

Research Article

Stability Switches and Hopf Bifurcation Induced by Nutrient Recycling Delay in a Reaction-Diffusion Nutrient-Phytoplankton Model

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In this paper, we study a reaction-diffusion nutrient-phytoplankton model with nutrient recycling delay and toxin-producing phytoplankton. For the delay-free case, we prove that the release rate of toxic chemicals by the toxin-producing phytoplankton can destabilize the system and cause periodic oscillation. For the time-delayed case, we observe that nutrient recycling delay may bring about stability switches and Hopf bifurcation. Moreover, we derive the formula for determining the direction and stability of the bifurcating periodic solutions. Finally, we give some numerical simulations to support the theoretical analyses.

1. Introduction

Aquatic ecosystem has a significant effect on human's living environment. It not only plays an important role in maintaining the global biological matter and water cycles but also undertakes the functions of water source, power source, transportation pollution purification place, and so on. Obviously, it is of great importance to investigate the dynamic behaviors of planktonic ecosystem. In the last few decades, lots of mathematical models have been constructed in terms of differential equations to describe the dynamical interactions between the dissolved nutrient and plankton population, see [1–5] and references cited therein. These models can deepen our knowledge of aquatic ecosystem cycle. Some interesting dynamical behaviors have been investigated, such as uniform persistence, asymptotic stability, Hopf bifurcation, Turing instability, and travelling waves.

In nature, the dead phytoplankton can be partly transformed into the nutrient. By taking into account the

recycling factor, Chakraborty et al. [6] proposed the following nutrient-phytoplankton model:

$$\begin{cases} \frac{dN}{dt} = a - bNP - dN + \beta P, \\ \frac{dP}{dt} = cNP - \alpha P - \frac{\theta P^2}{H^2 + P^2}, \end{cases} \quad (1)$$

where N and P denote the concentrations of nutrient and phytoplankton population, respectively. All the coefficients are positive constants, a is the constant nutrient input flow, b is the nutrient uptake rate of phytoplankton, c is the nutrient conversion rate of phytoplankton, d is the per capita-loss rate of nutrient, α is the per capita-mortality rate of phytoplankton, β is the portion of the phytoplankton recycled back to the nutrient concentration, θ is the release rate of toxic chemicals by the toxin-producing phytoplankton population, and H is the half-saturation constant. In view of

the conversion efficiency in ecological system, we always assume that $\alpha > \beta$ and $b > c$ throughout the paper.

For system (1), the limit cycle behavior is obtained by numerical simulations in [6]. It is found that inclusion of nutrient recycling enhances the chances of recurring bloom dynamics. After that, some expanded models were investigated by incorporating zooplankton population in [7–11].

As indicated in [12], no life is possible if spatial and temporal processes do not interact. Actually, the individuals can go anywhere in the spatial domain since a well-known fact about Fickian diffusion leading to a Laplacian term. In

the lakes or oceans, dissolved nutrient and phytoplankton population can move with the tide. The spatial diffusion has been considered in modelling the biological and ecological systems, such as [13–15]. Reaction-diffusion equations are widely used for modelling chemical reactions, biological systems, population dynamics, and nuclear reactor physics. This kind of models can better describe the spatial and temporal interactions. So, it is more realistic to use the reaction-diffusion equations to model the aquatic system. Then, we have the following model with spatial diffusion:

$$\left\{ \begin{array}{l} \frac{\partial N(x,t)}{\partial t} = d_1 \Delta N(x,t) + a - bN(x,t)P(x,t) - dN(x,t) + \beta P(x,t), \quad x \in \Omega, t > 0, \\ \frac{\partial P(x,t)}{\partial t} = d_2 \Delta P(x,t) + cN(x,t)P(x,t) - \alpha P(x,t) - \frac{\theta P^2(x,t)}{H^2 + P^2(x,t)}, \quad x \in \Omega, t > 0, \\ \frac{\partial N(x,t)}{\partial \nu} = \frac{\partial P(x,t)}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \\ N(x,0) = N_0(x) \geq 0, \quad P(x,0) = P_0(x) \geq 0, \quad x \in \bar{\Omega}, \end{array} \right. \quad (2)$$

where d_1 and d_2 are the diffusion coefficients of nutrient and phytoplankton, respectively. The initial functions $N_0(x)$ and $P_0(x)$ are nonnegative and nonidentical to zero. The homogeneous Neumann boundary conditions mean that the nutrient and phytoplankton do not leave the living region. For simplicity, we specify the spatial location as an interval $\Omega = (0, \pi)$, where π denotes the depth of the water column and can be changed into any depth through a suitable transformation. Moreover, some other nutrient-plankton models incorporating spatial effects are investigated in [16–21].

In models (1) and (2), it is assumed that the decomposition process of dead phytoplankton is instantaneous. However, it is not accurate enough. Usually, the interaction between nutrient and phytoplankton is not instantaneous and it does take time to transform dead phytoplankton into nutrient. Time delay can also bring rich dynamical behaviors, especially the stability switches, and hence it cannot be ignored [22–24]. Therefore, we further build the following model with nutrient recycling delay motivated by [25, 26]:

$$\left\{ \begin{array}{l} \frac{\partial N(x,t)}{\partial t} = d_1 \Delta N(x,t) + a - bN(x,t)P(x,t) - dN(x,t) + \beta P(x,t - \tau), \quad x \in \Omega, t > 0, \\ \frac{\partial P(x,t)}{\partial t} = d_2 \Delta P(x,t) + cN(x,t)P(x,t) - \alpha P(x,t) - \frac{\theta P^2(x,t)}{H^2 + P^2(x,t)}, \quad x \in \Omega, t > 0, \\ \frac{\partial N(x,t)}{\partial \nu} = \frac{\partial P(x,t)}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \\ N(x,t) = N_1(x,t) \geq 0, \quad P(x,t) = P_1(x,t) \geq 0, \quad x \in \bar{\Omega}, t \in [-\tau, 0], \end{array} \right. \quad (3)$$

where the nutrient recycling delay $\tau > 0$ is the decomposition time of dead phytoplankton population, $(N(x, t), P(x, t)) \in C(\bar{\Omega} \times [-\tau, \infty), \mathbb{R}_+^2)$. The initial functions $N_1(x, t)$ and $P_1(x, t)$ are nonnegative and nonidentical to zero. The aim of this paper is to investigate the joint effects of spatial diffusion and time delay on the spatiotemporal dynamics of the nutrient-phytoplankton system.

The highlights of this paper mainly contain two aspects. First, we synthetically consider the nutrient recycling delay and spatial diffusion in modelling the interactions between nutrient concentration and phytoplankton population, and the model is more closer to the real world. Second, by regarding nutrient recycling delay as the bifurcation parameter, we find that time delay may induce stability switches and sustained oscillations, which can help to understand the cyclical burst of algae bloom.

The rest of this paper is organized as follows. In Section 2, we establish the existence of nonnegative constant equilibria. In Section 3, we analyze the stability of positive equilibrium in the absence of time delay, the stability switches, and Hopf bifurcation in the presence of time delay, respectively. In Section 4, the algorithm for determining the Hopf bifurcating periodic solutions is derived. In Section 5, some numerical simulations are presented to verify the theoretical results. Finally, a brief conclusion is given in Section 6.

