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Research Article

Further Exploration on Bifurcation for Fractional-Order Bidirectional Associative Memory (BAM) Neural Networks concerning Time Delay*

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This work principally considers the stability issue and the emergence of Hopf bifurcation for a class of fractional-order BAM neural network models concerning time delays. Through the detailed analysis on the distribution of the roots of the characteristic equation of the involved fractional-order delayed BAM neural network systems, we set up a new delay-independent condition to guarantee the stability and the emergence of Hopf bifurcation for the investigated fractional-order delayed BAM neural network systems. The work indicates that delay is a significant element that has a vital impact on the stability and the emergence of Hopf bifurcation in fractional-order delayed BAM neural network systems. The simulation figures and bifurcation plots are clearly presented to verify the derived key research results. The established conclusions of this work have significant guiding value in regulating and optimizing neural networks.

1. Introduction

Neural networks have been found to have immense application prospect in a lot of subject areas such as modeling human brain, remote sensing, biological science, pattern recognition, artificial intelligence, and control technique [1, 2]. Usually, time delay often occurs in neural network systems due to the lag of the response of signal transmission of the neurons in neural networks. Thus, it is necessary for us to establish the delayed neural networks to describe the real situation of neural networks. Generally speaking, time delay often gives rise to the disappearance of stability, periodic oscillation, chaotic behavior, and so on [3, 4]. In order to grasp the effect of time delay on various dynamical properties of neural networks, miscellaneous delayed neural networks have been built and studied. Up to now, a great deal of valuable publications has been achieved. For instance, Aouiti et al. [5] investigated the existence and global exponential stability of pseudo almost periodic solution to delayed BAM neural networks involving leakage delays by virtue of fixed point theory and mathematical inequality skills. Yang et al. [6] studied the almost automorphic solution to high-order delayed BAM neural networks by means of the exponential dichotomy theory, Banach contraction mapping law, and differential inequality strategy. Maharajan et al. [7] set up a new global robust exponential stability condition for a class of uncertain BAM neural network systems involving mixed time delays. Popa [8] focused on the global μ-stability for impulsive complex-valued BAM neural networks concerning mixed delays. Sowmiya et al. [9] made a detailed analysis on mean-square asymptotic stability for impulsive discrete-time stochastic BAM neural networks involving Markovian jumping and multiple delays. For details, we refer the readers to [10, 11].

The general BAM networks are given by

$$\begin{cases} \dot{p}_i(t) = -\alpha_i p_i(t) + \sum_{j=1}^{m} a_{ji} h_i\left(q_j\left(t - \eta_{ji}\right)\right) + \mathcal{P}_i, \\ \dot{q}_j(t) = -\beta_j q_j(t) + \sum_{i=1}^{n} b_{ij} k_j\left(p_i\left(t - \zeta_{ij}\right)\right) + \mathcal{Q}_j, \end{cases} \quad (1)$$

where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$, α_i and β_j describe the stability of internal neuron processes on $\mathcal{P}$-layer and $\mathcal{Q}$-layer, respectively; a_{ji}, b_{ij} represent the connection weights; $p_i(t)$ and $q_j(t)$ denote the states of the neurons on $\mathcal{P}$-layer and $\mathcal{Q}$-layer, respectively; h_i and k_j are activation functions; $\mathcal{P}_i, \mathcal{Q}_j$ denote the inputs; η_{ji} and ζ_{ij} are time delays. The model (1) describes the change law of different neurons which lie in two layers. For details, please see [12, 13].

System (1) is a large-scale nonlinear dynamical model. It owns very complicated dynamical properties. In order to have a good command of the internal law of network system (1), many researchers pay much attention to some simplified versions of delayed neural network models. By the investigation on various dynamical peculiarities of the simplified neural network systems, we are able to grasp the potential dynamical properties for large-scale delayed neural network systems. During the past several years, a lot of works on the simplified neural network models have been published. For example, Hajihosseini et al. [14] discussed the bifurcation problem for recurrent neural networks involving three neurons. Kaslik and Balint [15] investigated the Neimark–Sacker bifurcation of a discrete-time delayed neural network system involving two neurons. Ge and Xu [16] obtained the sufficient condition to ensure the stability and the onset of Hopf bifurcation for delayed neural networks involving four neurons. Yang and Ye [17] dealt with the stability and bifurcation behavior for delayed BAM neural network involving five neurons. As to more concrete literatures on this theme, one can see [4, 18].

All above publications are only restricted to the integer-order dynamical equations. In recent years, fractional calculus has displayed wide application value in a lot of fields such as heat and mass transfer, electromagnetic and electrodynamics, control science, population systems, biophysics, and neural networks [19–27]. The study shows that fractional calculus can be regarded as a very useful tool to describe the object issues in the real world because it owns the memory property and hereditary function during the dynamic change process [28, 29]. Recently, fractional calculus has become the biggest concern of the present day world. In particular, fractional-order neural networks have also become one of the key hot issues in neural network area. Delay-induced Hopf bifurcation is a significant dynamical property in delayed dynamical models. However, it is a pity that a great deal of works is only concerned with delay-induced Hopf bifurcation for integer-order dynamical system concerning delays and few publications focus on the fractional-order case (see [30, 31]). In fractional-order neural networks, what is the effect of time delay and fractional-order on the stability and bifurcation? The solution of this problem is beneficial to the design of neural networks. Up to now, there are many bifurcation problems that are expected to be solved. This viewpoint stimulates us to deal with the delay-induced Hopf bifurcation of delayed neural networks involving multiple neurons.

Based on the neural networks (1), we consider the following fractional-order simplified delayed neural networks:

$$\begin{cases} \frac{d^{\xi} u_1(t)}{dt^{\xi}} = -k u_1(t) + a_{11} h\left(u_4(t-\zeta)\right) + a_{12} l\left(u_5(t-\zeta)\right) + a_{13} l\left(u_6(t-\zeta)\right), \\ \frac{d^{\xi} u_2(t)}{dt^{\xi}} = -k u_2(t) + a_{21} h\left(u_5(t-\zeta)\right) + a_{22} l\left(u_6(t-\zeta)\right) + a_{23} l\left(u_4(t-\zeta)\right), \\ \frac{d^{\xi} u_3(t)}{dt^{\xi}} = -k u_3(t) + a_{31} h\left(u_6(t-\zeta)\right) + a_{32} l\left(u_4(t-\zeta)\right) + a_{33} l\left(u_5(t-\zeta)\right), \\ \frac{d^{\xi} u_4(t)}{dt^{\xi}} = -k u_4(t) + a_{41} h\left(u_1(t-\zeta)\right) + a_{42} l\left(u_2(t-\zeta)\right) + a_{43} l\left(u_3(t-\zeta)\right), \\ \frac{d^{\xi} u_5(t)}{dt^{\xi}} = -k u_5(t) + a_{51} h\left(u_2(t-\zeta)\right) + a_{52} l\left(u_3(t-\zeta)\right) + a_{53} l\left(u_1(t-\zeta)\right), \\ \frac{d^{\xi} u_6(t)}{dt^{\xi}} = -k u_6(t) + a_{61} h\left(u_3(t-\zeta)\right) + a_{62} l\left(u_1(t-\zeta)\right) + a_{63} l\left(u_2(t-\zeta)\right), \end{cases} \quad (2)$$

where $\xi \in (0,1]$ is a real number; k describes the stability of internal neuron processes on $\mathscr{P}$-layer and $\mathscr{Q}$-layer; a_{ij} $(i = 1,2,3,4,5,6;\ j = 1,2,3)$ represents the connection weights; $u_i(t)$ $(i = 1,2,3)$ denotes the state of the i-neuron on $\mathscr{P}$-layer; $u_j(t)$ $(j = 4,5,6)$ denotes the state of the j-neuron on $\mathscr{Q}$-layer; h and l are activation functions. In order to establish the key results of this work, we make the following hypothesis:

$$\begin{aligned}&(H1)\, h,\\ &l \in C^1,\\ &h(0) = l(0) = 0.\end{aligned} \tag{3}$$

The remainder of this article is planned as follows. Section 2 presents the key theories on fractional calculus. Section 3 displays the main conclusions on stability and Hopf bifurcation for neural networks (2). Section 4 executes software simulations to illustrate the key conclusions of this article. Section 5 ends this work with a simple conclusion.

2. Indispensable Definitions and Lemmas

In this part, we give several necessary definitions and lemmas about fractional calculus which will be used in the next part.

Definition 1 (see [32]). Define Caputo fractional-order derivative as follows:

$$\mathscr{D}^{\xi} u(\varrho) = \frac{1}{\Gamma(l-\xi)} \int_{\varrho_0}^{\varrho} \frac{u^{(l)}(s)}{(\varrho - s)^{\xi - l + 1}} ds, \tag{4}$$

where $u(\varrho) \in ([\varrho_0, \infty), R), \Gamma(s) = \int_0^{\infty} \varrho^{s-1} e^{-\varrho} d\varrho, \quad \varrho \geq \varrho_0$, and$l \in Z^+, \xi \in [l-1, l)$.

Lemma 1 (see [33, 34]). *Consider the following model:*

$$\frac{d^{\xi} u(t)}{dt^{\xi}} = w(t, u(t)), u(0) = u_0, \tag{5}$$

where $\xi \in (0,1]$ and $w(t,u(t))$: $R^+ \times R^n \longrightarrow R^n, n \in Z^+$. Let u_* be the equilibrium point of system (5). If every eigenvalue (denoted by ς) of $(\partial w(t,u)/\partial u)|_{u=u_*}$ obeys $|arg(\varsigma)| > (\xi\pi/2)$, then we say that u_* is locally asymptotically stable.

Lemma 2 (see [35]). *Consider the following model:*

$$\begin{cases} \dfrac{d^{\xi_1} \mathscr{H}_1(t)}{dt^{\xi_1}} = e_{11}\mathscr{H}_1(t-\zeta_{11}) + e_{12}\mathscr{H}_2(t-\zeta_{12}) + \cdots + e_{1l}\mathscr{H}_l(t-\zeta_{1l}),\\ \dfrac{d^{\xi_2} \mathscr{H}_2(t)}{dt^{\xi_2}} = e_{21}\mathscr{H}_1(t-\zeta_{21}) + e_{22}\mathscr{H}_2(t-\zeta_{22}) + \cdots + e_{2l}\mathscr{H}_l(t-\zeta_{2l}),\\ \vdots\\ \dfrac{d^{\xi_l} \mathscr{H}(t)}{dt^{\xi_l}} = e_{l1}\mathscr{H}_1(t-\zeta_{l1}) + e_{l2}\mathscr{H}_2(t-\zeta_{l2}) + \cdots + e_{ll}\mathscr{H}_l(t-\zeta_{ll}), \end{cases} \tag{6}$$

where $\xi_i \in (0,1)$ $(i = 1,2,\ldots,l)$, the initial value $\mathscr{H}_i(t) = \omega_i(t) \in C[-\max_{i,h} \zeta_{ih}, 0]$, $t \in [-\max_{i,h} \zeta_{il}, 0]$, and $i,h = 1,2,\ldots,l$. Denote

$$\Delta(\eta) = \begin{bmatrix} \eta^{\xi_1} - e_{11}e^{-\eta\zeta_{11}} & -e_{12}e^{-\eta\zeta_{12}} & \cdots & -e_{1l}e^{-\eta\zeta_{1l}}\\ -e_{21}e^{-\eta\zeta_{12}} & \eta^{\xi_2} - e_{22}e^{-\eta\zeta_{22}} & \cdots & -e_{2l}e^{-\eta\zeta_{2l}}\\ \vdots & \vdots & \ddots & \vdots\\ -e_{l1}e^{-\eta\zeta_{l1}} & -e_{l2}e^{-\eta\zeta_{l2}} & \cdots & \eta^{\xi_l} - e_{ll}e^{-\eta\zeta_{ll}} \end{bmatrix}. \tag{7}$$

Then, the zero solution of system (6) is said to be asymptotically stable in Lyapunov sense provided that every root of $det(\Delta(\eta)) = 0$ owns negative real parts.

