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Linear Bifurcation Analysis with Applications to Relative Socio-Spatial Dynamics

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The objective of this research is the elaboration of elements of linear bifurcation analysis for the description the qualitative properties of orbits of the discrete autonomous iteration processes on the basis of linear approximation of the processes. The basic element of this analysis is the geometrical and numerical modification and application of the classical Routhian formalism, which is giving the description of the behavior of the iteration processes near the boundaries of the stability domains of equilibria. The use of the Routhian formalism is leading to the mapping of the domain of stability of equilibria from the space of control bifurcation parameters into the space of orbits of iteration processes. The study of the behavior of the iteration processes near the boundaries of stability domains can be achieved by the converting of coordinates of equilibria into control bifurcation parameters and by the movement of equilibria in the space of orbits. The crossing the boundaries of the stability domain reveals the plethora of the possible ways from stability, periodicity, the Arnold mode-locking tongues and quasi-periodicity to chaos. The numerical procedure of the description of such phenomena includes the spatial bifurcation diagrams in which the bifurcation parameter is the equilibrium itself. In this way the central problem of control of bifurcation can be solved: for each autonomous iteration process with big enough number of external parameters construct the realization of this iteration process with a preset combination of qualitative properties of equilibria. In this study the two-dimensional geometrical and numerical realizations of linear bifurcation analysis is presented in such a form which can be easily extended to multi-dimensional case. Further, a newly developed class of the discrete relative *m*-population/*n*-location Socio-Spatial dynamics is described. The proposed algorithm of linear bifurcation analyses is used for the detail analysis of the log–log-linear model of the one population/three location discrete relative dynamics.

Keywords: Control of bifurcations, Discrete non-linear dynamics, Discrete relative *m*-population/*n*-location socio-spatial dynamics

INTRODUCTION

In recent decades a new paradigm of bifurcations in behavior of non-linear systems appeared as a scientific approach and as a method to deal with manifestations of chaos and turbulence in different sciences. At present the essence of scientific efforts is shifted to further elaboration of conceptual framework of bifurcation analysis, to standardization of numerical methods and to the detailed description of the new important domains of applications. The central problem of the linear bifurcation analysis is the problem of control of bifurcations: to construct for each iteration process with a big enough number of the control bifurcation parameters the realization of this iteration process with a preset combination of qualitative properties of orbits. In the solution of this problem three main aspects are interwined: analytical and numerical aspects and the aspect of geometrical visualization.

The main objective of this research is two folded: to present the linear bifurcation analysis of the behavior of autonomous finite-dimensional discrete iteration processes and to apply the corresponding algorithm of analysis to the study of a new branch of non-linear dynamic systems studies: the Discrete Relative m-population/n-location Socio-Spatial Dynamics.

1 LINEAR BIFURCATION ANALYSIS

Let us start from the explicit form of the n-dimensional discrete time autonomous iteration processes (other explicit and implicit forms of the iteration processes can be considered also):

$$\begin{aligned} x_i(t+1) &= F_i(\mathbf{A}; \mathbf{x}(t)), \\ i &= 1, 2, \ldots, n; \quad t = 0, 1, 2, \ldots, \end{aligned} \tag{1}$$

where the vectors $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ represent the components of the iteration process in the time points $t = 0, 1, 2, \ldots$, $\mathbf{A}$ is the set of external constants (control bifurcation parameters) and the functions $F_i(\mathbf{A}; \mathbf{y})$, $i = 1, 2, \ldots, n$, are the differentiable functions of all their components $\mathbf{y} = (y_1, y_2, \ldots, y_n)$.

All possible equilibria $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ of the iteration process (1) are given by the system of equations

$$x_i^* = F_i(\mathbf{A}; \mathbf{x}^*), \quad i = 1, 2, \ldots, n. \tag{2}$$

In this paper we are presenting the analytical and numerial procedure of the bifurcation analysis in the following way: the essence of this procedure is the exchange of a part $\mathbf{A}_1$ of control bifurcation parameters from the set $\mathbf{A}$ by components of the equilibrium $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ with the help of Eqs. (2). In such a way the components of equilibria became control bifurcation parameters themselves.

As it will be shown further, the remaining part of parameters $\mathbf{A}_2 = \mathbf{A} \backslash \mathbf{A}_1$ give the description of the boundaries of the domain of stability of equilibria within the space of orbits. This means that it is possible to move the equilibrium without of movement of domain of its stability. The movement of equilibrium points can be placed on the segments of straight lines. This allows the complete computerized description of the appearance of different bifurcation phenomena in the space of orbits.