2. Existence of Equilibria

In this section, we give some preliminary results. For convenience, we make the following assumptions:

$$(H1) \quad b\theta > ac - \alpha d > 0.$$

$$(H2) \quad T_0 < 0, D_0 > 0.$$

$$(H3) \quad T_0 = 0 \text{ and } P^* - H \neq 0.$$

$$(H4) \quad d_2(bP^* + d) - d_1\theta P^*(P^{*2} - H^2)/(H^2 + P^{*2})^2 > 0.$$

$$(H5) \quad d_1\theta P^*(P^{*2} - H^2)/(H^2 + P^{*2})^2 - d_2(bP^* + d) > 2d_1d_2D_0.$$

$$(H6) \quad T_n^4 - 4A_nT_n^2 + 4B^2 > 0 \text{ and } T_n^2 - 2A_n < 0 \text{ for some } n \in \mathbb{N}_0.$$

Spatial diffusion and time delay do not affect the number and location of differential system, so systems (1)–(3) have the same nonnegative constant equilibria and the equilibria are the roots of the following algebraic equation:

$$\begin{pmatrix} \frac{\partial N(x, t)}{\partial t} \\ \frac{\partial P(x, t)}{\partial t} \end{pmatrix} = D \begin{pmatrix} \Delta N(x, t) \\ \Delta P(x, t) \end{pmatrix} + L_1 \begin{pmatrix} N(x, t) \\ P(x, t) \end{pmatrix} + L_2 \begin{pmatrix} N(x, t - \tau) \\ P(x, t - \tau) \end{pmatrix}, \quad (8)$$

where

$$\begin{cases} a - bNP - dN + \beta P = 0, \\ cNP - \alpha P - \frac{\theta P^2}{H^2 + P^2} = 0. \end{cases} \quad (4)$$

Obviously, there is always the boundary equilibrium $E_0(a/d, 0)$. Next, we establish the existence of positive constant equilibrium. From the second equation of (4), we have

$$N = \frac{\alpha}{c} + \frac{\theta P}{c(H^2 + P^2)}. \quad (5)$$

Substituting it into the first equation of (4) leads to

$$a + \beta P = (d + bP) \left[\frac{\alpha}{c} + \frac{\theta P}{c(H^2 + P^2)} \right], \quad (6)$$

that is,

$$a_3P^3 + a_2P^2 + a_1P + a_0 = 0, \quad (7)$$

where $a_3 = \alpha b - \beta c > 0$, $a_2 = \alpha d + b\theta - ac$, $a_1 = d\theta + \alpha bH^2 - \beta cH^2 > 0$, $a_0 = H^2(\alpha d - ac)$. Based on the Descartes' rule of signs, equation (7) has one or three positive roots and assumption (H1) can guarantee the existence of unique positive real root P^* of equation (7).

Thus, we can establish the existence of positive equilibrium.

Lemma 1. *If (H1) holds, then systems (1)–(3) have the unique positive equilibrium $E^*(N^*, P^*)$.*

Remark 1. From Lemma 1, if the nutrient uptake rate of phytoplankton and release rate of toxic chemicals by toxin-producing phytoplankton population are appropriately large, and the net growth rates of nutrient and phytoplankton are appropriately small, then the unique positive equilibrium of system may exist.

3. Stability Analysis and Occurrence of Hopf Bifurcation

In the following, we analyze the stability of E^* . Linearizing system (1) at E^* , we have

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} -bP^* - d & -bN^* \\ cP^* & \frac{\theta P^*(P^{*2} - H^2)}{(H^2 + P^{*2})^2} \end{pmatrix}, \quad (9)$$

$$L_2 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}.$$

Then, the characteristic equation of (8) is

$$\det(\lambda I - M_n - L_1 - L_2 e^{-\lambda\tau}) = 0, \quad (10)$$

where $I = \text{diag}\{1, 1\}$, $M_n = -n^2 \text{diag}\{1, 1\}$, $n \in \mathbb{N}_0$. Equation (10) can be turned into

$$\lambda^2 - T_n \lambda + A_n + B e^{-\lambda\tau} = 0, \quad (11)$$

where

$$T_n = -(d_1 + d_2)n^2 + \frac{\theta P^*(P^{*2} - H^2)}{(H^2 + P^{*2})^2} - bP^* - d,$$

$$A_n = d_1 d_2 n^4 + \left[d_2 (bP^* + d) - d_1 \frac{\theta P^*(P^{*2} - H^2)}{(H^2 + P^{*2})^2} \right] n^2 + bcN^* P^* - \frac{\theta P^*(P^{*2} - H^2)(bP^* + d)}{(H^2 + P^{*2})^2}, \quad (12)$$

$$B = -\beta cN^*.$$

3.1. The Case of $\tau = 0$. If $\tau = 0$ and $n = 0$, then characteristic equation (11) can be simplified to

$$\lambda^2 - T_0 \lambda + D_0 = 0, \quad (13)$$

which corresponds to ordinary differential system (1) and $D_0 = A_0 + B$. By Routh–Hurwitz criterion, the positive equilibrium E^* of (1) is asymptotically stable if the condition (H2) is satisfied.

Next, we explore the influence of parameter θ on the stability of E^* . Let $T_0 = 0$, then $\theta^* = (bP^* + d)(H^2 + P^{*2})^2 / P^*(P^{*2} - H^2)$ is the potential Hopf bifurcation value. In addition, we have

$$\frac{d\text{Re}\lambda(\theta)}{d\theta} \Big|_{\theta=\theta^*} = \frac{P^*(P^{*2} - H^2)}{2(H^2 + P^{*2})^2}. \quad (14)$$

From above, we can conclude the stability results of system (1).

Theorem 1

- (i) If (H1) and (H2) are satisfied, then the positive equilibrium E^* of (1) is asymptotically stable.
- (ii) If (H1) and (H3) are satisfied, then the positive equilibrium E^* of (1) is asymptotically stable when $\theta < \theta^*$ and unstable when $\theta > \theta^*$ for $P^* > H$. Conversely, E^* of (1) is asymptotically stable when $\theta > \theta^*$

and unstable when $\theta < \theta^*$ for $P^* < H$. Moreover, θ^* is the Hopf bifurcation value.

Remark 2. From Theorem 1, if the equilibrium state of phytoplankton P^* is smaller than the half-saturation constant, then condition (H2) can easily be satisfied and the nutrient and phytoplankton can achieve a balanced coexistence in the absence of diffusion. Otherwise, the release rate of toxic chemicals may induce the periodic variations of nutrient and phytoplankton.