3. Exploration on Delay-Induced Hopf Bifurcation

In this part, by discussing the characteristic equation of system (2) and setting the time delay as bifurcation parameter, we will establish the delay-independent sufficient condition to guarantee the stability and the onset of Hopf bifurcation for system (2).

In view of $(H1)$, one can easily know that system (2) owns the unique equilibrium $\mathscr{U}(0,0,0,0,0,0)$. The linear system of system (2) at the zero equilibrium $\mathscr{U}(0,0,0,0,0,0)$ owns the expression:

$$\begin{cases} \frac{d^{\xi}u_1(t)}{dt^{\xi}} = -ku_1(t) + b_{11}(u_4(t-\zeta)) + b_{12}(u_5(t-\zeta)) + b_{13}(u_6(t-\zeta)), \\ \frac{d^{\xi}u_2(t)}{dt^{\xi}} = -ku_2(t) + b_{21}(u_5(t-\zeta)) + b_{22}(u_6(t-\zeta)) + b_{23}(u_4(t-\zeta)), \\ \frac{d^{\xi}u_3(t)}{dt^{\xi}} = -ku_3(t) + b_{31}(u_6(t-\zeta)) + b_{32}(u_4(t-\zeta)) + b_{33}(u_5(t-\zeta)), \\ \frac{d^{\xi}u_4(t)}{dt^{\xi}} = -ku_4(t) + b_{41}(u_1(t-\zeta)) + b_{42}(u_2(t-\zeta)) + b_{43}(u_3(t-\zeta)), \\ \frac{d^{\xi}u_5(t)}{dt^{\xi}} = -ku_5(t) + b_{51}(u_2(t-\zeta)) + b_{52}(u_3(t-\zeta)) + b_{53}(u_1(t-\zeta)), \\ \frac{d^{\xi}u_6(t)}{dt^{\xi}} = -ku_6(t) + b_{61}(u_3(t-\zeta)) + b_{62}(u_1(t-\zeta)) + b_{63}(u_2(t-\zeta)), \end{cases} \tag{8}$$

where $b_{i1} = a_{i1}h'_{i1}(0), b_{i2} = a_{i2}l'(0), b_{i3} = a_{i3}l'(0)$, $i = 1,2,3,4,5,6$. The characteristic equation for equation (8) owns the expression:

$$\det\begin{bmatrix} s^{\xi}+k & 0 & 0 & -b_{11}e^{-s\zeta} & -b_{12}e^{-s\zeta} & -b_{13}e^{-s\zeta} \\ 0 & s^{\xi}+k & 0 & -b_{23}e^{-s\eta} & -b_{21}e^{-s\zeta} & -b_{22}e^{-s\zeta} \\ 0 & 0 & s^{\xi}+k & -b_{32}e^{-s\eta} & -b_{33}e^{-s\zeta} & -b_{31}e^{-s\zeta} \\ -b_{41}e^{-s\zeta} & -b_{42}e^{-s\zeta} & -b_{43}e^{-s\zeta} & s^{\xi}+k & 0 & 0 \\ -b_{53}e^{-s\zeta} & -b_{51}e^{-s\zeta} & -b_{52}e^{-s\zeta} & 0 & s^{\xi}+k & 0 \\ -b_{62}e^{-s\zeta} & -b_{63}e^{-s\zeta} & -b_{61}e^{-s\zeta} & 0 & 0 & s^{\xi}+k \end{bmatrix} = 0. \tag{9}$$

By equation (9), we get

$$\mathscr{U}_1(s) + \mathscr{U}_2(s)e^{-2s\zeta} + \mathscr{U}_3(s)e^{-4s\zeta} + \mathscr{U}_4(s)e^{-6s\zeta} = 0, \tag{10}$$

where

$$\begin{cases} \mathcal{U}_1(s) = s^{6\xi} + \mu_5 s^{5\xi} + \mu_4 s^{4\xi} + \mu_3 s^{3\xi} + \mu_2 s^{2\xi} + \mu_1 s^{\xi} + \mu_0, \\ \mathcal{U}_2(s) = \nu_4 s^{4\xi} + \nu_3 s^{3\xi} + \nu_2 s^{2\xi} + \nu_1 s^{\xi} + \nu_0, \\ \mathcal{U}_3(s) = \chi_2 s^{2\xi} + \chi_1 s^{\xi} + \chi_0, \\ \mathcal{U}_4(s) = \rho_0 = -c_{11}c_{22}c_{33}, \end{cases} \tag{11}$$

where

$$\begin{cases} \mu_0 = k^6 - k^4(c_{13}c_{31} + c_{12}c_{21} + c_{23}c_{32}) - k^3(c_{11}c_{23}c_{31} + c_{13}c_{21}c_{32}), \\ \mu_1 = 6k^2 - 4k^3(c_{13}c_{31} + c_{12}c_{21} + c_{23}c_{32}) - 3k^2(c_{11}c_{23}c_{31} + c_{13}c_{21}c_{32}) \\ +2k[c_{33}(c_{11} + c_{22}) + c_{11}c_{22}], \\ \mu_2 = 9k^4 - 6k^2(c_{13}c_{31} + c_{12}c_{21} + c_{23}c_{32}) - 3k(c_{11}c_{23}c_{31} + c_{13}c_{21}c_{32}) \\ +c_{33}(c_{11} + c_{22}) + c_{11}c_{22}, \\ \mu_3 = 26k^3 - 4k(c_{13}c_{31} + c_{12}c_{21} + c_{23}c_{32}) - (c_{11}c_{23}c_{31} + c_{13}c_{21}c_{32}), \\ \mu_4 = 15k^2 - (c_{13}c_{31} + c_{12}c_{21} + c_{23}c_{32}), \\ \mu_5 = 6k, \\ \nu_0 = k^2(c_{11}c_{31}c_{22} + c_{12}c_{21}c_{33} + c_{11}c_{23}c_{32}) - k^4(c_{11} + c_{22} + c_{33}), \\ \nu_1 = 2k(c_{11}c_{31}c_{22} + c_{12}c_{21}c_{33} + c_{11}c_{23}c_{32}) - 4k^3(c_{11} + c_{22} + c_{33}), \\ \nu_2 = c_{11}c_{31}c_{22} + c_{12}c_{21}c_{33} + c_{11}c_{23}c_{32} - 6k^2, \\ \nu_3 = -4k(c_{11} + c_{22} + c_{33}), \\ \nu_4 = -(c_{11} + c_{22} + c_{33}), \\ \chi_0 = k^2[c_{33}(c_{11} + c_{22}) + c_{11}c_{22}], \\ \chi_1 = 2k[c_{33}(c_{11} + c_{22}) + c_{11}c_{22}], \\ \chi_2 = c_{33}(c_{11} + c_{22}) + c_{11}c_{22}, \end{cases} \tag{12}$$

where

$$\begin{cases} c_{11} = b_{41}b_{11} + b_{42}b_{23} + b_{43}b_{32}, \\ c_{12} = b_{41}b_{12} + b_{42}b_{21} + b_{43}b_{33}, \\ c_{13} = b_{41}b_{13} + b_{42}b_{22} + b_{43}b_{31}, \\ c_{21} = b_{53}b_{11} + b_{51}b_{23} + b_{52}b_{32}, \\ c_{22} = b_{53}b_{12} + b_{51}b_{21} + b_{52}b_{33}, \\ c_{23} = b_{53}b_{13} + b_{51}b_{22} + b_{52}b_{31}, \\ c_{31} = b_{62}b_{11} + b_{63}b_{23} + b_{61}b_{32}, \\ c_{32} = b_{62}b_{12} + b_{63}b_{21} + b_{61}b_{33}, \\ c_{33} = b_{62}b_{13} + b_{63}b_{22} + b_{61}b_{31}. \end{cases} \tag{13}$$

By virtue of (10), we get

$$\mathcal{U}_1(s)e^{4s\zeta} + \mathcal{U}_2(s)e^{2s\zeta} + \mathcal{U}_3(s) + \mathcal{U}_4(s)e^{-2s\zeta} = 0. \tag{14}$$

Assume that $s = i\vartheta = \vartheta(\cos(\pi/2) + i\sin(\pi/2))$ is the root of (14) and denote the real parts and imaginary parts of $\mathcal{U}_j(s)\,(j = 1, 2, 3, 4)$ by $\mathcal{U}_{jR}(s)$ and $\mathcal{U}_{jI}(s)\,(j = 1, 2, 3, 4)$, respectively. It follows from (14) that

$$\begin{cases} \mathcal{U}_{1R}(\vartheta)\cos 4\vartheta\zeta - \mathcal{U}_{1I}(\vartheta)\sin 4\vartheta\zeta + [\mathcal{U}_{2R}(\vartheta) + \mathcal{U}_{4R}(\vartheta)]\cos 2\vartheta\zeta \\ +[\mathcal{U}_{4I}(\vartheta) - \mathcal{U}_{2I}(\vartheta)]\sin 2\vartheta\zeta = -\mathcal{U}_{3R}(\vartheta), \\ \mathcal{U}_{1I}(\vartheta)\cos 4\vartheta\zeta + \mathcal{U}_{1R}(\vartheta)\sin 4\vartheta\zeta + [\mathcal{U}_{2I}(\vartheta) + \mathcal{U}_{4I}(\vartheta)]\cos 2\vartheta\zeta \\ +[\mathcal{U}_{2R}(\vartheta) - \mathcal{U}_{4R}(\vartheta)]\sin 2\,\vartheta\zeta = -\mathcal{U}_{3I}(\vartheta), \end{cases} \tag{15}$$

where

$$\begin{cases} \mathscr{U}_{1R}(\vartheta) = \vartheta^{6\xi}\cos 3\,\xi\pi + \mu_5\vartheta^{5\xi}\cos\dfrac{5\xi\pi}{2} + \mu_4\vartheta^{4\xi}\cos 2\,\xi\pi \\ +\mu_3\vartheta^{3\xi}\cos\dfrac{3\xi\pi}{2} + \mu_2\vartheta^{2\xi}\cos\xi\pi + \mu_1\vartheta^{\xi}\cos\dfrac{\xi\pi}{2} + \mu_0, \\ \mathscr{U}_{1I}(\vartheta) = \vartheta^{6\xi}\sin 3\,\xi\pi + \mu_5\vartheta^{5\xi}\sin\dfrac{5\xi\pi}{2} + \mu_4\vartheta^{4\xi}\sin 2\,\xi\pi \\ +\mu_3\vartheta^{3\xi}\sin\dfrac{3\xi\pi}{2} + \mu_2\vartheta^{2\xi}\sin\xi\pi + \mu_1\vartheta^{\xi}\sin\dfrac{\xi\pi}{2}, \\ \mathscr{U}_{2R}(\vartheta) = \nu_4\vartheta^{4\xi}\cos 2\,\xi\pi + \nu_3\vartheta^{3\xi}\cos\dfrac{3\xi\pi}{2} + \nu_2\vartheta^{2\xi}\cos\xi\pi + \nu_1\vartheta^{\xi}\cos\dfrac{\xi\pi}{2} + \nu_0, \\ \mathscr{U}_{2I}(\vartheta) = \nu_4\vartheta^{4\xi}\sin 2\,\xi\pi + \nu_3\vartheta^{3\xi}\sin\dfrac{3\xi\pi}{2} + \nu_2\vartheta^{2\xi}\sin\xi\pi + \nu_1\vartheta^{\xi}\sin\dfrac{\xi\pi}{2}, \\ \mathscr{U}_{3R}(\vartheta) = \chi_2\vartheta^{2\xi}\cos\xi\pi + \chi_1\vartheta^{\xi}\cos\dfrac{\xi\pi}{2} + \chi_0, \\ \mathscr{U}_{3I}(\vartheta) = \chi_2\vartheta^{2\xi}\sin\xi\pi + \chi_1\vartheta^{\xi}\sin\dfrac{\xi\pi}{2}, \\ \mathscr{U}_{4R}(\vartheta) = \rho_0, \\ \mathscr{U}_{4I}(\vartheta) = 0. \end{cases} \tag{16}$$

In view of (16), we can rewrite (15) as

$$\begin{cases} \mathscr{U}_{1R}(\vartheta)\cos 4\,\vartheta\zeta - \mathscr{U}_{1I}(\vartheta)\sin 4\,\vartheta\zeta + \left[\mathscr{U}_{2R}(\vartheta) + \mathscr{U}_{4R}(\vartheta)\right]\cos 2\,\vartheta\zeta - \mathscr{U}_{2I}(\vartheta)\sin 2\,\vartheta\zeta = -\mathscr{U}_{3R}(\vartheta), \\ \mathscr{U}_{1I}(\vartheta)\cos 4\,\vartheta\zeta + \mathscr{U}_{1R}(\vartheta)\sin 4\,\vartheta\zeta + \mathscr{U}_{2I}(\vartheta) + \cos 2\,\vartheta\zeta + \left[\mathscr{U}_{2R}(\vartheta) - \mathscr{U}_{4R}(\vartheta)\right]\sin 2\,\vartheta\zeta = -\mathscr{U}_{3I}(\vartheta), \end{cases} \tag{17}$$

According to $\sin 2\,\vartheta\zeta = \pm\sqrt{1 - \cos^2 2\vartheta\zeta}$, we are to deal with two cases.