Thus, the geometrical content of the proposed bifurcation analysis includes the travels of equilibria in the space of orbits which reveal the qualitative features of the behavior of the trajectories of the iteration process near the boundaries of domain of stability of equilibria.

The linear bifurcation analysis is based on the construction of the Jacobi matrix of the linear approximation of the iteration process

$$J_{t;t+1} = \|s_{ij}(t+1, t)\|, \tag{3}$$

where

$$s_{ij}(t+1, t) = \frac{\partial x_i(t+1)}{\partial x_j(t)}, \quad i, j = 1, 2, \ldots, n. \tag{4}$$

The following analytical expressions are of use:

1. the value of Jacobi matrix $J^* = \|s_{ij}^*\|$ at the equilibrium $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ and
2. the characteristic polynomial of the Jacobi matrix J^*:

$$P(\mu) = \mu^n + a_1\mu^{n-1} + \cdots + a_{n-1}\mu + a_n.$$

As is well known the construction of the analytical forms of the coefficients of the characteristic polynomial $P(\mu)$ can be done with the help of the principal minors of the Jacobi matrix J^*. Thus, the following analytical objects should be computed:

3. Principal minors of the Jacobi matrix J^*.

By the well-known theorem of von Neumann the equilibrium $\mathbf{x}^*$ is asymptotically stable iff for all its eigenvalues μ the following condition holds:

$$|\mu| < 1. \tag{5}$$

Consider the space P of all coefficients of the characteristic polynomials of the order n. Condition (5) defines in this space the geometrical domain of asymptotical stability. The analytical description of this stability domain can be constructed with the help of the classical Routh–Hurwitz procedure in the form of the non-linear inequalities. This procedure can be described as follows (see, Samuelson, 1983, pp. 435–437).

First of all, construct the parameters

$$b_0 = \sum_{i=0}^{n} a_i;$$

$$b_1 = \sum_{i=0}^{n} a_i(n-2i), \quad \text{where } a_0 = 1;$$

$$\vdots$$

$$b_r = \sum_{i=0}^{n} a_i \sum_{k=0}^{n} (-1)^k \binom{n-i}{r-k}\binom{i}{k},$$

$$\text{where} \quad \binom{i}{k} = \begin{cases} i!/(k!(i-k)!), & i \geq k;\ k \geq 0, \\ 0, & i < k, \\ 0, & k < 0, \end{cases}$$

$$\vdots$$

$$b_n = 1 - a_1 + a_2 - \cdots + (-1)^{n-1} a_{n-1} + (-1)^n a_n. \tag{6}$$

Further, construct the matrix

$$\begin{pmatrix} b_1 & b_3 & b_5 & \cdot & \cdot & \cdot \\ b_0 & b_2 & b_4 & \cdot & \cdot & \cdot \\ 0 & b_1 & b_3 & \cdot & \cdot & \cdot \\ 0 & b_0 & b_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{7}$$

and its principal minors $\Delta_1, \Delta_2, \ldots, \Delta_n$.

The conditions of asymptotical stability are:

$$b_0 > 0; \qquad \Delta_r > 0, \quad r = 1, 2, \ldots, n \tag{8}$$

and the boundaries of the stability domain in the space P determined with the help of described above Routhian procedure by the non-linear equalities:

$$b_0 = 0; \qquad \Delta_r = 0, \quad r = 1, 2, \ldots, n. \tag{9}$$

On the boundaries (9) the absolute values of some eigenvalues of the Jacobi matrix are equal 1 and the plethora of different bifurcation phenomena exist.

In two- and three-dimensional cases the domains of stability can be visualized in the following form: for $n = 2$

$$\begin{aligned} b_0 &= 1 + a_1 + a_2; \quad b_1 = 2 - 2a_2; \\ b_2 &= 1 - a_1 + a_2; \end{aligned} \tag{10}$$

and the stability domain in the space of parameters a_1, a_2 is defined by the linear inequalities

$$-1 \pm a_1 < a_2 < 1. \tag{11}$$

Geometrically, these inequalities represent a triangle of stability with the vertices $\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]$.