Here, we mainly investigate the Turing instability induced by spatial diffusion. The linearization of system (2) at the positive equilibrium E^* is

$$\begin{cases} \frac{\partial N}{\partial t} = d_1 \Delta N - (bP^* + d)N + (\beta - bN^*)P, \\ \frac{\partial P}{\partial t} = d_2 \Delta P + cP^* N + \frac{\theta P^*(P^{*2} - H^2)}{(H^2 + P^{*2})^2} P, \end{cases} \quad (15)$$

and the corresponding characteristic equation is

$$\lambda^2 - T_n \lambda + D_n = 0, \quad (16)$$

where

$$T_n = -(d_1 + d_2)n^2 + \frac{\theta P^*(P^{*2} - H^2)}{(H^2 + P^{*2})^2} - bP^* - d, \quad (17)$$

$$D_n = d_1 d_2 n^4 + \left[d_2 (bP^* + d) - d_1 \frac{\theta P^*(P^{*2} - H^2)}{(H^2 + P^{*2})^2} \right] n^2 + cP^* (bN^* - \beta) - \frac{\theta P^*(P^{*2} - H^2)(bP^* + d)}{(H^2 + P^{*2})^2}.$$

It is obvious that $T_n < 0$ when $T_0 < 0$. Then, the positive equilibrium E^* of diffusive system (2) is stable if $D_n > 0$ for any n and unstable if $D_n < 0$ for some $n > 0$. It is not difficult to determine the sign of D_n by using the properties of parabola.

Following Turing's idea in [27], different diffusion rates of two chemical reactants could sometimes destabilize an otherwise stable equilibrium of the reaction-diffusion system and lead to nonuniform spatial patterns. This kind of diffusion-driven instability is usually called Turing instability.

Theorem 2. *Suppose (H1) and (H2) hold, we have the following statements:*

- (i) *If (H4) is satisfied, then the positive equilibrium E^* of (2) is locally asymptotically stable.*
- (ii) *If (H5) is satisfied, then the positive equilibrium E^* of (2) is unstable. It implies that Turing instability occurs.*

Remark 3. From Theorem 2, the diffusion coefficients have significant effect on the stability of positive equilibrium in the presence of spatial diffusion. If the diffusion rate of phytoplankton is larger, then the aquatic system is stable. On the contrary, the system may be unstable.

3.2. *The Case of $\tau > 0$.* For $\tau > 0$, let $\lambda = i\omega$ ($\omega > 0$) be a root of (11). We have

$$-\omega^2 - i\omega T_n + A_n + B(\cos \omega\tau - i \sin \omega\tau) = 0. \quad (18)$$

Separating the real and imaginary parts can lead to

$$\begin{cases} B \cos \omega\tau = \omega^2 - A_n, \\ B \sin \omega\tau = -\omega T_n. \end{cases} \quad (19)$$

Squaring both sides of the above equations, we get

$$\omega^4 + (T_n^2 - 2A_n)\omega^2 + A_n^2 - B^2 = 0. \quad (20)$$

From $D_n = A_n + B$ and $B < 0$, it can be found that $D_n > 0$ implies $A_n - B > 0$. Moreover, equation (20) has two positive real roots under the assumption (H6). Thus, characteristic equation (11) has two pairs of purely imaginary roots $\lambda = \pm i\omega_n^\pm$, where

$$\omega_n^\pm = \sqrt{\frac{(2A_n - T_n) \pm \sqrt{T_n^4 - 4A_n T_n^2 + 4B^2}}{2}}, \quad n \in \mathbb{N}_0,$$

$$\tau_{nj}^\pm = \frac{1}{\omega_n^\pm} \left\{ 2\pi - \arccos \frac{\omega_n^{\pm 2} - A_n}{B} + 2j\pi \right\}, \quad j = 0, 1, 2, \dots \quad (21)$$

By direct computation, we have $\omega_n^- < \omega_n^+$ and $\omega_n^{-2} - A_n/B > \omega_n^{+2} - A_n/B > 0$. Thus, $\tau_{nj}^- > \tau_{nj}^+$.

In what follows, we shall verify the transversality condition.

Lemma 2. *$Re(d\lambda/d\tau)_{\tau=\tau_{nj}^+}^{-1} > 0$ and $Re(d\lambda/d\tau)_{\tau=\tau_{nj}^-}^{-1} < 0$.*

Proof. Differentiating both sides of equation (11) with respect to τ yields

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{2\lambda - T_n}{B\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda} = \frac{(2\lambda - T_n)e^{\lambda\tau}}{B\lambda} - \frac{\tau}{\lambda}. \quad (22)$$

Then,

$$\begin{aligned} Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_{nj}^\pm} &= Re \frac{(2i\omega_n^\pm - T_n)(\cos \omega_n^\pm \tau_{nj}^\pm + i \sin \omega_n^\pm \tau_{nj}^\pm)}{iB\omega_n^\pm} \\ &= \frac{2\omega_n^\pm \cos \omega_n^\pm \tau_{nj}^\pm - T_n \sin \omega_n^\pm \tau_{nj}^\pm}{B\omega_n^\pm} \\ &= \frac{2\omega_n^\pm (\omega_n^{\pm 2} - A_n) + T_n^2 \omega_n^\pm}{B^2 \omega_n^\pm} \\ &= \frac{2(\omega_n^{\pm 2} - A_n) + T_n^2}{B^2} \\ &= \frac{\pm \sqrt{T_n^4 - 4A_n T_n^2 + 4B^2}}{B^2}. \end{aligned} \quad (23)$$

Through the above analysis and motivated by Corollary 2.4 of Ruan and Wei in [28], we can establish the distribution of roots of characteristic equation (11) in the case of $n = 0$.

Lemma 3. *If (H1), (H2), (H4), and (H6) are satisfied, then we have the following:*

- (i) If $\tau_{01}^+ < \tau_{00}^-$, then all roots of (11) have negative real parts for $\tau \in [0, \tau_{00}^+)$; equation (11) has a pair of purely imaginary roots and other roots have negative real parts for $\tau = \tau_{00}^+$; equation (11) has at least a root with positive real part for $\tau > \tau_{00}^+$.
- (ii) If $\tau_{01}^+ > \tau_{00}^-$, then there exists a positive integer k such that all roots of (11) have negative real parts for $\tau \in [0, \tau_{00}^+) \cup (\tau_{00}^-, \tau_{01}^+) \cup \dots \cup (\tau_{0,k-1}^-, \tau_{0k}^+)$, and equation (11) has at least one root with positive real part for $\tau \in (\tau_{00}^+, \tau_{00}^-) \cup (\tau_{01}^+, \tau_{01}^-) \cup \dots \cup (\tau_{0,k-1}^+, \tau_{0,k-1}^-) \cup (\tau_{0k}^+, +\infty)$.

As time delay changes, the stability of the equilibrium solution may also change. The stability may transit from stable to unstable at some critical value and may also transit from unstable to stable at some other critical value. Such phenomena are often referred to as stability switches, i.e., the state can be switched from stable to unstable and back to stable and so on just by progressive increase of the time delay. Then, we can obtain the stability switches and existence of Hopf bifurcation for E^* in (3).

