta - 1}{2} \leq \sqrt{2|E(G)| - \delta n + \frac{(\delta+1)^2}{4}}, \end{aligned} \tag{76}$$

which yields

$$|E(G)| \geq \delta^2 + \lfloor \frac{1}{4}(n-\delta)^2 \rfloor. \tag{77}$$

By Theorem 20, G is super-κ. □

Remark 4. The lower bound on m given in Theorem 20 is sharp. For example, let $n = 3\delta + 3$, $V(G) = X \cup S \cup Y$, $G[X] = K_{1,\delta}$, $G[Y] = K_{1,\delta+1}$, and S is a minimum vertex-cut of G with $G[S] = \overline{K_\delta}$, $G[X \cup S] = K_{\delta,\delta+1}$, and $G[Y \cup S] = K_{\delta+1,\delta+1}$. It is easy to check that

$$|E(G)| = \delta(\delta+1) + (\delta+1)^2 = \delta^2 + \lfloor \frac{1}{4}(n-\delta)^2 \rfloor - 1. \tag{78}$$

However, $G - S = K_{1,\delta} \cup K_{1,\delta+1}$, which yields that G is not super-connected.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare no conflicts of interest.

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