For $n = 3$:

$$\begin{aligned} b_0 &= 1 + a_1 + a_2 + a_3; \\ b_1 &= 3 + a_1 - a_2 - 3a_3; \\ b_2 &= 3 - a_1 - a_2 + 3a_3; \\ b_3 &= 1 - a_1 + a_2 - a_3; \end{aligned} \tag{12}$$

and the stability domain is defined by the linear and quadratic inequalities:

$$\begin{aligned} &1 + a_1 + a_2 + a_3 > 0; \\ &1 - a_1 + a_2 - a_3 > 0; \\ &1 - a_2 + a_1 a_3 - a_3^2 > 0. \end{aligned} \tag{13}$$

In the three-dimensional space of the coefficients a_1, a_2, a_3 this domain has three boundary surfaces: two planes and a saddle (parabolic hyperboloid). More precisely, the plane $1 + a_1 + a_2 + a_3 = 0$ touches the domain of stability of equilibria by the triangle ABC with the vertices

$$A = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix};$$

the plane $1 - a_1 + a_2 - a_3 = 0$ touches the domain of stability of equilibria by the triangle ABD with the vertices

$$A = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix}.$$

The straight lines generated by segments AC, BC, AD, BD lie on the saddle $1 - a_2 + a_1 a_3 - a_3^2 = 0$.

Next, because the components of the Jacobi matrix J^* are the functions of the coordinates of the equlibrium $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$, it is possible to construct the analytical and geometrical images of the boundaries of the domain of stability in the space of orbits. It is important to underline, that because the parameters from $\mathbf{A}_1$ can be analytically presented with the help of the coordinates of the fixed points, the boundaries of the domain of stability in the space of orbits depend only on the parameters from $\mathbf{A}_2$. Therefore, it is possible to move an equilibrium to the preset given point of the boundary with known bifurcation effect.

In conclusion, the mapping of the domain of stability of equilibria from the space P of all coefficients of the characteristic polynomials eigenvalues into the space of orbits together with the immovability of the boundaries of the domain of stability in the space of orbits give the possibility to describe all admissible qualitative features of the behavior of the iteration process near the boundaries of the stability domain. The travels of the equilibrium in the space of orbits on the segments of straight lines and the crossing the boundaries of the stability domain reveal the plethora of the possible ways from stability, periodicity, Arnold horns and quasi-periodicity to chaos. It is important to stress that the travels of equilibria also reveal geometrically and numerically the mechanism by which the mode-locking areas of periodic resonances destroy quasi-periodic orbits without using the elaborate analytical techniques. The numerical procedure of the description of such phenomena includes the construction of spatial bifurcation diagrams in which the bifurcation parameter is the equilibrium itself.

The organization of the travels of equilibria in the space of orbits on the segments of straight lines can be done in the following way: it is possible to parametrize the segment of the straight line between the equilibria $\mathbf{x}$ and $\mathbf{y}$ as

$$\mathbf{x}(j) = \mathbf{x}\left(1 - \frac{j}{T}\right) + \mathbf{y}\frac{j}{T}, \quad j = 0, 1, \ldots, T, \tag{14}$$

where j is a bifurcation parameter and T is a number of bifurcation steps. In such a way a planar bifurcation diagram can be constructed. The usual (linear or one-dimensional) bifurcation diagram can be obtained from (14) by the fixation of some coordinate of the vectors $\mathbf{x}(j)$.

Thus, for each iteration process with a big enough number of the control bifurcation parameters it is possible to construct the realization of this iteration process with a preset combination of qualitative properties of orbits (cf. Sonis, 1990; 1993; 1994).

2 REALIZATION OF THE LINEAR BIFURCATION ANALYSIS FOR TWO-DIMENSIONAL AUTONOMOUS ITERATION PROCESSES

In this section we present in brief a two-dimensional realization of the linear bifurcation analysis. The form of this realization can be extended in the same manner to a multi-dimensional case.

Let us start with the iterations of the type

$$\begin{aligned} x(t+1) &= G(x(t),\, y(t)), \\ y(t+1) &= H(x(t),\, y(t)). \end{aligned} \tag{15}$$

The standard linear stability analysis of the general two-dimensional discrete map (15) is based on the consideration of the general Jacobi matrix

$$J(t+1,t) = \begin{bmatrix} \partial G/\partial x & \partial G/\partial y \\ \partial H/\partial x & \partial H/\partial y \end{bmatrix} \tag{16}$$

and its value J^* on the fixed point x^*, y^*

$$J^* = \begin{bmatrix} \partial G^*/\partial x^* & \partial G^*/\partial y^* \\ \partial H^*/\partial x^* & \partial H^*/\partial y^* \end{bmatrix}, \tag{17}$$

where $G^* = G(x^*, y^*),\ H^* = H(x^*, y^*)$.