Theorem 3. *If (H1), (H2), (H4), and (H6) are satisfied, then the following conclusions are true:*

- (i) *If $\tau_{01}^+ < \tau_{00}^-$, then positive equilibrium E^* of (3) is asymptotically stable for $\tau \in [0, \tau_{00}^+)$ and is unstable for $\tau > \tau_{00}^+$. System (3) undergoes Hopf bifurcations near E^* when $\tau = \tau_{00}^+$.*

- (ii) *If $\tau_{01}^+ > \tau_{00}^-$, then there exists a positive integer k such that E^* switches k times from stability to instability to stability and so on such that E^* is asymptotically stable for $\tau \in [0, \tau_{00}^+) \cup (\tau_{00}^-, \tau_{01}^+) \cup \dots \cup (\tau_{0,k-1}^-, \tau_{0k}^+)$ and is unstable for $\tau \in (\tau_{00}^+, \tau_{00}^-) \cup (\tau_{01}^+, \tau_{01}^-) \cup \dots \cup (\tau_{0,k-1}^+, \tau_{0,k-1}^-) \cup (\tau_{0k}^+, +\infty)$. System (3) undergoes Hopf bifurcation around E^* for every $\tau = \tau_{0j}^\pm$.*

4. Properties of Hopf Bifurcation

Here, we derive the Hopf bifurcation properties by using the center manifold theorem and normal form theory for partial functional differential equations in [29, 30]. For simplicity, we denote the Hopf bifurcation value by τ^* and the corresponding simply purely imaginary characteristic roots by $\pm i\omega^*$, respectively.

Let $X = \{N, P \in W^{2,2}(\Omega) : \partial N(x, t)/\partial x = \partial P(x, t)/\partial x = 0, x = 0, \pi\}$, $\tilde{N}(\cdot, t) = N(\cdot, \tau t)$, $\tilde{P}(\cdot, t) = P(\cdot, \tau t)$, $\tilde{U}(t) = (\tilde{N}(\cdot, t), \tilde{P}(\cdot, t))$, and $\tau = \tau^* + \mu$ with $\mu \in \mathbb{R}$, then $\mu = 0$ is the Hopf bifurcation value of system (3). For simplicity, we drop the tilde and rewrite system (3) in the form

$$\frac{dU(t)}{dt} = \tau D\Delta U(t) + L(\mu)(U_t) + f(U_t, \mu), \quad (24)$$

where $D = \text{diag}\{d_1, d_2\}$, $\varphi = (\varphi_1, \varphi_2)^T \in \mathcal{C} = C([-1, 0], X)$, and $L(\mu)(\cdot) : \mathcal{C} \rightarrow X$, $f : \mathcal{C} \times \mathbb{R} \rightarrow X$ are given by

$$L(\mu)(\varphi) = (\tau^* + \mu) \begin{pmatrix} (-bP^* - d)\varphi_1(0) - bN^*\varphi_2(0) + \beta\varphi_2(-1) \\ cP^*\varphi_1(0) + \frac{\theta P^*(P^{*2} - H^2)}{(H^2 + P^{*2})^2}\varphi_2(0) \end{pmatrix},$$

$$f(\varphi, \mu) = (\tau^* + \mu) \begin{pmatrix} \sum_{i+j+l \geq 2} \frac{1}{i!j!l!} F_{ijl} \varphi_1^i(0) \varphi_2^j(0) \varphi_2^l(-1) \\ \sum_{i+j \geq 2} \frac{1}{i!j!} G_{ij} \varphi_1^i(0) \varphi_2^j(0) \end{pmatrix}, \quad (25)$$

and

$$F(u, v, w) = a - buv - du + \beta w,$$

$$G(u, v) = cuv - \alpha v - \frac{\theta v^2}{H^2 + v^2},$$

$$F_{ijl} = \frac{\partial^{i+j+l} F}{\partial u^i \partial v^j \partial w^l} (N^*, P^*, P^*),$$

$$G_{ij} = \frac{\partial^{i+j} G}{\partial u^i \partial v^j} (N^*, P^*). \quad (26)$$

From the previous section, we know that $\pm i\omega^* \tau^*$ is a pair of simple purely imaginary eigenvalues of the following linear differential equation:

$$\dot{U}(t) = \tau D\Delta U(t) + L(\mu)(U_t). \quad (27)$$

Due to the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\vartheta, \mu)$ ($-1 \leq \vartheta \leq 0$), whose elements are of bounded variation functions such that

$$-\tau^* Dn^2 \varphi(0) + L(\mu)(\varphi) = \int_{-1}^0 d\eta(\vartheta, \tau^*) \varphi(\vartheta) \quad \text{for } \varphi \in \mathcal{C}([-1, 0], \mathbb{R}^2), \quad (28)$$

where

$$\eta(\vartheta, \tau^*) = \begin{cases} \tau^* \begin{pmatrix} -bP^* - d - d_1 n^2 & -bN^* \\ cP^* & \frac{\theta P^*(P^{*2} - H^2)}{(H^2 + P^{*2})^2} - d_2 n^2 \end{pmatrix}, & \vartheta = 0, \\ 0, & \vartheta \in (-1, 0), \\ -\tau^* \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, & \vartheta = -1. \end{cases} \quad (29)$$

For $\Phi \in C^1([-1, 0], \mathbb{R}^2)$, $\Psi \in C^1([0, 1], \mathbb{R}^2)$, we define

$$A_1(\Phi(\vartheta)) = \begin{cases} \frac{d\Phi(\vartheta)}{d\vartheta}, & \vartheta \in [-1, 0), \\ \int_{-1}^0 d\eta(\vartheta, \tau^*) \Phi(\vartheta), & \vartheta = 0, \end{cases} \quad (30)$$

$$A_1^*(\Psi(s)) = \begin{cases} \frac{d\Psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^0 d\eta^T(\vartheta, \tau^*) \Psi(-\vartheta), & s = 0. \end{cases}$$

Then, A_1^* and A_1 are adjoint operators under the bilinear form

$$\begin{aligned} (\Psi, \Phi)_0 &= \overline{\Psi}(0)\Phi(0) - \int_{-1}^0 \int_0^\vartheta \overline{\Psi}(\zeta - \vartheta) d\eta(\vartheta, 0) \Phi(\zeta) d\zeta \\ &= \overline{\Psi}(0)\Phi(0) + \tau^* \int_{-1}^0 \overline{\Psi}(\zeta + 1) \begin{pmatrix} 0 & 0 \\ r & -r \end{pmatrix} \Phi(\zeta) d\zeta. \end{aligned} \quad (31)$$

It is calculated that $q(\vartheta) = (1, q_1)^T e^{i\omega^* \tau^* \vartheta}$ ($\vartheta \in [-1, 0]$) and $q^*(s) = M(q_2, 1)^T e^{i\omega^* \tau^* s}$ ($s \in [[0, 1])$) are eigenvectors of A_1 and A_1^* corresponding to $i\omega^* \tau^*$ and $-i\omega^* \tau^*$, respectively, where

$$\begin{aligned} q_1 &= \frac{i\omega^* + d_1 n^2 + bP^* + d}{bN^* - \beta e^{-i\omega^* \tau^*}}, \\ q_2 &= \frac{cP^*}{-i\omega^* + d_1 n^2 + bP^* + d}, \end{aligned} \quad (32)$$

$$M = [\eta + \bar{\xi} + \beta \eta \bar{\xi} \tau^* e^{-i\omega^* \tau^*}]^{-1}.$$

Then, $Q = \text{span}\{q(\vartheta), \overline{q(\vartheta)}\}$ is the center subspace of system (3), and $Q^* = \text{span}\{q^*(s), \overline{q^*(s)}\}$ is the adjoint subspace.