(i) If $\sin 2\,\vartheta\zeta = \sqrt{1 - \cos^2 2\vartheta\zeta}$, it follows from the first equation of (17) that

$$2\mathscr{U}_{1R}(\vartheta)\left(2\cos^2 2\vartheta\zeta - 1\right) - 2\mathscr{U}_{1I}(\vartheta)\cos 2\,\vartheta\zeta\sqrt{1 - \cos^2 2\vartheta\zeta} + \left[\mathscr{U}_{2R}(\vartheta) + \mathscr{U}_{4R}(\vartheta)\right]\cos 2\,\vartheta\zeta - \mathscr{U}_{2I}(\vartheta)\sqrt{1 - \cos^2 2\vartheta\zeta} = -\mathscr{U}_{3R}(\vartheta), \tag{18}$$

which leads to

$$\left[2\mathcal{U}_{1R}(\vartheta)\left(2\cos^2 2\vartheta\zeta-1\right)+\left[\mathcal{U}_{2R}(\vartheta)+\mathcal{U}_{4R}(\vartheta)\right]\cos 2\vartheta\zeta+\mathcal{U}_{3R}(\vartheta)\right]^2-\left[2\mathcal{U}_{1I}(\vartheta)\cos 2\vartheta\zeta-\mathcal{U}_{2I}(\vartheta)\right]^2\left(1-\cos^2 2\vartheta\zeta\right). \tag{19}$$

Then, one gets

$$\rho_1\cos^4 2\vartheta\zeta+\rho_2\cos^3 2\vartheta\zeta+\rho_3\cos^2 2\vartheta\zeta+\rho_4\cos 2\vartheta\zeta+\rho_5=0, \tag{20}$$

where

$$\begin{cases}\rho_1=16\mathcal{U}_{1R}^2(\vartheta)+4\mathcal{U}_{1I}^2(\vartheta),\\ \rho_2=8\mathcal{U}_{1R}(\vartheta)\left(\mathcal{U}_{2R}(\vartheta)+\mathcal{U}_{4R}(\vartheta)\right)+4\mathcal{U}_{1I}(\vartheta)\mathcal{U}_{2I}(\vartheta),\\ \rho_3=\left(\mathcal{U}_{2R}(\vartheta)+\mathcal{U}_{4R}(\vartheta)\right)^2+8\mathcal{U}_{1R}(\vartheta)\left(\mathcal{U}_{3R}(\vartheta)-2\mathcal{U}_{1R}(\vartheta)\right)+\mathcal{U}_{2I}^2(\vartheta)-4\mathcal{U}_{1R}^2(\vartheta),\\ \rho_4=2\left(\mathcal{U}_{2R}(\vartheta)+\mathcal{U}_{4R}(\vartheta)\right)\left(\mathcal{U}_{3R}(\vartheta)-\mathcal{U}_{1R}(\vartheta)\right)-4\mathcal{U}_{1I}(\vartheta)\mathcal{U}_{2I}(\vartheta),\\ \rho_5=\left(\mathcal{U}_{3R}(\vartheta)-2\mathcal{U}_{1R}(\vartheta)\right)^2-\mathcal{U}_{2I}^2(\vartheta).\end{cases} \tag{21}$$

Suppose that $\cos 2\vartheta\zeta=\eta$ and set

$$h(\eta)=\eta^4+\frac{\rho_2}{\rho_1}\eta^3+\frac{\rho_3}{\rho_1}\eta^2+\frac{\rho_4}{\rho_1}\eta+\frac{\rho_5}{\rho_1}, \tag{22}$$

then

$$\frac{dh(\eta)}{d\eta}=4\eta^3+\frac{3\rho_2}{\rho_1}\eta^2+\frac{2\rho_3}{\rho_1}\eta+\frac{\rho_4}{\rho_1}. \tag{23}$$

Let

$$4\eta^3+\frac{3\rho_2}{\rho_1}\eta^2+\frac{2\rho_3}{\rho_1}\eta+\frac{\rho_4}{\rho_1}=0. \tag{24}$$

Assume that $y=\eta+(\rho_2/4\rho_1)$, then (24) can be expressed as

$$y^3+r_1y+r_2=0, \tag{25}$$

where

$$r_1=\frac{\rho_3}{2\rho_1}-\frac{3\rho_2^2}{16\rho_1^2},\qquad r_2=\frac{\rho_2^3}{32\rho_1^3}-\frac{\rho_2\rho_3}{8\rho_1^2}+\frac{\rho_4}{4\rho_1}. \tag{26}$$

Denote

$$\delta_1=\left(\frac{r_2}{2}\right)^2+\left(\frac{r_1}{3}\right)^3,\qquad \delta_2=\frac{-1+i\sqrt{3}}{2}. \tag{27}$$

By (25), one gets

$$\begin{cases}y_1=\sqrt[3]{-\frac{r_2}{2}+\sqrt{\delta_1}}+\sqrt[3]{-\frac{r_2}{2}-\sqrt{\delta_1}},\\ y_2=\sqrt[3]{-\frac{r_2}{2}+\sqrt{\delta_1}}\,\delta_2+\sqrt[3]{-\frac{r_2}{2}-\sqrt{\delta_1}}\,\delta_2^2,\\ y_3=\sqrt[3]{-\frac{r_2}{2}+\sqrt{\delta_1}}\,\delta_2^2+\sqrt[3]{-\frac{r_2}{2}-\sqrt{\delta_1}}\,\delta_2.\end{cases} \tag{28}$$

According to the analysis above, one can obtain the expression of $\cos 2\vartheta\zeta$. Then, one can derive the expression of $\sin 2\vartheta\zeta$. Here, we suppose that

$$\cos 2\vartheta\zeta=\varphi_1(\vartheta),\qquad \sin 2\vartheta\zeta=\varphi_2(\vartheta). \tag{29}$$

Hence,

$$\varphi_1^2(\vartheta)+\varphi_2^2(\vartheta)=1. \tag{30}$$

By virtue of computer software, one can easily derive the root (say ϑ) of (30). Thus, one has

$$\zeta^{1l}=\frac{1}{2\vartheta}\left[\arccos\varphi_1(\vartheta)+2l\pi\right],\quad l=0,1,2,\ldots. \tag{31}$$

(ii) If $\sin 2\vartheta\zeta=-\sqrt{1-\cos^2 2\vartheta\zeta}$, by means of the same method, one can also derive

$$\cos 2\vartheta\zeta=\psi_1(\vartheta),\qquad \sin 2\vartheta\zeta=\psi_2(\vartheta). \tag{32}$$

Then,

$$\psi_1^2(\vartheta) + \psi_2^2(\vartheta) = 1. \tag{33}$$

By virtue of computer software, we can derive the root (say ϑ) of (33). Then,

$$\zeta^{2k} = \frac{1}{2\vartheta}\left[\arccos\ \psi_1(\vartheta) + 2k\pi\right], \quad k = 0, 1, 2, \ldots. \tag{34}$$

Define

$$\zeta_0 = \min\left\{\zeta^{1l}, \zeta^{2l}\right\}, \quad l = 0, 1, 2, \ldots. \tag{35}$$

In the sequel, we are to verify the transversality condition to ensure the onset of Hopf bifurcation. The following hypothesis is needed. $(H2)$ $\mathscr{A}_R^*\mathscr{B}_R^* + \mathscr{A}_I^*\mathscr{B}_I^* > 0$, where