The eigenvalues of the Jacobi matrix J^* are the solutions of the characteristic polynomial

$$\mu^2 + a_1\mu + a_2 = \mu^2 - \mathrm{Tr}\ J^*\mu + \det J^* = 0, \tag{18}$$

where

$$\begin{aligned} -a_1 &= \mathrm{Tr}\ J^* = \frac{\partial G^*}{\partial x^*} + \frac{\partial H^*}{\partial y^*}; \\ a_2 &= \det\ J^* = \begin{vmatrix} \partial G^*/\partial x^* & \partial G^*/\partial y^* \\ \partial H^*/\partial x^* & \partial H^*/\partial y^* \end{vmatrix}. \end{aligned} \tag{19}$$

Next we will summarize the qualitative properties of the behavior of discrete map which are the results of the standard linear stability analysis (see, for example, Hsu, 1977; Thompson and Stewart, 1986, pp. 150–161; Sonis, 1990).

By the well-known von Neumann theorem, the equilibrium (x^*, y^*) is asymptotically stable if and only if for all its eigenvalues μ_1, μ_2 the following conditions hold:

$$|\mu_1| < 1, \quad |\mu_2| < 1. \tag{20}$$

The outcome of the general Routh–Hurwitz stability conditions (11) in the case $n = 2$ for the polynomial $\mu^2 + a_1\mu + a_2$ is

$$\begin{aligned} &b_0 = 1 + a_1 + a_2; \quad b_1 = 2 - 2a_2; \\ &b_2 = 1 - a_1 + a_2; \end{aligned} \tag{21}$$

and the stability domain in the space P of parameters a_1, a_2 is defined by the linear inequalities:

$$-1 \pm a_1 < a_2 < 1. \tag{22}$$

In the plane of the coefficients a_1, a_2 the domain of stability defined by the conditions (22) is the triangle ABC with the vertices (see Fig. 1):

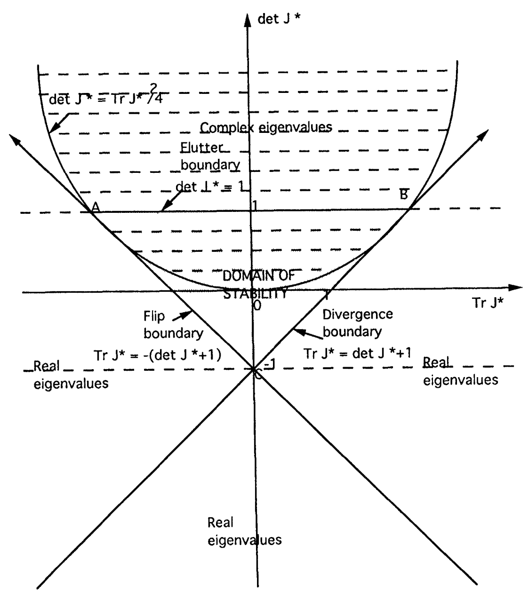

FIGURE 1 Domain of stability of discrete two-dimensional non-linear dynamics.

$$A = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

The parabola $a_2 = \frac{1}{4}a_1^2$ divides the triangle into two major domains: the eigenvalues are real outside of the parabola and are complex conjugate inside of this parabola. On the parabola itself the eigenvalues are equal.
The sides of the triangle of stability are generated by the following straight lines:

– the *divergence boundary* under the equation

$$1 - a_1 + a_2 = 0, \tag{23}$$

– the *flip boundary* under the equation

$$1 + a_1 + a_2 = 0, \tag{24}$$

– the *flutter boundary* under the equation

$$a_2 = 1. \tag{25}$$

On the divergence boundary at least one of the eigenvalues is equal to 1. Crossing of this boundary allows for orbits to be repelled from the equilibrium. Such divergence starts from within the domain of stability; this domain is the divergence-locking domain.

On the flip boundary at least one of the eigenvalues is equal to -1. Each point on the flip boundary corresponds to a two-periodic cycle, and movement outside the domain of stability generates the Feigenbaum type period doubling sequence, leading to chaos (Feigenbaum, 1978).

On the flutter boundary $|\mu_1| = |\mu_2| = 1$. It is easy to describe the type of bifurcations in all points on the flutter boundary. The condition $|\mu_1| = |\mu_2| = 1$ means that $\mu_1 = e^{i2\pi\Omega}$, $\mu_2 = e^{-i2\pi\Omega}$, $0 \leq \Omega \leq 1$, and therefore,

$$-a_1 = \mathrm{Tr}\, J^* = \mu_1 + \mu_2 = 2\cos 2\pi\Omega. \tag{26}$$

If Ω is a rational fraction: $\Omega = p/q$, then we have q-periodic (resonance) fixed points; between them there are fixed points of strong resonance with $\Omega = \frac{1}{3}$, $\Omega = \frac{1}{4}$. Other rational fractions $\Omega = p/q$ represent points of weak resonance.