Let $h \cdot f_n^H = h_1 \beta_n^1 + h_2 \beta_n^2$, $f_n = (\beta_n^1, \beta_n^2)$, and $\beta_n^1 = (\cos nx, 0)^T$, $\beta_n^2 = (0, \cos nx)^T$. The complex-valued L^2 inner product on Hilbert space X_C is

$$\langle U_1, U_2 \rangle = \frac{1}{\pi} \int_0^\pi (u_1 \overline{v_1} + u_2 \overline{v_2}) dx, \quad (33)$$

for $U_1 = (u_1, u_2), U_2 = (v_1, v_2) \in X_C$, and $\langle \beta_0^i, \beta_0^i \rangle = 1$,
 $\langle \beta_n^i, \beta_n^i \rangle = 1/2, i = 1, 2, n = 1, 2, \dots$,

$$\langle \Phi, f_n \rangle = (\langle \Phi, \beta_n^1 \rangle, \langle \Phi, \beta_n^2 \rangle), \quad (34)$$

where $\Phi \in C([-1, 0], X)$. Then, the center subspace of system (27) at $\mu = 0$ is

$$P_{CN}\mathcal{L} = \{(q(\vartheta)z + \overline{q(\vartheta)\bar{z}}) \cdot f_n, z \in \mathbb{C}\}. \quad (35)$$

If $\mu = 0$, then we can obtain

$$W(z, \bar{z}, \theta) = W_{20} \frac{z^2}{2} + W_{11} z \bar{z} + W_{02} \frac{\bar{z}^2}{2} + \dots \quad (36)$$

The flow of system (24) on the center manifold can be written as

$$U_t = (q(\theta)z(t) + \overline{q(\theta)\bar{z}(t)}) \cdot f_n + W(z(t), \bar{z}(t), \theta). \quad (37)$$

Moreover, for $U_t \in C_0$ of (24) at $\tau = \tau^*$, we have $\dot{z} = i\omega^* \tau^* z + g(z, \bar{z})$, where

$$g(z, \bar{z}) = \overline{q^*(0)} \langle f(U_t, 0), f_n \rangle = g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2 \bar{z}}{2} + \dots \quad (38)$$

Following the calculation procedures in [29, 30], we can get

$$\begin{aligned} g_{20} &= \tau^* \overline{M} [-2b\xi\bar{\eta} + (2c\xi + G_{22}\xi^2)], \\ g_{11} &= \tau^* \overline{M} [-2b\text{Re}\{\xi\}\bar{\eta} + 2c\text{Re}\{\xi\} + G_{22}|\xi|^2], \\ g_{02} &= \overline{g_{20}}, \\ g_{21} &= -\frac{\overline{M}\tau^*}{\pi} \int_0^\pi b\bar{\eta}\cos^2 nx \cdot (W_{20}^{(2)}(0) + \bar{\xi}W_{20}^{(1)}(0) + 2\xi W_{11}^{(1)}(0) + 2W_{11}^{(2)}(0)) dx \\ &\quad + \frac{\overline{M}\tau^*}{\pi} \int_0^\pi [c \cos^2 nx \cdot (W_{20}^{(2)}(0) + \bar{\xi}W_{20}^{(1)}(0) + 2\xi W_{11}^{(1)}(0) + 2W_{11}^{(2)}(0)) + G_{22}\cos^2 nx \cdot (\bar{\xi}W_{20}^{(2)}(0) + 2\xi W_{11}^{(2)}(0))] dx, \end{aligned} \quad (39)$$

where

$$\begin{aligned} W_{20}(\vartheta) &= \left(\frac{ig_{20}}{\omega^* \tau^*} q(\vartheta) + \frac{i\overline{g_{02}}}{3\omega^* \tau^*} \overline{q(\vartheta)} \right) \cdot f_n + E_1 e^{2i\omega^* \tau^* \vartheta}, \\ W_{11}(\vartheta) &= \left(-\frac{ig_{11}}{\omega^* \tau^*} q(\vartheta) + \frac{i\overline{g_{11}} \overline{q(\vartheta)}}{\omega^* \tau^*} \right) \cdot f_n + E_2, \\ E_1 &= E_1' \times \begin{pmatrix} -2b\xi\bar{\eta} \\ 2c\xi + G_{22}\xi^2 \end{pmatrix} \cos^2 nx, \\ E_1' &= \begin{pmatrix} 2i\omega^* + d_1 n^2 + bP^* + d & bN^* - \beta e^{-2i\omega^* \tau^*} \\ -cP^* & 2i\omega^* + d_2 n^2 - \frac{\theta P^* (P^{*2} - H^2)}{(H^2 + P^{*2})^2} \end{pmatrix}^{-1}, \\ E_2 &= E_2' \times \begin{pmatrix} -2b\text{Re}\{\xi\} \\ 2c\text{Re}\{\xi\} + G_{22}|\xi|^2 \end{pmatrix} \cos^2 nx, \\ E_2' &= \begin{pmatrix} d_1 n^2 + bP^* + d & bN^* - \beta \\ -cP^* & d_2 n^2 - \frac{\theta P^* (P^{*2} - H^2)}{(H^2 + P^{*2})^2} \end{pmatrix}^{-1}. \end{aligned} \quad (40)$$

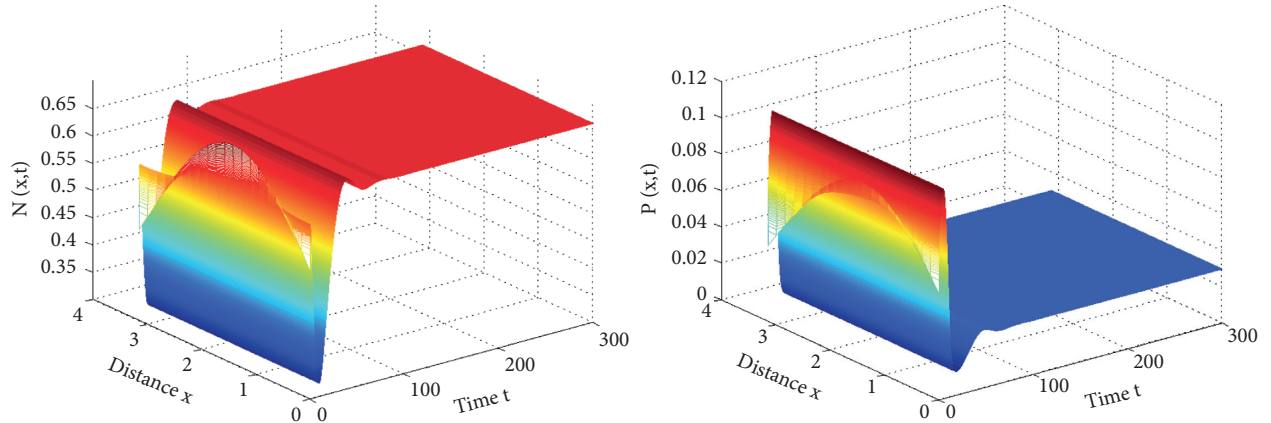


FIGURE 1: The positive equilibrium E^* of (2) is asymptotically stable when $\theta = 0.02$.