$$\begin{aligned}
\mathscr{A}_R^* ={}& \left[6\xi\vartheta_0^{6\xi-1}\cos\frac{(6\xi-1)\pi}{2} + 5\xi\mu_5\vartheta_0^{5\xi-1}\cos\frac{(5\xi-1)\pi}{2} + 4\xi\mu_4\vartheta_0^{4\xi-1} \times \cos\frac{(4\xi-1)\pi}{2} + 3\xi\mu_3\vartheta_0^{3\xi-1}\cos\frac{(3\xi-1)\pi}{2}\right.\\
&\left. + 2\xi\mu_2\vartheta_0^{2\xi-1}\cos\frac{(2\xi-1)\pi}{2} + \xi\mu_1\vartheta_0^{\xi-1}\cos\frac{(\xi-1)\pi}{2}\right]\\
&+\left[4\xi\nu_4\vartheta_0^{4\xi-1}\cos\frac{(4\xi-1)\pi}{2} + 3\xi\nu_3\vartheta_0^{3\xi-1}\cos\frac{(3\xi-1)\pi}{2} + 2\xi\nu_2\vartheta_0^{2\xi-1}\cos\frac{(2\xi-1)\pi}{2} + \xi\nu_1\vartheta_0^{\xi-1}\cos\frac{(\xi-1)\pi}{2}\right]\cos 2\vartheta_0\zeta_0\\
&+\left[4\xi\nu_4\vartheta_0^{4\xi-1}\sin\frac{(4\xi-1)\pi}{2} + 3\xi\nu_3\vartheta_0^{3\xi-1}\sin\frac{(3\xi-1)\pi}{2} + 2\xi\nu_2\vartheta_0^{2\xi-1}\sin\frac{(2\xi-1)\pi}{2} + \xi\nu_1\vartheta_0^{\xi-1}\sin\frac{(\xi-1)\pi}{2}\right]\sin 2\vartheta_0\zeta_0\\
&+\left[2\xi\chi_2\vartheta_0^{2\xi-1}\cos\frac{(2\xi-1)\pi}{2} + \xi\chi_1\vartheta_0^{\xi-1}\cos\frac{(\xi-1)\pi}{2}\right]\cos 4\vartheta_0\zeta_0 + \left[2\xi\chi_2\vartheta_0^{2\xi-1}\sin\frac{(2\xi-1)\pi}{2} + \xi\chi_1\vartheta_0^{\xi-1}\sin\frac{(\xi-1)\pi}{2}\right]\sin 4\vartheta_0\zeta_0,\\
\mathscr{A}_I^* ={}& \left[6\xi\vartheta_0^{6\xi-1}\sin\frac{(6\xi-1)\pi}{2} + 5\xi\mu_5\vartheta_0^{5\xi-1}\sin\frac{(5\xi-1)\pi}{2} + 4\xi\mu_4\vartheta_0^{4\xi-1} \times \sin\frac{(4\xi-1)\pi}{2} + 3\xi\mu_3\vartheta_0^{3\xi-1}\sin\frac{(3\xi-1)\pi}{2}\right.\\
&\left. + 2\xi\mu_2\vartheta_0^{2\xi-1}\sin\frac{(2\xi-1)\pi}{2} + \xi\mu_1\vartheta_0^{\xi-1}\sin\frac{(\xi-1)\pi}{2}\right]\\
&+\left[4\xi\nu_4\vartheta_0^{4\xi-1}\cos\frac{(4\xi-1)\pi}{2} + 3\xi\nu_3\vartheta_0^{3\xi-1}\cos\frac{(3\xi-1)\pi}{2} + 2\xi\nu_2\vartheta_0^{2\xi-1}\cos\frac{(2\xi-1)\pi}{2} + \xi\nu_1\vartheta_0^{\xi-1}\cos\frac{(\xi-1)\pi}{2}\right]\sin 2\vartheta_0\zeta_0\\
&-\left[4\xi\nu_4\vartheta_0^{4\xi-1}\sin\frac{(4\xi-1)\pi}{2} + 3\xi\nu_3\vartheta_0^{3\xi-1}\sin\frac{(3\xi-1)\pi}{2} + 2\xi\nu_2\vartheta_0^{2\xi-1}\sin\frac{(2\xi-1)\pi}{2} + \xi\nu_1\vartheta_0^{\xi-1}\sin\frac{(\xi-1)\pi}{2}\right]\cos 2\vartheta_0\zeta_0\\
&-\left[2\xi\chi_2\vartheta_0^{2\xi-1}\cos\frac{(2\xi-1)\pi}{2} + \xi\chi_1\vartheta_0^{\xi-1}\cos\frac{(\xi-1)\pi}{2}\right]\sin 4\vartheta_0\zeta_0 + \left[2\xi\chi_2\vartheta_0^{2\xi-1}\sin\frac{(2\xi-1)\pi}{2} + \xi\chi_1\vartheta_0^{\xi-1}\sin\frac{(\xi-1)\pi}{2}\right]\cos 4\vartheta_0\zeta_0,\\
\mathscr{B}_R^* ={}& 2\vartheta_0\left(\nu_4\vartheta_0^{4\xi}\cos 2\ \xi\pi + \nu_3\vartheta_0^{3\xi}\cos\frac{3\xi\pi}{2} + \nu_2\vartheta_0^{2\xi}\cos\ \xi\pi + \nu_1\vartheta_0^{\xi}\cos\frac{\xi\pi}{2} + \nu_0\right)\sin 2\vartheta_0\zeta_0\\
&+ 2\vartheta_0\left(\nu_4\vartheta_0^{4\xi}\sin 2\ \xi\pi + \nu_3\vartheta_0^{3\xi}\sin\frac{3\xi\pi}{2} + \nu_2\vartheta_0^{2\xi}\sin\ \xi\pi + \nu_1\vartheta_0^{\xi}\sin\frac{\xi\pi}{2} + \nu_0\right)\cos 2\vartheta_0\zeta_0,\\
\mathscr{B}_I^* ={}& 2\vartheta_0\left(\nu_4\vartheta_0^{4\xi}\cos 2\ \xi\pi + \nu_3\vartheta_0^{3\xi}\cos\frac{3\xi\pi}{2} + \nu_2\vartheta_0^{2\xi}\cos\ \xi\pi + \nu_1\vartheta_0^{\xi}\cos\frac{\xi\pi}{2} + \nu_0\right)\cos 2\vartheta_0\zeta_0\\
&+ 2\vartheta_0\left(\nu_4\vartheta_0^{4\xi}\sin 2\ \xi\pi + \nu_3\vartheta_0^{3\xi}\sin\frac{3\xi\pi}{2} + \nu_2\vartheta_0^{2\xi}\sin\ \xi\pi + \nu_1\vartheta_0^{\xi}\sin\frac{\xi\pi}{2} + \nu_0\right)\sin 2\vartheta_0\zeta_0.
\end{aligned} \tag{36}$$

Lemma 3. *Assume that $s(\zeta) = \phi_1(\zeta) + i\phi_2(\zeta)$ is the root of (10) at $\zeta = \zeta_0$ and $\phi_1(\zeta_0) = 0, \phi_2(\zeta_0) = \vartheta_0$, then $Re[(ds/d\zeta)]_{\zeta=\zeta_0,\vartheta=\vartheta_0} > 0$.*

Proof. By virtue of (10), we get

$$\begin{aligned}&\frac{d\mathcal{U}_1(s)}{d\zeta} + \frac{d\mathcal{U}_2(s)}{d\zeta}e^{-2s\zeta} - 2e^{-2s\zeta}\left(\frac{ds}{d\zeta}\zeta + s\right)\mathcal{U}_2(s) + \frac{d\mathcal{U}_3(s)}{d\zeta}e^{-4s\zeta},\\ &-4e^{-4s\zeta}\left(\frac{ds}{d\zeta}\zeta + s\right)\mathcal{U}_3(s) + \frac{d\mathcal{U}_4(s)}{d\zeta}e^{-6s\zeta} - 6e^{-6s\zeta}\left(\frac{ds}{d\zeta}\zeta + s\right)\mathcal{U}_4(s) = 0.\end{aligned} \tag{37}$$

Since

$$\begin{cases}\dfrac{d\mathcal{U}_1(s)}{d\zeta} = \left[6\xi s^{6\xi-1} + 5\xi\mu_5 s^{5\xi-1} + 4\xi\mu_4 s^{4\xi-1} + 3\xi\mu_3 s^{3\xi-1} + 2\xi\mu_2 s^{2\xi-1} + \xi\mu_1 s^{\xi-1}\right]\dfrac{ds}{d\zeta},\\ \dfrac{d\mathcal{U}_2(s)}{d\zeta} = \left[4\xi\nu_4 s^{4\xi-1} + 3\xi\nu_3 s^{3\xi-1} + 2\xi\nu_2 s^{2\xi-1} + \xi\nu_1 s^{\xi-1}\right]\dfrac{ds}{d\xi},\\ \dfrac{d\mathcal{U}_3(s)}{d\zeta} = \left[2\xi\chi_2 s^{2\xi-1} + \xi\chi_1 s^{\xi-1}\right]\dfrac{ds}{d\zeta},\\ \dfrac{d\mathcal{U}_4(s)}{d\zeta} = 0,\end{cases} \tag{38}$$

then by (37) and (38), one gets

$$\left[\frac{ds}{d\zeta}\right]^{-1} = \frac{\mathcal{A}(s)}{\mathcal{B}(s)} - \frac{\zeta}{s}, \tag{39}$$

where

$$\begin{cases}\mathcal{A}(s) = 6\xi s^{6\xi-1} + 5\xi\mu_5 s^{5\xi-1} + 4\xi\mu_4 s^{4\xi-1} + 3\xi\mu_3 s^{3\xi-1} + 2\xi\mu_2 s^{2\xi-1}\\ +\xi\mu_1 s^{\xi-1} + \left(4\xi\nu_4 s^{4\xi-1} + 3\xi\nu_3 s^{3\xi-1} + 2\xi\nu_2 s^{2\xi-1} + \xi\nu_1 s^{\xi-1}\right)e^{-2s\zeta} + \left[2\xi\chi_2 s^{2\xi-1} + \xi\chi_1 s^{\xi-1}\right]e^{-4s\zeta},\\ \mathcal{B}(s) = 2se^{-2s\zeta}\left(\nu_4 s^{4\xi} + \nu_3 s^{3\xi} + \nu_2 s^{2\xi} + \nu_1 s^{\xi} + \nu_0\right)\\ +4s\left(\chi_2 s^{2\xi} + \chi_1 s^{\xi} + \chi_0\right)e^{-4s\zeta} + 6s\rho_0 e^{-6s\zeta}.\end{cases} \tag{40}$$

It follows from $(H2)$ that

$$\operatorname{Re}\left\{\left[\frac{ds}{d\zeta}\right]^{-1}\right\}\Big|_{\zeta=\zeta_0,\vartheta=\vartheta_0} = \frac{\mathscr{A}_R^*\mathscr{B}_R^* + \mathscr{A}_I^*\mathscr{B}_I^*}{(\mathscr{A}_R^*)^2 + (\mathscr{B}_I^*)^2} > 0. \quad (41)$$

This completes the proof.
Let

$$\begin{cases} \tau_1 = \mu_5, \\ \tau_2 = \mu_4 + \nu_4, \\ \tau_3 = \mu3 + \nu_3, \\ \tau_4 = \mu_2 + \nu_2 + \chi_2, \\ \tau_5 = \mu_1 + \nu_1 + \chi_1, \\ \tau_6 = \mu_0 + \nu_0 + \chi_0 + \rho_0. \end{cases} \quad (42)$$

Next, the following assumption is needed:
$(H3)$ the following inequalities are true:

$$\begin{cases} \mathscr{G}_1 = \tau_1 > 0, \\ \mathscr{G}_2 = \det\begin{bmatrix} \tau_1 & 1 \\ \tau_3 & \tau_2 \end{bmatrix} > 0, \\ \mathscr{G}_3 = \det\begin{bmatrix} \tau_1 & 1 & 0 \\ \tau_3 & \tau_2 & \tau_1 \\ \tau_5 & \tau_4 & \tau_3 \end{bmatrix} > 0, \\ \mathscr{G}_4 = \det\begin{bmatrix} \tau_1 & 1 & 0 & 0 \\ \tau_3 & \tau_2 & \tau_1 & 1 \\ \tau_5 & \tau_4 & \tau_3 & \tau_2 \\ 0 & \tau_6 & \tau_5 & \tau_4 \end{bmatrix} > 0, \\ \mathscr{G}_5 = \det\begin{bmatrix} \tau_1 & 1 & 0 & 0 & 0 \\ \tau_3 & \tau_2 & \tau_1 & 1 & 0 \\ \tau_5 & \tau_4 & \tau_3 & \tau_2 & \tau_1 \\ 0 & \tau_6 & \tau_5 & \tau_4 & \tau_3 \\ 0 & 0 & 0 & \tau_6 & \tau_5 \end{bmatrix} > 0, \\ \mathscr{G}_6 = \tau_6 > 0. \end{cases} \quad (43)$$

Lemma 4. *If $\zeta = 0$ and $(\mathscr{H}3)$ is fulfilled, then system (2) is locally asymptotically stable.*

Proof. Obviously, (10) with $\zeta = 0$ owns the following expression:

$$\mathscr{U}_1(s) + \mathscr{U}_2(s) + \mathscr{U}_3(s) + \mathscr{U}_4(s) = 0. \quad (44)$$

Namely,

$$\lambda^6 + \tau_1\lambda^5 + \tau_2\lambda^4 + \tau_3\lambda^3 + \tau_4\lambda^2 + \tau_5\lambda + \tau_6 = 0. \quad (45)$$

By means of $(H3)$, one knows that every root λ_i of (45) satisfies $|\arg(\lambda_i)| > (\xi\pi/2)(i = 1, 2, \ldots, 6)$. So, we can obtain that Lemma 3 holds. This ends the proof.

According to the study above, the following result is built.