The same periodic behavior is also observed in a small domain of Ω near p/q. This domain, the mode-locking domain, is the image of the Arnold tongue from the corresponding domain of change in eigenvalues in the complex plane (Arnold, 1977). For strong resonance, the mode-locking domain starts within the domain of stability (Kogan, 1991).

If Ω is not rational, the quasi-periodic motion of orbits appears.

Presenting $a_1 = -\mathrm{Tr}\, J^*$ and $a_2 = \det J^*$ through the coordinates x^*, y^* of the equilibrium one obtains in the space of orbits the domain of stability of equilibria; boundaries of this domain are the following curves:
– the *divergence boundary* with the equation

$$\mathrm{Tr}\ J^* = \Delta^* + 1; \tag{27}$$

– the *flip boundary* with the equation

$$\mathrm{Tr}\ J^* = -(\Delta^* + 1); \tag{28}$$

– the *flutter boundary* with the equation

$$\det J^* = 1. \tag{29}$$

(It should be mentioned that in three-dimensional case we will have the divergence plane, the flip plane and the flutter saddle. It is important to note that for the higher dimensions the invariant tori including periodic and quasi-periodic motion appear. This issue will be considered elsewhere.)

3 DISCRETE RELATIVE *m* POPULATION/*n* LOCATION SOCIO-SPATIAL DYNAMICS

In the next sections the ideas of bifurcation analysis will be applied for the specific cases of a new general model of discrete relative multiple population/multiple location socio-spatial dynamics (see Dendrinos and Sonis, 1990).

We will start from one population (stock)/n locations case. Let the vector

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)), \quad t = 0, 1, 2, \ldots$$

be the relative population size distribution at time t between n locations. Such a formulation could be specified for any socio-economic quantity, normalized over a regional or national total.

The one population/multiple location relative discrete socio-spatial dynamics then is given by:

$$x_i(t+1) = F_i(\mathbf{x}(t)) / \sum_{j=1}^{n} F_j(\mathbf{x}(t)),$$
$$i = 1, 2, \ldots, n; \; t = 0, 1, 2, \ldots; \qquad (30)$$

$$F_i(\mathbf{x}(t)) > 0, \quad i = 1, 2, \ldots, n;$$
$$0 < x_i(0) < 1, \; i = 1, 2, \ldots, n;$$
$$\sum_j x_j(0) = 1. \qquad (31)$$

The expression $F_i(\mathbf{x}(t))$ is the locational comparative advantages enjoyed by the population at (i, t). Functions F_i depend on the relative distribution of the population in all locations, and on other environmental parameters.

A specific log-linear formulation for the functions F_i with the universality properties may be represented by the following:

$$F_i(x(t)) = A_i \prod_j x_j(t)^{a_{ij}};$$
$$-\infty < a_{ij} + \infty; \quad A_i > 0, \; i = 1, 2, \ldots, n; \qquad (32)$$

where $A_1, A_2, \ldots, A_n$ are the composite locational advantages of the locations 1, 2, ..., n, and the matrix $\|a_{ij}\|$ is the matrix of the composite elasticities of relative population growth. This iteration process can reproduce each preset dynamic behavior including stability, periodic motion, quasi-periodicity and various forms of chaotic movement.

A specific log–log-linear formulation for the functions F_i may be represented by the following functions $F_i(\mathbf{x}(t))$ of the exponential form:

$$F_i(\mathbf{x}(t)) = \exp w_i \prod_j x_j(t)^{a_{ij}};$$
$$-\infty < a_{ij} < +\infty; \quad i = 1, 2, \ldots, n; \qquad (33)$$

where the matrix $\|a_{ij}\|$ is the matrix of the spatio-temporal composite elasticities.

It is important to stress that the relative dynamics (30) can be generated by the following extreme principle (cf. Gontar, 1981; Sonis and Gontar, 1992): the relative Socio-Spatial dynamics proceed in such a way that in the transfer from time t to time $t+1$ the information functional

$$I(t, t+1) = \sum_{i=1}^{n} x_i(t+1)$$
$$\times [\ln x_i(t+1) - \ln f_i(\mathbf{x}(t)) - 1] \qquad (34)$$

reaches its minimum in the space of vectors $\mathbf{x}(t+1)$ subject to the conservation condition:

$$\sum_{i=1}^{n} x_i(t+1) = 1.$$

This extreme principle defines a new law of *collective* non-local population redistribution behavior which is a meso-level counterpart of the utility optimization individual behavior.