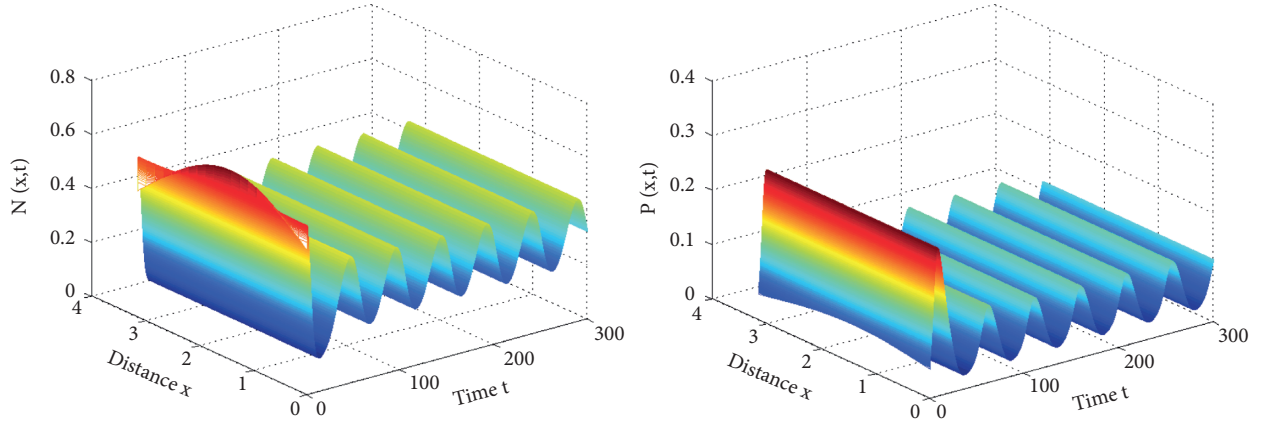


FIGURE 2: The positive equilibrium E^* of (2) is unstable when $\theta = 0.02$.

Thus, we can compute

$$\begin{aligned}
 c_1(0) &= \frac{i}{2\omega^*\tau^*} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{g_{21}}{2}, \\
 \ell_2 &= \frac{\text{Re}(c_1(0))}{\text{Re}(\lambda'(\tau^*))}, \\
 \iota_2 &= 2\text{Re}(c_1(0)), \\
 \gamma_2 &= -\frac{1}{\omega^*\tau^*} (\text{Im}(c_1(0)) + \ell_2\text{Im}(\lambda'(\tau^*))).
 \end{aligned} \tag{41}$$

Then, by the Hopf bifurcation theory in [29, 30], we have the following conclusions about the properties of the Hopf bifurcation.

Theorem 4. For system (3),

- (i) ℓ_2 determines the direction of Hopf bifurcation: if $\ell_2 > 0$, then the supercritical bifurcation bifurcates stable limit cycles; if $\ell_2 < 0$, then the subcritical bifurcation bifurcates unstable limit cycles.

- (ii) ι_2 determines the stability of bifurcating periodic solutions: the periodic solutions are orbitally asymptotically stable if $\iota_2 < 0$ and are unstable if $\iota_2 > 0$.

- (iii) γ_2 determines the period of the bifurcating periodic solutions: the period is monotonically increasing at the time delay τ when $\gamma_2 > 0$ and is monotonically decreasing at the time delay τ when $\gamma_2 < 0$.

5. Numerical Simulations

In this section, we mainly verify the theoretical results obtained in the previous sections by some numerical examples. For system (3), we set $\Omega = (0, \pi)$, $d_1 = d_2 = 1$, $a = 0.03$, $b = 1.5$, $c = 1$, $d = 0.003$, $k = 0.05$, $H = 0.035$, $\alpha = 0.1$, and $\beta = 0.05$.

Example 1. If $\theta = 0.04$ and $\tau = 0$ with the initial conditions $N_1(x, t) = 0.5 + 0.2 \sin(x + t)$ and $P_1(x, t) = 0.05 + 0.05 \sin(x + t)$, then the positive constant equilibrium E^* (0.6636, 0.0296) of system (2) is asymptotically stable, as shown in Figure 1.

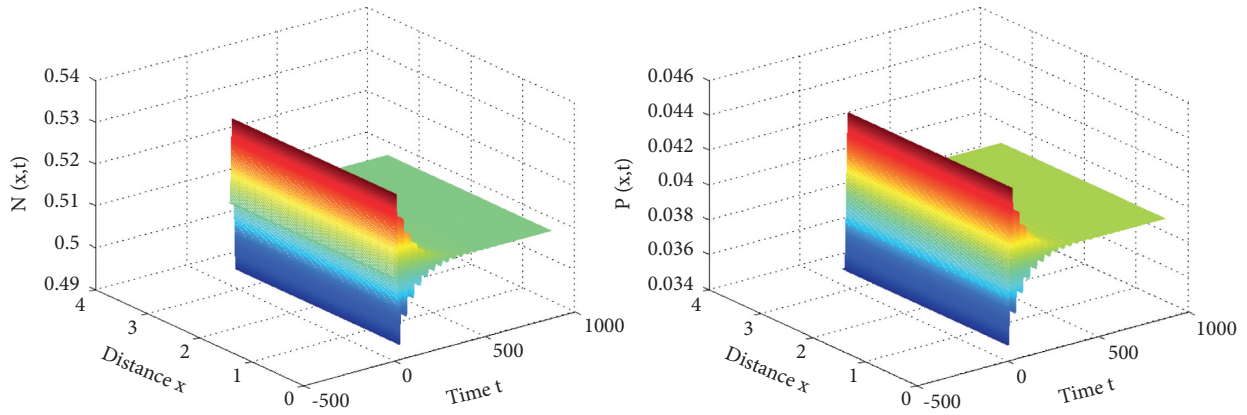


FIGURE 3: The positive equilibrium E^* of (3) is stable when $\tau = 10$.

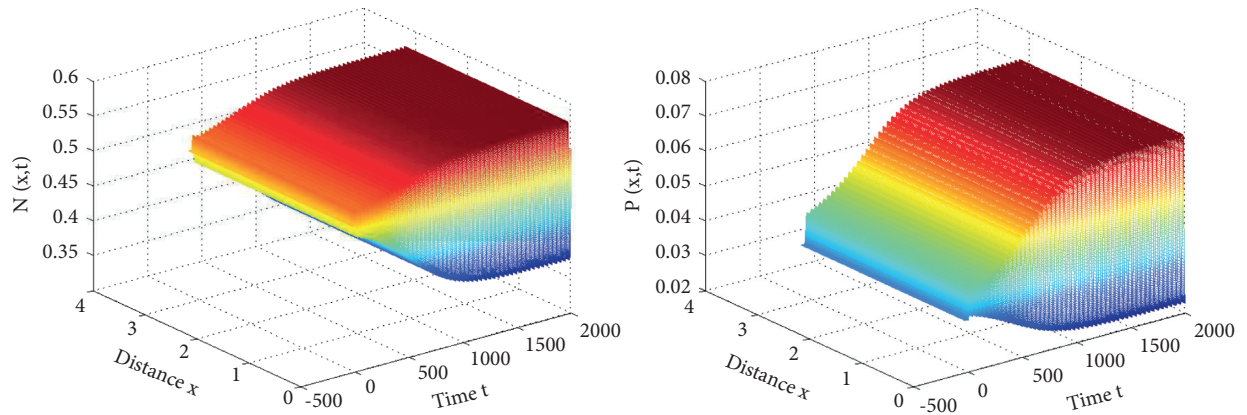


FIGURE 4: The positive equilibrium E^* of (3) is unstable when $\tau = 30$.