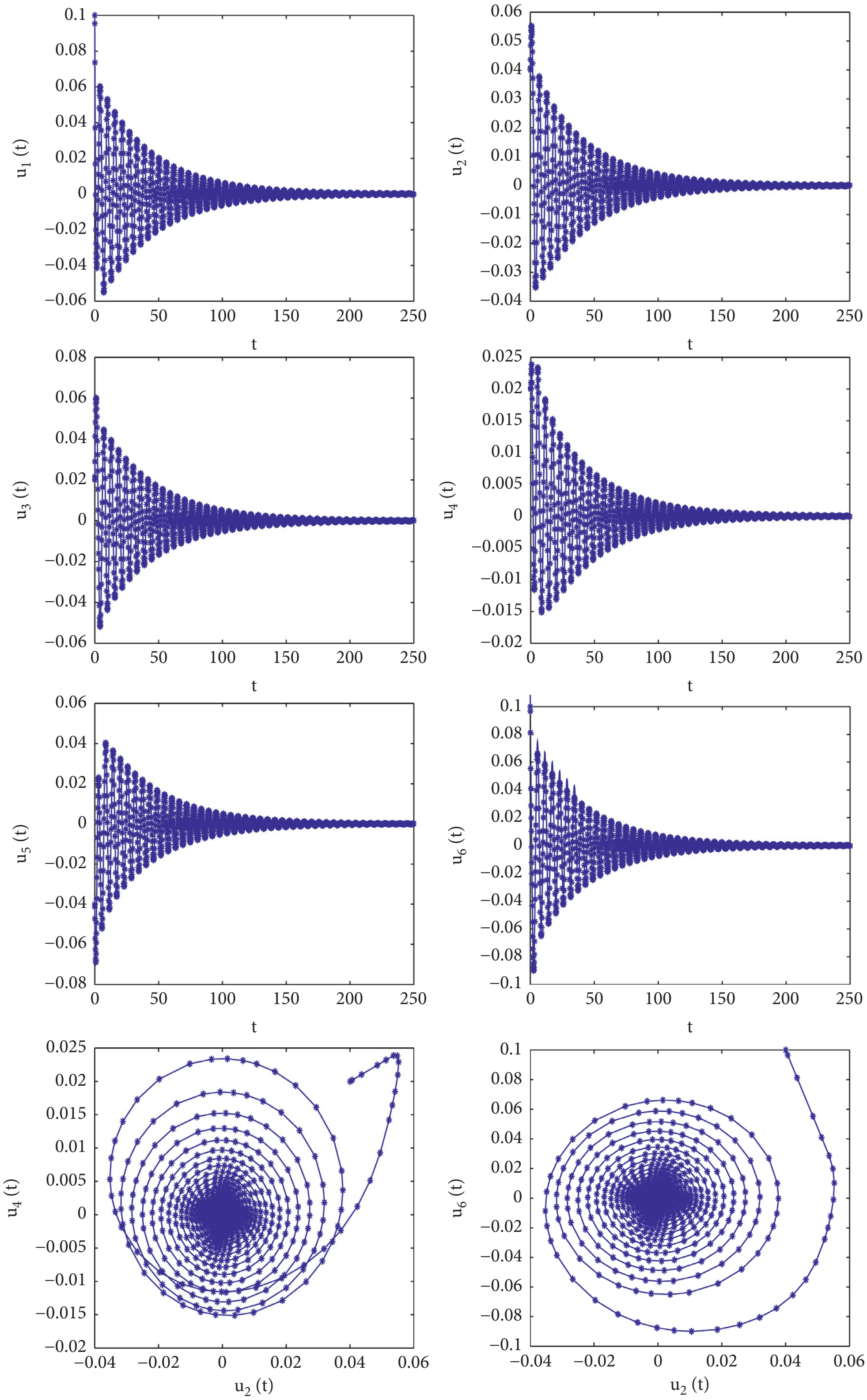
Theorem 1. *If $(H1)$–$(H3)$ hold true, then the equilibrium point $\mathscr{U}(0,0,0,0,0,0)$ of system (2) is locally asymptotically stable proved that $\zeta \in [0, \zeta_0)$ and a Hopf bifurcation is to arise around $\mathscr{U}(0,0,0,0,0,0)$ if $\zeta = \zeta_0$.*

Remark 1. Theorem 1 shows that ζ_0 is a critical value which determines whether system (2) is stable or unstable. If $\zeta < \zeta_0$, then system (2) is stable, and if $\zeta > \zeta_0$, then system (2) becomes unstable and a family of periodic solutions will appear near $\mathscr{U}(0,0,0,0,0,0)$.

Remark 2. In [3, 4], Cheng et al. studied the stability and Hopf bifurcation of integer-order delayed neural networks. They obtain the characteristic equation by applying integer-order differential equation theory and determinant knowledge. In this work, we investigate the stability and Hopf bifurcation of fractional-order delayed neural networks. We obtain the characteristic equation by applying fractional-order differential equation theory, Laplace transform, and determinant knowledge. The investigation on the distribution of the characteristic roots for characteristic equation of fractional-order neural networks is more difficult that of integer-order case. From this viewpoint, we think that our work replenishes and improves the earlier works of Cheng et al. [3, 4].

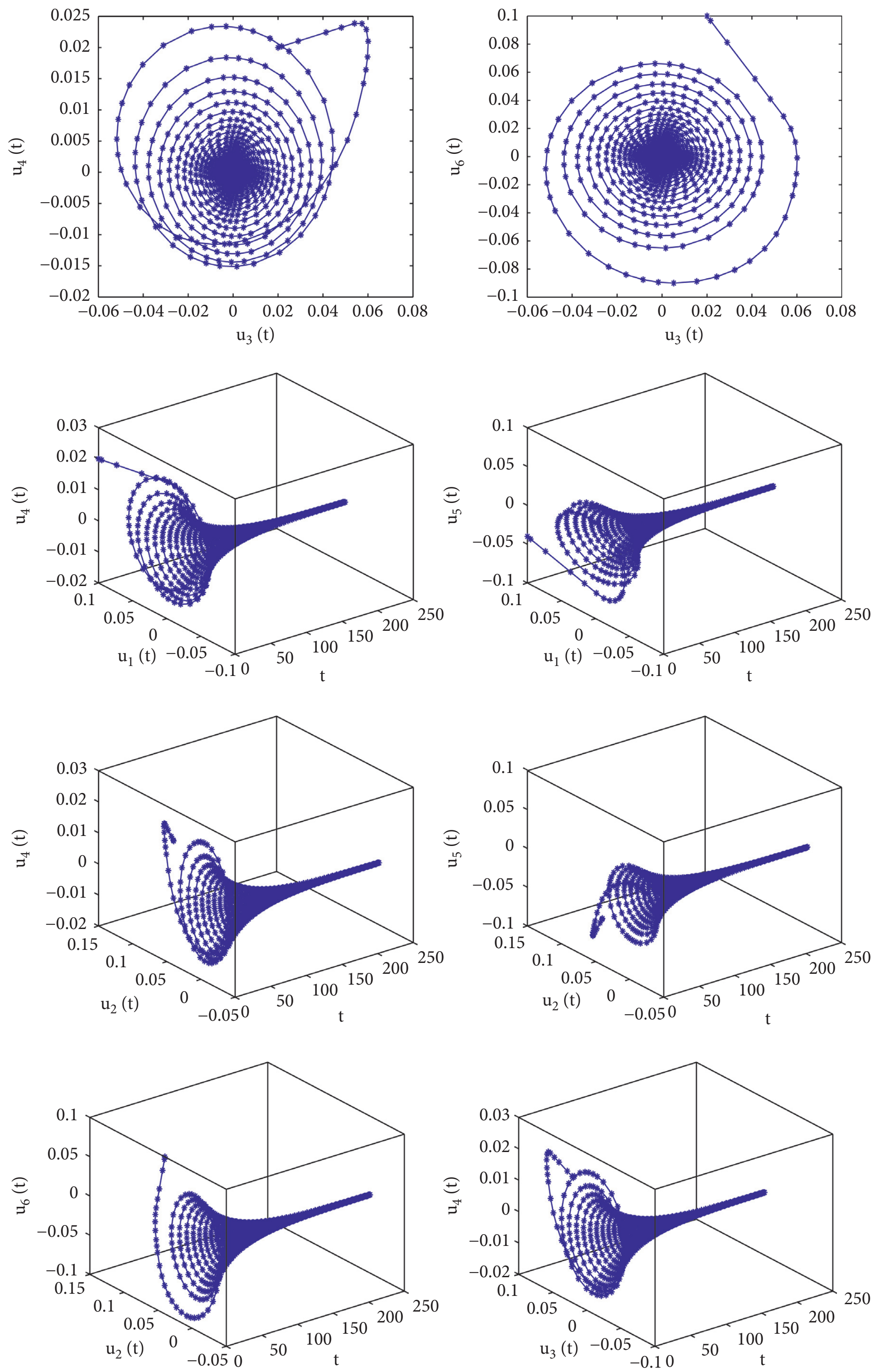
4. Software Simulation Plots

Give the fractional-order neural network system:



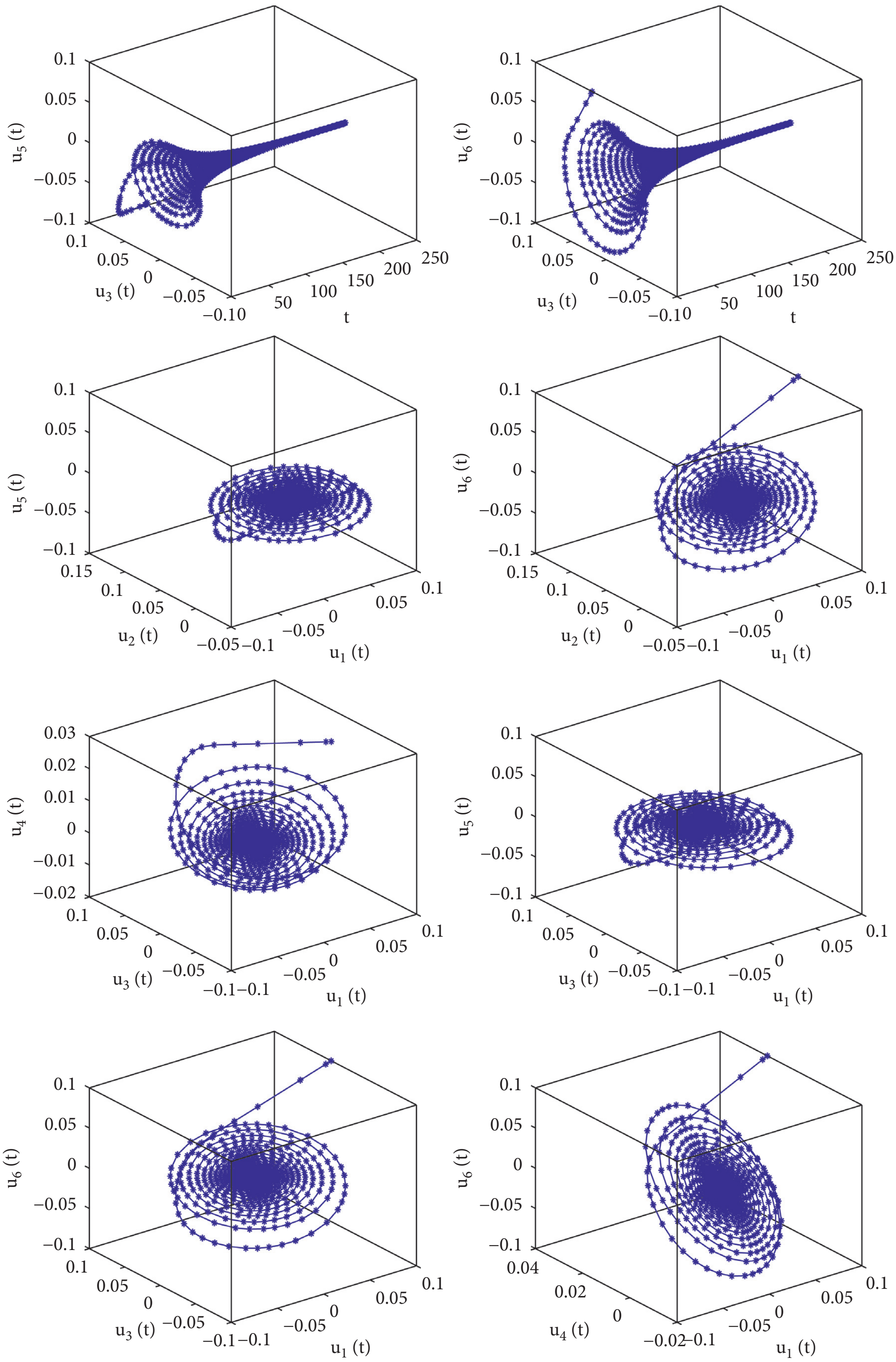
(a)

Figure 1: Continued.