Moreover, it is possible to formulate a more general extreme principle which will generate the multinomial relative socio-spatial dynamics as well as an arbitrary iteration process with the help of informational functionals of the universal analytical form. Such a principle represents the collective local and non-local synergetic interactions between the constituencies of an arbitrary autonomous iteration process (Sonis and Gontar, 1992). It should be mentioned that the information minimization principle is the discrete analogue of the problem stated and solved by Vito

Volterra in 1939; to construct the Hamilton variational principle for the logistic type system of differential equations describing the "struggle for existence". The analytical form of the information minimization principle is similar to the generalization of the Volterra principle in modern Innovation Diffusion theory (Sonis, 1992).

We now assume that there exist m different populations (stocks) located in n different locations. Examples of such populations (stocks) could be m distinct population (or labor) types; distinct capital stocks (classified, for example, according to vintage; stocks of financial capital (currencies); different types of economic outputs (products); or any other economic, social, political and other types of socio-spatial variables, or a combination of them. The general model of the relative distribution of such stocks in space–time can be presented in the following form:

$$\begin{aligned}
&x_{ji}(t+1) = F_{ji}(\mathbf{x}(t)) \Big/ \sum_{s=1}^{n} F_{js}(\mathbf{x}_j(t)),\\
&\quad i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m;\\
&\quad t = 0, 1, 2, \ldots;\\
&F_{ji}(\mathbf{x}_j(t)) > 0\\
&\quad \text{for } \mathbf{x}_j(t) = (x_{j1}(t), x_{j2}(t), \ldots, x_{jn}(t)),\\
&\quad t = 0, 1, 2, \ldots,\quad i = 1, 2, \ldots, n;\\
&\quad j = 1, 2, \ldots, m;
\end{aligned} \tag{35}$$

such that $0 < x_{ji}(t) < 1$, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$; and

$$\sum_i x_{ji}(t) = 1. \tag{36}$$

4 APPLICATION OF THE CONTROL OF BIFURCATIONS ALGORITHM TO THE STUDY OF THE ONE POPULATION/THREE LOCATION RELATIVE DYNAMICS

Consider the following one population/three location log–log-linear model:

$$\begin{aligned}
x_1(t+1) &= 1/[1 + A_2\exp(\mu_{23}x_3(t))\\
&\qquad + A_3\exp(\mu_{31}x_1(t))],\\
x_2(t+1) &= A_2\exp(\mu_{23}x_3(t))/\\
&\qquad [1 + A_2\exp(\mu_{23}x_3(t))\\
&\qquad + A_3\exp(\mu_{31}x_1(t))],\\
x_3(t+1) &= A_3\exp(\mu_{31}x_1(t))/\\
&\qquad [1 + A_2\exp(\mu_{23}x_3(t))\\
&\qquad + A_3\exp(\mu_{31}x_1(t))],\\
A_2, A_3 &> 0,\quad -\infty < \mu_{23}, \mu_{31} < +\infty
\end{aligned} \tag{37}$$

describing the changes in relative population shares $x_1(t)$, $x_2(t)$, $x_3(t)$ distributed between three locations (or between three alternatives of choice).

First of all let us describe the space of orbits of the dynamics (37). For this purpose the barycentric coordinates within the Moebius triangle will be used.

1. *A Moebius plane as a space of orbits* Moebius plane is the two-dimensional space (plane) defined by three barycentric coordinates x_1, x_2, x_3, $x_1 + x_2 + x_3 = 1$, of each point within it. The scale element of this plane is the Moebius equilateral triangle with the unit scale on its sides. This triangle is generated by three coordinate axes (Fig. 2). It is possible to measure the barycentric coordinates of each point in Moebius plane by projecting it (parallel to the sides) onto the sides of the Moebius triangle. If the point P lies within the Moebius triangle, then its barycentric coordinates x_1, x_2, x_3 must be between 0 and 1:

$$x_1 + x_2 + x_3 = 1;\quad 0 \le x_1, x_2, x_3 \le 1. \tag{38}$$

If the point Q lies outside the Moebius triangle, then one of the barycentric coordinates must be negative and other to be greater than 1, but the condition $x_1 + x_2 + x_3 = 1$ always hold. The vertices of the Moebius triangle are:

$$\begin{aligned}
X&:\quad x_1 = 1;\ x_2 = 0;\ x_3 = 0,\\
Y&:\quad x_1 = 0;\ x_2 = 1;\ x_3 = 0,\\
Z&:\quad x_1 = 0;\ x_2 = 0;\ x_3 = 1.
\end{aligned} \tag{39}$$