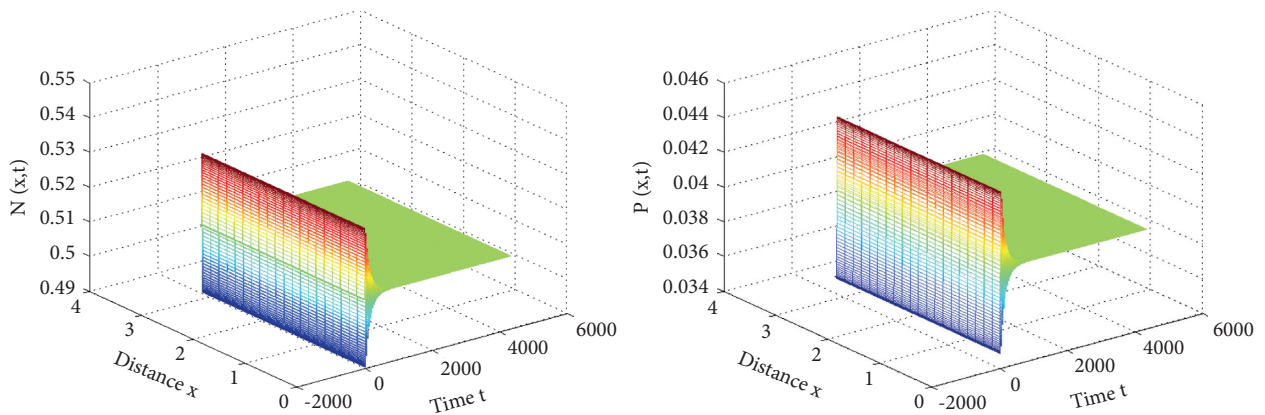


FIGURE 5: The positive equilibrium E^* of (3) is stable when $\tau = 50$.

Example 2. If $\theta = 0.02$ and $\tau = 0$ with the initial conditions $N_1(x, t) = 0.5 + 0.2 \sin(x + t)$ and $P_1(x, t) = 0.05 + 0.02 \sin(x + t)$, then the positive constant equilibrium $E^*(0.6636, 0.0296)$ of system (2) is unstable and Hopf bifurcation occurs, see Figure 2.

For $\tau > 0$, we fix $\theta = 0.029$. Then, system (3) has the unique positive equilibrium solution $E^*(0.5110, 0.0397)$. Moreover, equation (20) has two positive roots $\omega_0^+ \approx 0.2294$ and $\omega_0^- = 0.0408$, $\tau_{00}^+ \approx 14 < \tau_{00}^- \approx 41 < \tau_{01}^+ \approx 69 < \tau_{01}^- \approx 96 < \dots$. According to Theorem 3, if time delay τ varies, then the

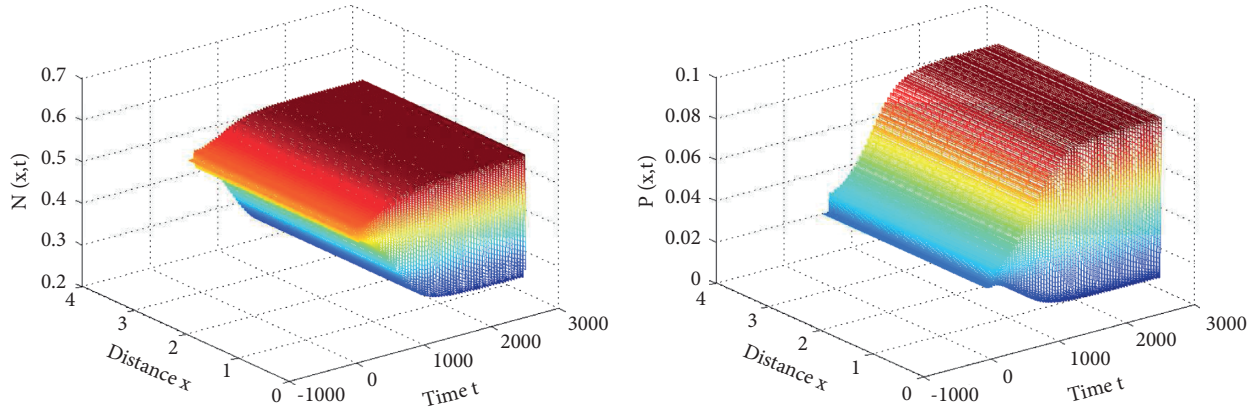


FIGURE 6: The positive equilibrium E^* of (3) is unstable when $\tau = 75$.

stability of positive equilibrium E^* (0.5110, 0.0397) of system (3) would change.

Example 3. For $\tau_{00}^+ \approx 14$, we can compute $c_1(0) \approx -2.3543 - 1.2902i$, then $\ell_2 > 0$ and $\iota_2 < 0$. From Theorem 4, we can know that the Hopf bifurcation is supercritical and the spatially homogeneous periodic solutions are stable. The numerical solutions with initial conditions $N_1(x, t) = 0.51$ and $P_1(x, t) = 0.035$ are presented in Figures 3 and 4.

Example 4. For $\tau_{01}^+ \approx 69$, we can compute $c_1(0) \approx -3.8745 - 1.6482i$, then $\ell_2 > 0$ and $\iota_2 < 0$. The Hopf bifurcation is supercritical and the spatially homogeneous periodic solutions are stable. The numerical solutions with initial conditions $N_1(x, t) = 0.51$ and $P_1(x, t) = 0.035$ are presented in Figures 5 and 6.

It can be found that nutrient recycling delay can bring about the stability switches in the nutrient-phytoplankton system.

6. Conclusion

In this paper, we have expanded the model proposed in [6] and investigated a novel reaction-diffusion nutrient-phytoplankton ecological model incorporating nutrient recycling delay. We proved that the release rate of toxic chemicals by the toxin-producing phytoplankton population θ can destabilize the positive constant equilibrium and may generate time-periodic solutions. We also revealed the effect of nutrient recycling delay on our model. By taking time delay as the bifurcation parameter, multiple stability switches of the positive constant equilibrium and Hopf bifurcation phenomena can be observed under some conditions, which contrast sharply with the case in the absence of time delay. The results can help to understand the periodic outbreak of algae.