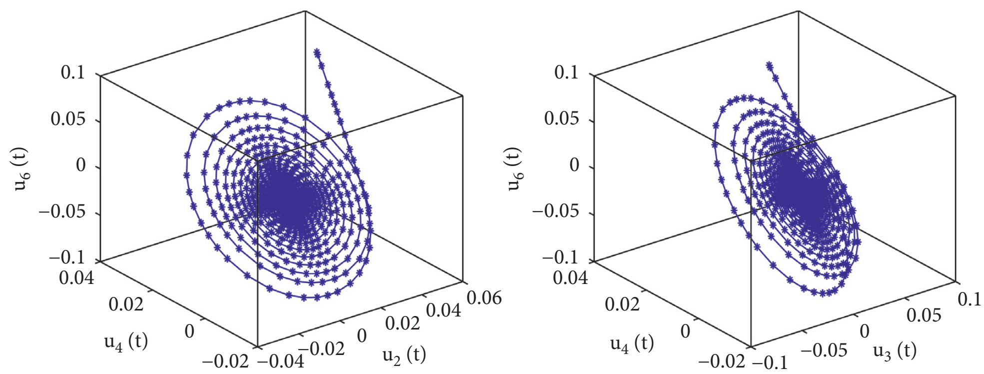
(b)

Figure 1: Continued.



(c)

FIGURE 1: Continued.



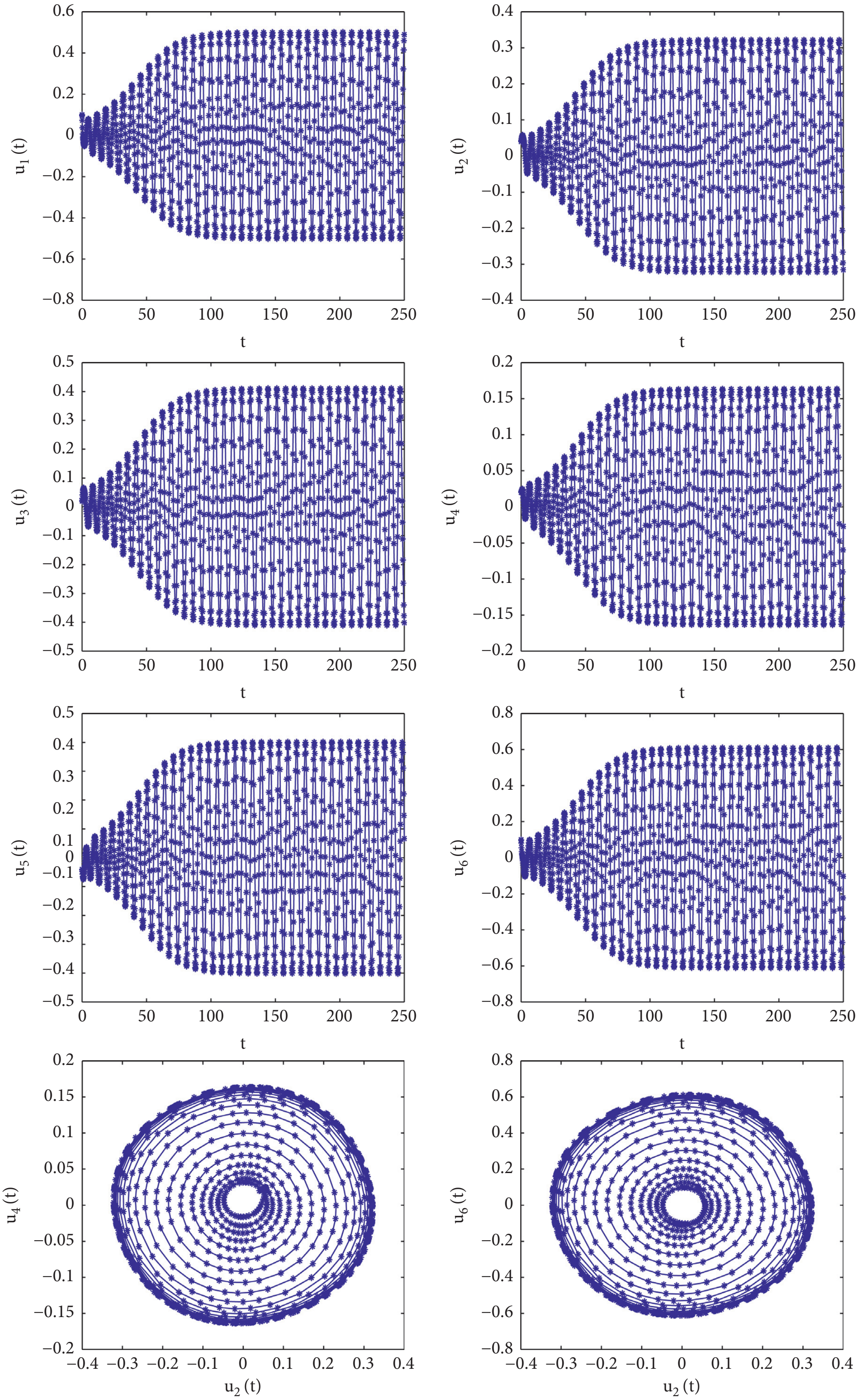
(d)

Figure 1: Stability property for neural network model (46) involving $\zeta = 0.65 < \zeta_0 = 0.75$.

$$\begin{cases} \dfrac{d^{\xi}u_1(t)}{dt^{\xi}} = -u_1(t) + 0.8\tanh(u_4(t-\zeta)) + 0.8\tanh(u_5(t-\zeta)) - 0.9\tanh(u_6(t-\zeta)), \\ \dfrac{d^{\xi}u_2(t)}{dt^{\xi}} = -u_2(t) + 0.5\tanh(u_5(t-\zeta)) - 0.2\tanh(u_6(t-\zeta)) + 0.5\tanh(u_4(t-\zeta)), \\ \dfrac{d^{\xi}u_3(t)}{dt^{\xi}} = -u_3(t) - 0.5\tanh(u_6(t-\zeta)) - 0.5\tanh(u_4(t-\zeta)) + 0.8\tanh(u_5(t-\zeta)), \\ \dfrac{d^{\xi}u_4(t)}{dt^{\xi}} = -u_4(t) + 0.2\tanh(u_1(t-\zeta)) + 0.6\tanh(u_2(t-\zeta)) - 0.8\tanh(u_3(t-\zeta)), \\ \dfrac{d^{\xi}u_5(t)}{dt^{\xi}} = -u_5(t) - 0.9\tanh(u_2(t-\zeta)) - 0.5\tanh(u_3(t-\zeta)) + 0.7\tanh(u_1(t-\zeta)), \\ \dfrac{d^{\xi}u_6(t)}{dt^{\xi}} = -u_6(t) + 0.2\tanh(u_3(t-\zeta)) - 1.2\tanh(u_1(t-\zeta)) - 0.9\tanh(u_2(t-\zeta)). \end{cases} \tag{46}$$

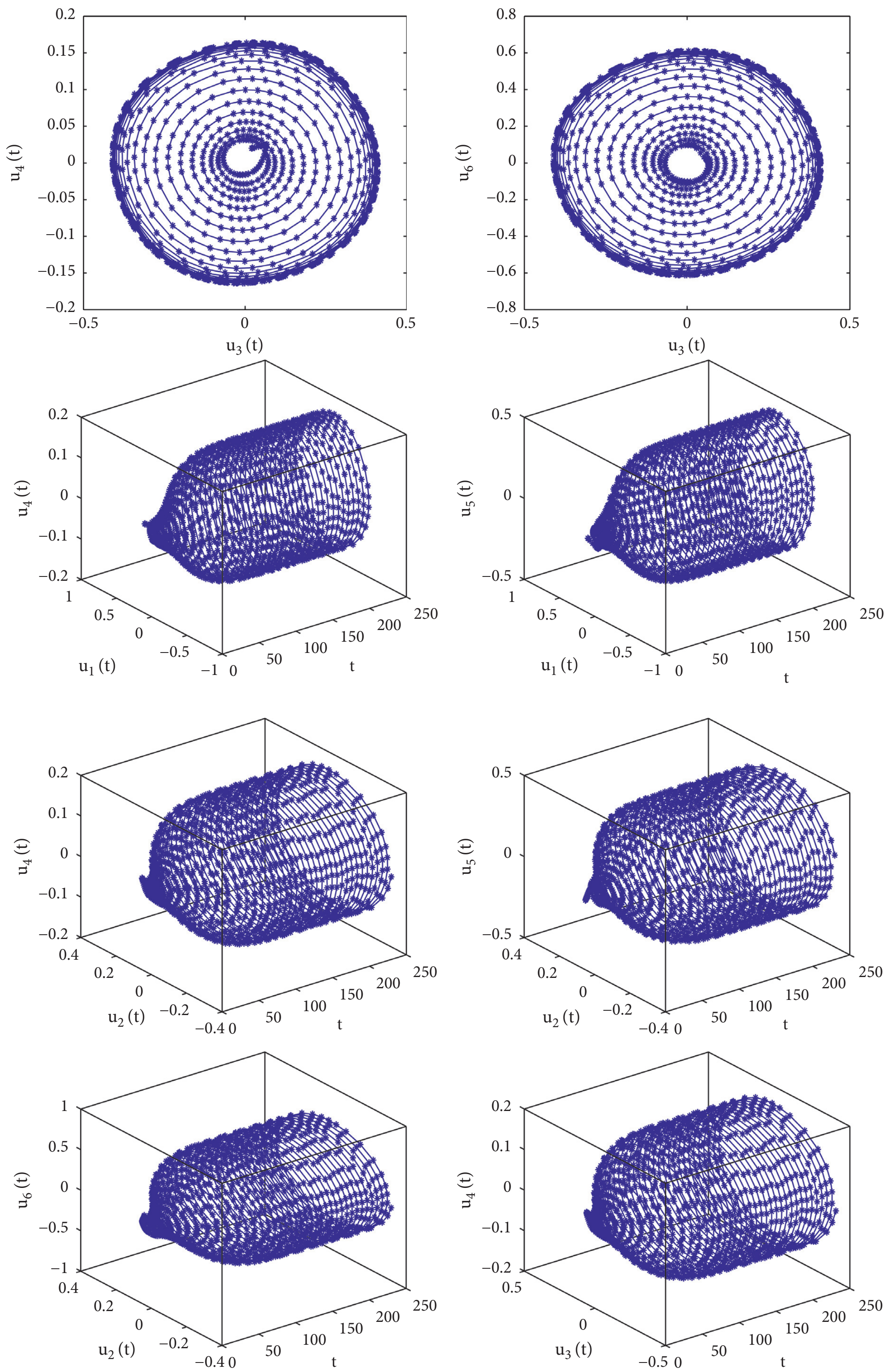
Apparently, neural network system (46) owns the unique zero equilibrium point $\mathscr{U}(0,0,0,0,0,0)$. Let $\xi = 0.94$. By means of computer software, one can derive $\zeta_0 = 0.75$ and $\vartheta_0 = 2.0922$. By virtue of algebraic computation with computer, one can verify that the assumptions $(H1)$–$(H3)$ of Theorem 1 hold. Then, one can conclude that the zero equilibrium point $\mathscr{U}(0,0,0,0,0,0)$ of neural network system (46) is locally asymptotically stable provided that $\zeta \in [0, 0.75)$. To illustrate this fact, we carry out computer simulations. We carry out numerical discretizations of model (46) by Adams–Bashforth–Moulton numerical algorithm. The integration algorithm starts with the solutions of system (46) in terms of R-L integral. The implicit discretization approach is applied to construct the tactics. We select $\zeta = 0.67 < \zeta_0 = 0.75$. The computer simulation figures are presented in Figure 1 which shows the locally asymptotically stable behavior of the neural network system (46). When ζ passes through the critical value $\zeta_0 = 0.75$, then the delay-induced Hopf bifurcation of neural network system (46) will arise in the vicinity of $\mathscr{U}(0,0,0,0,0,0)$. To explain this fact, we select $\zeta = 0.9 > \zeta_0 = 0.75$. The computer simulation figures are presented in Figure 2 which shows the Hopf bifurcation phenomenon of neural network system (46). The initial conditions are $(0.09, 0.038, 0.02, 0.02, -0.039, 0.08)$ and the time step is 0.0035 and the time of simulation is 250 seconds. To display the Hopf bifurcation phenomenon of neural network system (46) intuitively, we also draw the bifurcation plots which can be seen in Figures 3–8. From Figures 3–8, one can easily know that the bifurcation value of neural network system is 0.75.