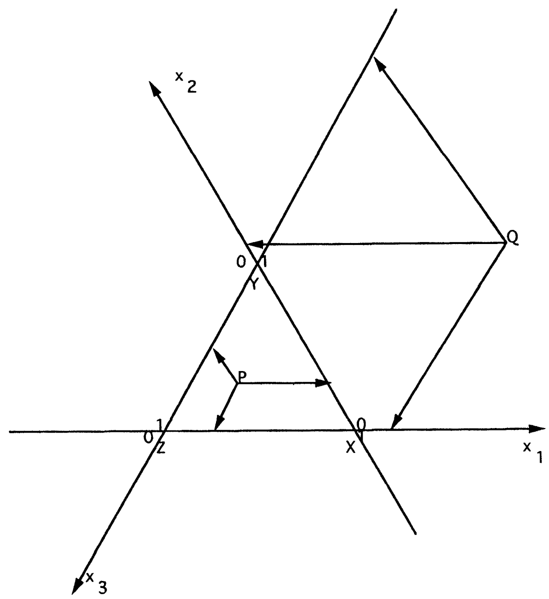

FIGURE 2 Barycentric coordinates in Meobius plane.

For the dynamics (37) the Moebius triangle gives the natural way to present the orbits of dynamics and their fixed points. Moreover, because of conditions A_2, $A_3 > 0$ the orbits of the relative dynamics occur within the Moebius triangle itself.

2. *Fixed points* Now we will concentrate ourselves on the graphical representation of the behavior of the non-periodic fixed point x_1^*, x_2^*, x_3^* of the dynamics (37) within the Moebius triangle under arbitrary changes in the parameters A_2, $A_3 > 0$ and $-\infty < \mu_{23}$, $\mu_{31} < +\infty$.

Eqs. (37) and (38) imply that the coordinates of the non-periodic fixed point x_1^*, x_2^*, x_3^* satisfy the system of equations:

$$\begin{aligned} x_2^*/x_1^* &= A_2 \exp(\mu_{23} x_3^*); \\ x_3^*/x_1^* &= A_3 \exp(\mu_{31} x_1^*); \\ x_1^* + x_2^* + x_3^* &= 1. \end{aligned} \tag{40}$$

This system implies that

$$\begin{aligned} x_2^* &= x_1^* \exp[\mu_{23} A_3 x_1^* \exp(\mu_{31} x_1^*)], \\ x_3^* &= x_1^* A_3 \exp(\mu_{31} x_1^*), \end{aligned} \tag{41}$$

and

$$\begin{aligned} x_1^* + x_1^* A_2 \exp[\mu_{23} A_3 x_1^* \exp(\mu_{31} x_1^*)] \\ + x_1^* A_3 \exp(\mu_{31} x_1^*) = 1. \end{aligned} \tag{42}$$

The dynamics (37) have only one non-periodic fixed point. For the proof consider a function

$$\begin{aligned} f(x_1^*) = x_1^* + x_1^* A_2 \exp[\mu_{23} A_3 x_1^* \exp(\mu_{31} x_1^*)] \\ + x_1^* A_3 \exp(\mu_{31} x_1^*) - 1. \end{aligned}$$

It is easy to see that the derivative of this function is positive, and $f(x_1^*)$ tends to -1 if x_1^* tends to $0 + 0$, and $f(x_1^*)$ tends to some positive value C if x_1^* tends to $1 - 0$. Thus, the function $f(x_1^*)$ increases monotonically from -1 to $C > 0$, and, therefore, there is only one point x_1^* between 0 and 1 such that $f(x_1^*) = 0$.

This fixed point it is easy to calculate from Eqs. (41), (42) with the help of the computation of the values of the left part of Eq. (42) in two points of the x_1^*-axis. Refinement of the mesh size near suspected fixed point by dividing it in two makes it possible to pin down the location of any fixed point.

3. *Changes in the model parameters and linear bifurcation analysis* Consider now all models (37) with the fixed positive parameters A_2, A_3 and changeable parameters μ_{23}, μ_{31}. It will be shown further, that the position of the domain of stability and the flip, flutter and divergence boundaries are prescribed by the values A_2, A_3 only, while the position of the equilibrium depends on all parameters $A_2, A_3, \mu_{23}, \mu_{31}$. By changing the appropriate parameters μ_{23}, μ_{31} one can put the non-periodic equilibrium into an arbitrary place within the domain of stability. Thus, the parameters μ_{23}, μ_{31} are plying a role of external bifurcation parameters. Eqs. (40) give the following dependence of these external bifurcation parameters on the coordinates of the fixed point:

$$\begin{aligned} \mu_{23} &= (1/x_3^*) \ln(x_2^*/x_1^* A_2), \\ \mu_{31} &= (1/x_1^*) \ln(x_3^*/x_1^* A_3). \end{aligned} \tag{43}$$

These relationships allow to convert the fixed point of the dynamics (37) into the internal bifurcation parameters. The preset choice of the movement of fixed point in the space of orbits (for example, on the straight line between two points of the Moebius triangle) can be converted with the help of the formulas (43) into the change of the external parameters μ_{23}, μ_{31} controlling the model bifurcations.