It is observed that the model is more comprehensive by taking account of the zooplankton population, and rich dynamical behaviors have been presented in [3, 8, 10, 26], such as uniform persistence, Turing bifurcation, and global Hopf bifurcation. However, these models do not investigate

the joint effect of spatial diffusion and time delay. In addition, in a reaction-diffusion system, when the Hopf bifurcation curve and Turing bifurcation curve intersect, the Turing-Hopf bifurcation will occur and generate spatiotemporal patterns, see [31–33]. We will do further research on spatiotemporal patterns via Turing-Hopf bifurcation in the delayed reaction-diffusion model incorporating three components in the future following the aforementioned references.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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References

- [1] X.-Z. He and S. Ruan, “Global stability in chemostat-type plankton models with delayed nutrient recycling,” *Journal of Mathematical Biology*, vol. 37, no. 3, pp. 253–271, 1998.
- [2] O. Pardo, “Global stability for a phytoplankton-nutrient system,” *Journal of Biological Systems*, vol. 08, no. 02, pp. 195–209, 2000.
- [3] S. R.-J. Jang and J. Baglama, “Nutrient-plankton models with nutrient recycling,” *Computers and Mathematics with Applications*, vol. 49, pp. 375–387, 2005.
- [4] M. Gao, H. Shi, and Z. Li, “A planktonic resource-consumer model with a temporal delay in nutrient recycling,” *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 511–516, 2008.
- [5] A. Sharma, A. K. Sharma, and K. Agnihotri, “The dynamic of plankton-nutrient interaction with delay,” *Applied Mathematics and Computation*, vol. 231, pp. 503–515, 2014.

- [6] S. Chakraborty, S. Chatterjee, E. Venturino, and J. Chattopadhyay, "Recurring plankton bloom dynamics modeled via toxin-producing phytoplankton," *Journal of Biological Physics*, vol. 33, no. 4, pp. 271–290, 2007.
- [7] B. Ghanbari and J. F. Gómez-Aguilar, "Modeling the dynamics of nutrient-phytoplankton-zooplankton system with variable-order fractional derivatives," *Chaos, Solitons and Fractals*, vol. 116, pp. 114–120, 2018.
- [8] Z. Qiu and H. Zhu, "Complex dynamics of a nutrient-plankton system with nonlinear phytoplankton mortality and allelopathy," *Discrete and Continuous Dynamical Systems-Series B*, vol. 21, no. 8, pp. 2703–2728, 2016.
- [9] R. P. Kaur, A. Sharma, and A. K. Sharma, "The impact of additional food on plankton dynamics in the absence and presence of toxicity," *Biosystems*, vol. 202, Article ID 104359, 2021.
- [10] N. K. Thakur, A. Ojha, P. K. Tiwari, and R. K. Upadhyay, "An investigation of delay induced stability transition in nutrient-plankton systems," *Chaos, Solitons and Fractals*, vol. 142, Article ID 110474, 2021.
- [11] N. K. Thakur, A. Ojha, D. Jana, and R. K. Upadhyay, "Modeling the plankton-fish dynamics with top predator interference and multiple gestation delays," *Nonlinear Dynamics*, vol. 100, no. 4, pp. 4003–4029, 2020.
- [12] R. K. Upadhyay and S. R. K. Iyengar, *Spatial Dynamics and Patten Formations in Biological Populations*, CRC Press, Boca Raton, Florida, 2021.
- [13] Q. Song, R. Yang, C. Zhang, and L. Wang, "Bifurcation analysis of a diffusive predator-prey model with Beddington-DeAngelis functional response," *Journal of Applied Analysis and Computation*, vol. 11, no. 2, pp. 920–936, 2021.
- [14] H. Jiang, "Turing bifurcation in a diffusive predator-prey model with schooling behavior," *Applied Mathematics Letters*, vol. 96, pp. 230–235, 2019.
- [15] C.-C. Wu, "The spreading speed for a predator-prey model with one predator and two preys," *Applied Mathematics Letters*, vol. 91, pp. 9–14, 2019.
- [16] S. Kumari, R. K. Upadhyay, P. Kumar, and V. Rai, "Dynamics and patterns of species abundance in ocean A mathematical modeling study," *Nonlinear Analysis: Real World Applications*, vol. 60, Article ID 103303, 2021.
- [17] S. Kumari and R. K. Upadhyay, "Dynamics comparison between non-spatial and spatial systems of the plankton-fish interaction model," *Nonlinear Dynamics*, vol. 99, no. 3, pp. 2479–2503, 2020.
- [18] S. Mishra and R. K. Upadhyay, "Strategies for the existence of spatial patterns in predator-prey communities generated by cross-diffusion," *Nonlinear Analysis: Real World Applications*, vol. 51, Article ID 103018, 2020.
- [19] R. K. Upadhyay, S. Kumari, P. Kumar, and V. Rai, "Spatial distribution of microalgae in marine systems A reaction-diffusion model," *Ecological Complexity*, vol. 39, Article ID 100771, 2019.
- [20] N. K. Thakur, S. T. Tiwari, B. Dubey, and R. K. Upadhyay, "Diffusive three species plankton model in the presence of toxic prey: application to Sundarban mangrove wetland," *Journal of Biological Systems*, vol. 25, pp. 1–22, 2017.
- [21] S. Yan, X. Lian, W. Wang, and R. K. Upadhyay, "Spatio-temporal dynamics in a delayed diffusive predator model," *Applied Mathematics and Computation*, vol. 224, pp. 524–534, 2013.
- [22] J. Zhao and J. Wei, "Dynamics in a diffusive plankton system with delay and toxic substances effect," *Nonlinear Analysis: Real World Applications*, vol. 22, pp. 66–83, 2015.
- [23] V. Kumar, J. Dhar, and H. S. Bhatti, "Stability switching dynamics of a food chain system incorporating gestation delays," *Differential Equations and Dynamical Systems*, vol. 28, no. 3, pp. 685–705, 2020.
- [24] Z. Jiang, Y. Zhao, X. Bai, and Z. Zhang, "Bifurcation and control of a planktonic ecological system with double delays by delayed feedback control," *Journal of the Franklin Institute*, vol. 358, no. 7, pp. 3609–3632, 2021.
- [25] S. Ruan, "Turing instability and travelling waves in diffusive plankton models with delayed nutrient recycling," *IMA Journal of Applied Mathematics*, vol. 61, no. 1, pp. 15–32, 1998.
- [26] Y. Wang, H. Wang, and W. Jiang, "Stability switches and global Hopf bifurcation in a nutrient-plankton model," *Nonlinear Dynamics*, vol. 78, no. 2, pp. 981–994, 2014.
- [27] A. M. Turing, "The chemical basis of morphogenesis," *Philosophical Transactions of the Royal Society of London. Series B*, vol. 237, pp. 37–72, 1952.
- [28] S. Ruan and J. Wei, "On the zeros of transcendental functions with applications to stability of delay differential equations with two delays," *Dynamics of Continuous, Discrete and Impulsive Systems, Ser. A: Mathematical Analysis*, vol. 10, pp. 863–874, 2003.
- [29] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer, New York, NY, USA, 1996.
- [30] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge, MA, USA, 1981.
- [31] W. Jiang, Q. An, and J. Shi, "Formulation of the normal form of Turing-Hopf bifurcation in partial functional differential equations," *Journal of Differential Equations*, vol. 268, no. 10, pp. 6067–6102, 2020.
- [32] Y. Song and X. Zou, "Spatiotemporal dynamics in a diffusive ratio-dependent predator-prey model near a Hopf-Turing bifurcation point," *Computers and Mathematics with Applications*, vol. 67, no. 10, pp. 1978–1997, 2014.
- [33] M. Chen, R. Wu, and L. Chen, "Spatiotemporal patterns induced by Turing and Turing-Hopf bifurcations in a predator-prey system," *Applied Mathematics and Computation*, vol. 380, Article ID 125300, 2020.