Remark 3. In Figure 1, the subfigures 1–10 stand for the relation of the variable in horizontal axis and vertical axis. The subfigures 11–26 stand for the relation of the variable in horizontal axis, vertical axis, and vertical axis. In Figure 2, the subfigures 1–10 stand for the relation of the variable in horizontal axis and vertical axis. The subfigures 11–26 stand



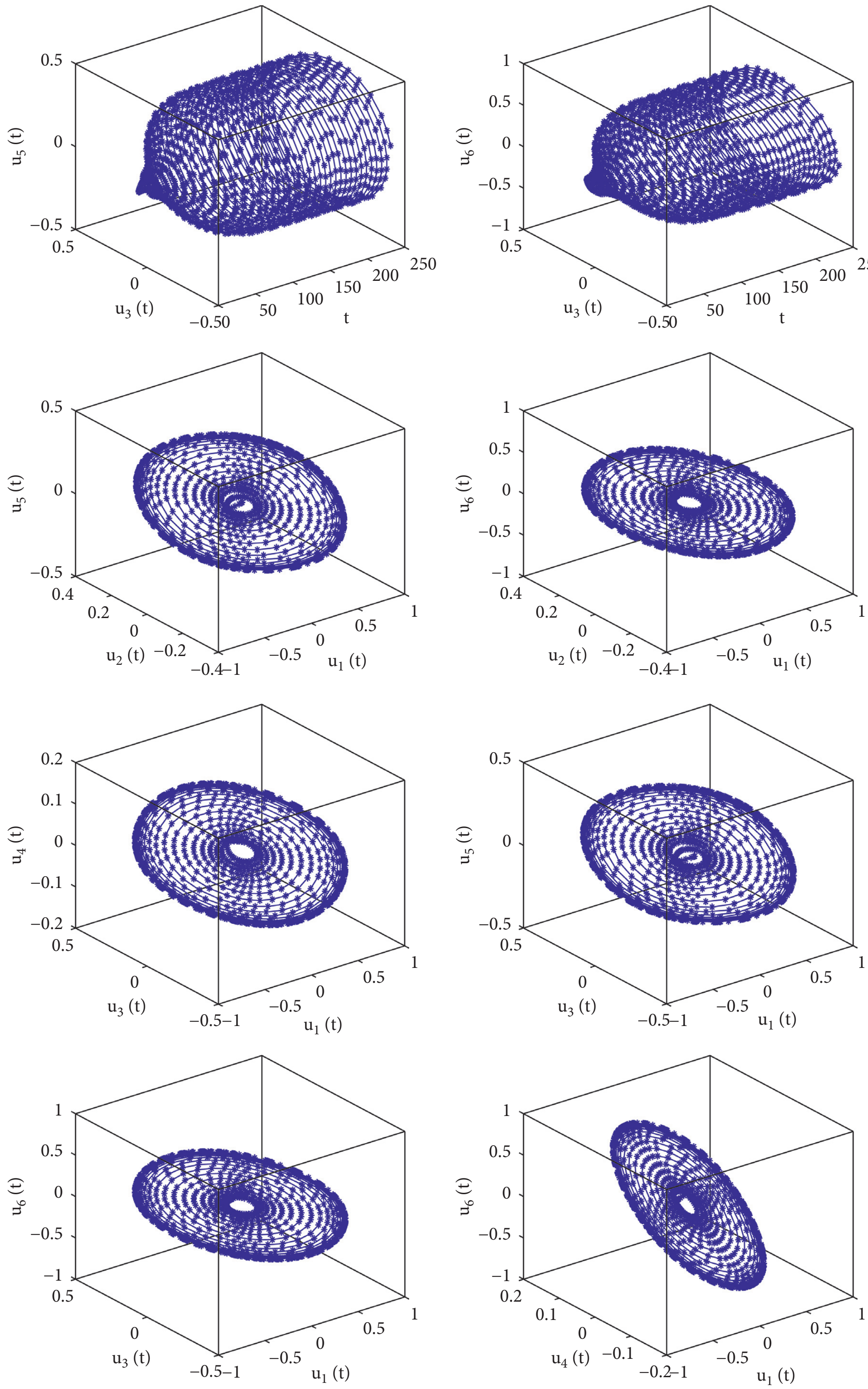
(a)

Figure 2: Continued.



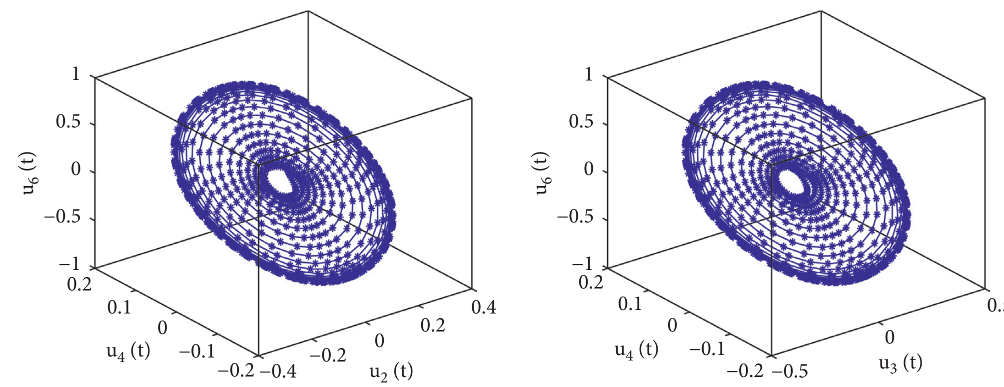
(b)

Figure 2: Continued.



(c)

Figure 2: Continued.



(d)

Figure 2: Hopf bifurcation for neural network model (46) involving $\zeta = 0.9 > \zeta_0 = 0.75$.

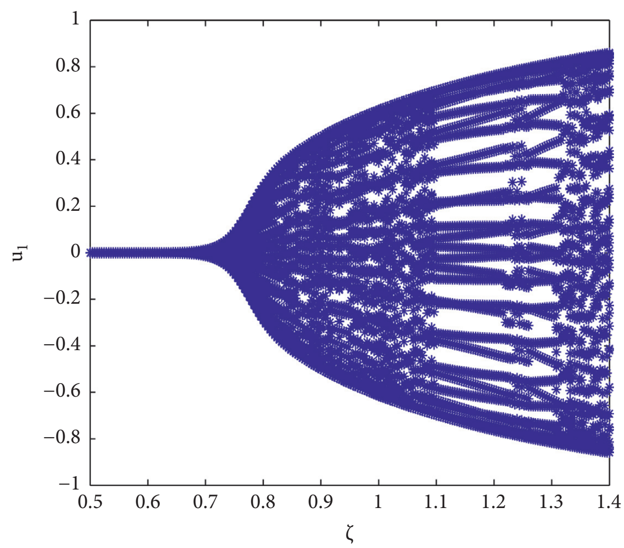


Figure 3: Bifurcation figure of neural network model (46): ζ-u_1.

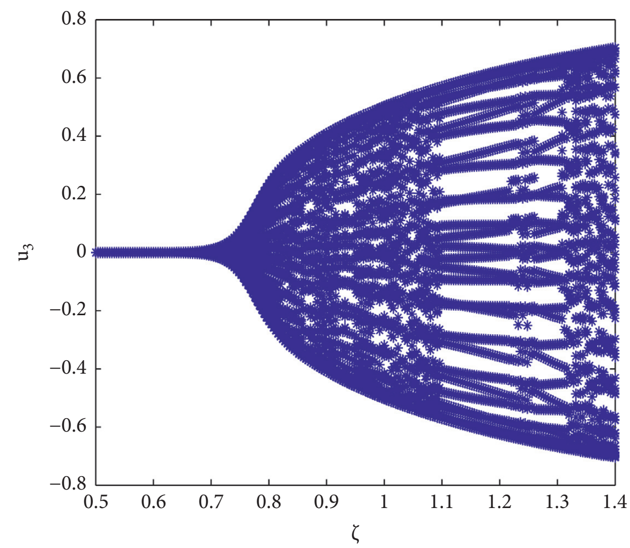


Figure 5: Bifurcation figure of neural network model (4.2): ζ-u_3.

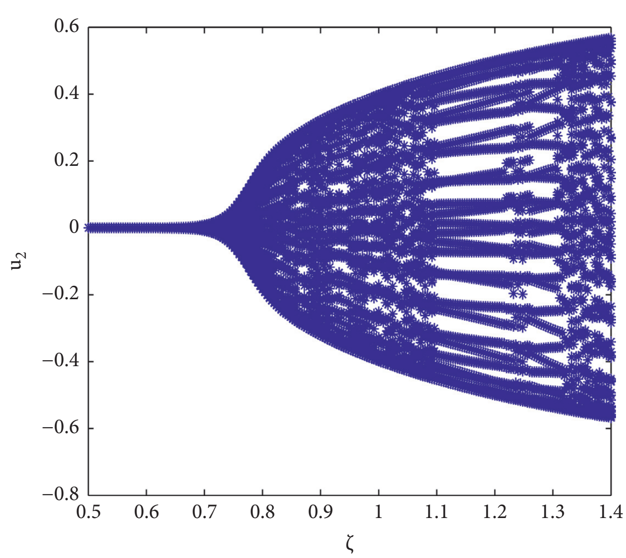


Figure 4: Bifurcation figure of neural network model (46): ζ-u_2.

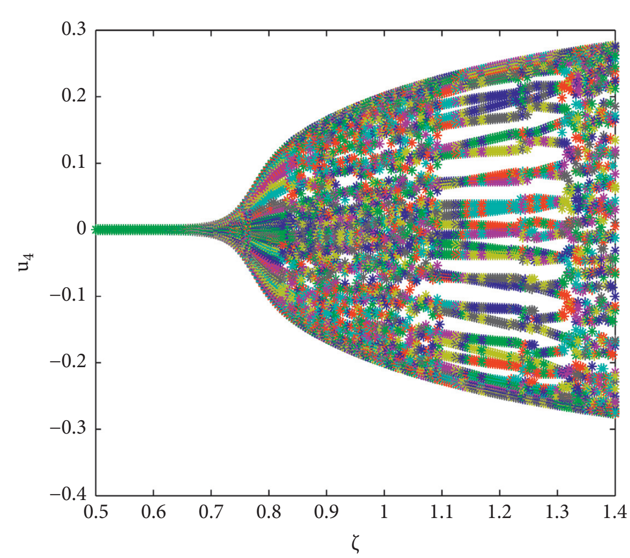


Figure 6: Bifurcation figure of neural network model (4.2): ζ-u_4.

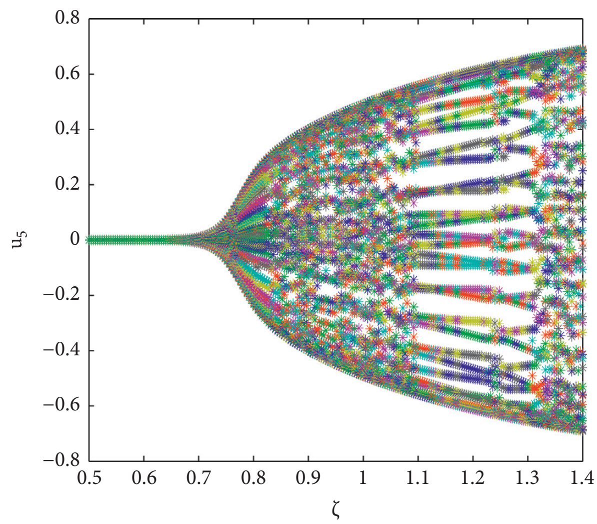


FIGURE 7: Bifurcation figure of neural network model (46): ζ-u_5.

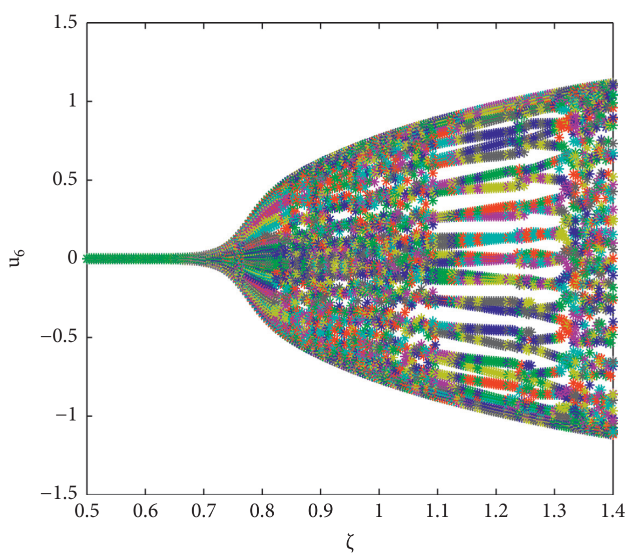


FIGURE 8: Bifurcation figure of neural network model (46): ζ-u_6.

for the relation of the variable in horizontal axis, vertical axis, and vertical axis.

5. Conclusions

Delay-induced Hopf bifurcation phenomenon is a significant dynamical behavior in delayed dynamical models. In particular, delay-induced Hopf bifurcation in neural network area has attracted much attention from good many scholars in the whole world. During the past decades, some researchers have investigated the Hopf bifurcation problem of fractional-order delayed neural networks. However, the major works are only concerned with the low-dimensional delayed fractional-order delayed neural networks; few works focus on the high-dimensional fractional-order ones. In this work, we mainly focus on the stability problem and the appearance of Hopf bifurcation of high-dimensional fractional-order delayed BAM neural network systems. The study shows that when the time delay keeps in a suitable range, the neural network systems will remain a stable state and if the time delay passes the critical value, then a Hopf bifurcation will take place around the equilibrium point of the involved neural networks. Thus, the time delay is a momentous factor that affects the stability and Hopf bifurcation for the investigated neural networks. In the end, the software simulation results and bifurcation diagrams efficaciously illustrate the effectiveness of the crucial analytical conclusions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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References

[1] B. Wróbel, A. Abdelmotaleb, N. Davey, and V. Steuber, "Evolving small spiking neural networks to work as state machines for temporal pattern recognition," *BMC Neuroscience*, vol. 16, 2015 Article number: P238.

[2] I. Rojas, J. Cabestany, and A. Catala, "Advances in artificial neural networks and computational intelligence," *Neural Processing Letters*, vol. 42, no. 1, pp. 1–3, 2015.

[3] Z. Cheng, K. Xie, T. Wang, and J. Cao, "Stability and Hopf bifurcation of three-triangle neural networks with delays," *Neurocomputing*, vol. 322, pp. 206–215, 2018.

[4] Z. Cheng, D. Li, and J. Cao, "Stability and Hopf bifurcation of a three-layer neural network model with delays," *Neurocomputing*, vol. 175, pp. 355–370, 2016.

[5] C. Aouiti, I. Ben Gharbia, J. Cao, M. Salah M'hamdi, and A. Alsaedi, "Existence and global exponential stability of pseudo almost periodic solution for neutral delay BAM neural networks with time-varying delay in leakage terms," *Chaos, Solitons & Fractals*, vol. 107, pp. 111–127, 2018.

[6] W. Yang, W. Yu, J. Cao, F. E. Alsaadi, and T. Hayat, "Almost automorphic solution for neutral type high-order Hopfield BAM neural networks with time-varying leakage delays on time scales," *Neurocomputing*, vol. 267, pp. 241–260, 2017.

[7] C. Maharajan, R. Raja, J. Cao, and G. Rajchakit, "Novel global robust exponential stability criterion for uncertain inertial-type BAM neural networks with discrete and distributed time-varying delays via Lagrange sense," *Journal of the Franklin Institute*, vol. 355, no. 11, pp. 4727–4754, 2018.

[8] C.-A. Popa, "Global μ-stability of neutral-type impulsive complex-valued BAM neural networks with leakage delay and unbounded time-varying delays," *Neurocomputing*, vol. 376, pp. 73–94, 2020.

[9] C. Sowmiya, R. Raja, Q. Zhu, and G. Rajchakit, "Further mean-square asymptotic stability of impulsive discrete-time stochastic BAM neural networks with Markovian jumping

and multiple time-varying delays," *Journal of the Franklin Institute*, vol. 356, no. 1, pp. 561–591, 2019.

[10] F. Lin and Z. Zhang, "Global asymptotic synchronization of a class of BAM neural networks with time delays via integrating inequality techniques," *Journal of Systems Science and Complexity*, vol. 33, no. 2, pp. 366–382, 2020.

[11] N. Belmahi and N. Shawagfeh, "A new mathematical model for the glycolysis phenomenon involving Caputo fractional derivative: well posedness, stability and bifurcation," *Chaos, Solitons & Fractals*, vol. 142, Article ID 110520, 2021.

[12] M. Liu, X. Xu, and C. Zhang, "Stability and global Hopf bifurcation for neutral BAM neural network," *Neurocomputing*, vol. 145, pp. 122–130, 2014.

[13] K. Gopalsamy and X. Xue-Zhong He, "Delay-independent stability in bidirectional associative memory networks," *IEEE Transactions on Neural Networks*, vol. 5, no. 6, pp. 998–1002, 1994.

[14] A. Hajihosseini, F. Maleki, and G. R. Rokni Lamooki, "Bifurcation analysis on a generalized recurrent neural network with two interconnected three-neuron components," *Chaos, Solitons & Fractals*, vol. 44, no. 11, pp. 1004–1019, 2011.

[15] E. Kaslik and S. Balint, "Bifurcation analysis for a discrete-time Hopfield neural network of two neurons with two delays and self-connections," *Chaos, Solitons & Fractals*, vol. 39, no. 1, pp. 83–91, 2009.

[16] J. Ge and J. Xu, "Stability and Hopf bifurcation on four-neuron neural networks with inertia and multiple delays," *Neurocomputing*, vol. 287, pp. 34–44, 2018.

[17] Y. Yang and J. Ye, "Stability and bifurcation in a simplified five-neuron BAM neural network with delays," *Chaos, Solitons & Fractals*, vol. 42, no. 4, pp. 2357–2363, 2009.

[18] B. Wang and J. Jian, "Stability and Hopf bifurcation analysis on a four-neuron BAM neural network with distributed delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 2, pp. 189–204, 2010.

[19] R. Amin, K. Shah, M. Asif, and I. Khan, "A computational algorithm for the numerical solution of fractional order delay differential equations," *Applied Mathematics and Computation*, vol. 402, Article ID 125863, 2021.

[20] A. Si-Ammour, S. Djennoune, and M. Bettayeb, "A sliding mode control for linear fractional systems with input and state delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 2310–2318, 2009.

[21] C. Xu, Z. Liu, L. Yao, and C. Aouiti, "Further exploration on bifurcation of fractional-order six-neuron bi-directional associative memory neural networks with multi-delays," *Applied Mathematics and Computation*, vol. 410, Article ID 126458, 2021.

[22] P. Mani, R. Rajan, L. Shanmugam, and Y. Hoon Joo, "Adaptive control for fractional order induced chaotic fuzzy cellular neural networks and its application to image encryption," *Information Sciences*, vol. 491, pp. 74–89, 2019.

[23] L. Li, X. Liu, M. Tang, S. Zhang, and X.-M. Zhang, "Asymptotical synchronization analysis of fractional-order complex neural networks with non-delayed and delayed couplings," *Neurocomputing*, vol. 445, pp. 180–193, 2021.

[24] C. Xu, M. Liao, P. Li, and S. Yuan, "Impact of leakage delay on bifurcation in fractional-order complex-valued neural networks," *Chaos, Solitons & Fractals*, vol. 142, Article ID 110535, 2021.

[25] B. Cao and X. Nie, "Event-triggered adaptive neural networks control for fractional-order nonstrict-feedback nonlinear systems with unmodeled dynamics and input saturation," *Neural Networks*, vol. 142, pp. 288–302, 2021.

[26] C. J. Xu, W. Zhang, C. Aouiti, Z. X. Liu, M. X. Liao, and P. L. Li, "Further investigation on bifurcation and their control of fractional-order BAM neural networks involving four neurons and multiple delays," *Mathematical Methods in the Applied Sciences*, vol. 75. , 2021 in press.

[27] C. Xu, Z. Liu, M. Liao, P. Li, Q. Xiao, and S. Yuan, "Fractional-order bidirectional associate memory (BAM) neural networks with multiple delays: the case of Hopf bifurcation," *Mathematics and Computers in Simulation*, vol. 182, pp. 471–494, 2021.

[28] T. Kan, Z. Gao, C. Yang, and J. Jian, "Convolutional neural networks based on fractional-order momentum for parameter training," *Neurocomputing*, vol. 449, pp. 85–99, 2021.

[29] M. Xiao, W. X. Zheng, J. Lin, G. Jiang, L. Zhao, and J. Cao, "Fractional-order PD control at Hopf bifurcations in delayed fractional-order small-world networks," *Journal of the Franklin Institute*, vol. 354, no. 17, pp. 7643–7667, 2017.

[30] B. Tao, M. Xiao, Q. Sun, and J. Cao, "Hopf bifurcation analysis of a delayed fractional-order genetic regulatory network model," *Neurocomputing*, vol. 275, pp. 677–686, 2018.

[31] W. Hu, D. Ding, Y. Zhang, N. Wang, and D. Liang, "Hopf bifurcation and chaos in a fractional order delayed memristor-based chaotic circuit system," *Optik*, vol. 130, pp. 189–200, 2017.

[32] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.

[33] D. Matignon, "Stability results for fractional differential equations with applications to control processing," *Computational Engineering in Systems Applications*, vol. 2, pp. 963–968, 1996.

[34] X. Wang, Z. Wang, and J. Xia, "Stability and bifurcation control of a delayed fractional-order eco-epidemiological model with incommensurate orders," *Journal of the Franklin Institute*, vol. 356, no. 15, pp. 8278–8295, 2019.

[35] W. Deng, C. Li, and J. Lü, "Stability analysis of linear fractional differential system with multiple time delays," *Nonlinear Dynamics*, vol. 48, no. 4, pp. 409–416, 2007.