4. *The Jacobi matrix* Consider the slope-response functions

$$s_{ij}(t+1,\, t) = \frac{\partial x_i(t+1)}{\partial x_j(t)}, \quad i, j = 1,\, 2,\, 3,$$

which are the entries of the Jacobi slope-matrix

$$J(t+1,\, t) = \|s_{ij}(t+1,\, t)\|.$$

The direct calculation gives:

$$\begin{aligned}
&s_{11}(t+1,\, t) = -\mu_{31}[x_1(t+1)\ x_3(t+1)];\\
&s_{21}(t+1,\, t) = -\mu_{31}[x_2(t+1)\ x_3(t+1)];\\
&s_{31}(t+1,\, t) = \mu_{31}[1 - x_3(t+1)]\ x_3(t+1);\\
&s_{12}(t+1,\, t) = s_{22}(t+1,\, t) = s_{32}(t+1,\, t) = 0;\\
&s_{13}(t+1,\, t) = -\mu_{23}[x_1(t+1)\ x_2(t+1)];\\
&s_{23}(t+1,\, t) = \mu_{23}[1 - x_2(t+1)]\ x_2(t+1)];\\
&s_{33}(t+1,\, t) = -\mu_{23}[x_2(t+1)\ x_3(t+1)].
\end{aligned} \tag{44}$$

Obviously the determinant of the Jacobi matrix (Jacobian) is equal to zero: $\det J(t+1, t) = 0$. At the fixed point x_1^*, x_2^*, x_3^* the Jacobi slope-matrix $J^* = \|s_{ij}^*\|$ has the form

$$J^* = \begin{vmatrix} -\mu_{31}x_1^*x_3^* & 0 & -\mu_{23}x_1^*x_2^* \\ -\mu_{31}x_2^*x_3^* & 0 & \mu_{23}x_2^*(1-x_2^*) \\ \mu_{31}x_3^*(1-x_3^*) & 0 & -\mu_{23}x_2^*x_3^* \end{vmatrix} \tag{45}$$

such that the Jacobian at the fixed point $\det J^* = 0$.

The characteristic equation of the Jacobi matrix:

$$\mu^3 - \mathrm{Tr}\ J^*\mu^2 + \Delta^*\mu - \det J^* = 0, \tag{46}$$

where

$$\mathrm{Tr}\, J^* = \sum_{i=1}^{3} s_{ij}^* = -x_3^*(\mu_{31}x_1^* + \mu_{23}x_2^*) \tag{47}$$

and

$$\begin{aligned}\Delta^* &= \begin{vmatrix} s_{11}^* & s_{12}^* \\ s_{21}^* & s_{22}^* \end{vmatrix} + \begin{vmatrix} s_{11}^* & s_{13}^* \\ s_{31}^* & s_{33}^* \end{vmatrix} + \begin{vmatrix} s_{22}^* & s_{23}^* \\ s_{32}^* & s_{33}^* \end{vmatrix}\\
&= \mu_{23}\mu_{31}x_1^*x_2^*x_3^*.\end{aligned} \tag{48}$$

Since $\det J^* = 0$ then the non-zero eigenvalues of the Jacobi matrix J^* are the solutions of the quadrate equation

$$\mu^2 - \mathrm{Tr}\, J^*\mu + \Delta^* = 0. \tag{49}$$

5. *Flip, flutter and divergence boundaries in the space of orbits* Substituting (43) into (48) and (49) one obtains:

$$\begin{aligned}\mathrm{Tr}\, J^* &= -x_2^*\ \ln(x_2^*/x_1^*A_2) - x_3^*\ \ln(x_3^*/x_1^*A_3),\\ \Delta^* &= x_2^*\ \ln(x_2^*/x_1^*A_2)\ln(x_3^*/x_1^*A_3).\end{aligned} \tag{50}$$

These formulas allow to construct in the space of orbits the images of the triangle of stability ABC and its sides – the flip, flutter and divergence boundaries. The domain of stability, defined by the inequalities $-1 \pm \mathrm{Tr}\, J^* < \Delta^* < 1$, becomes:

$$\begin{aligned}&-1 \pm [x_2^*\ \ln(x_2^*/x_1^*A_2) + x_3^*\ \ln(x_3^*/x_1^*A_3)]\\ &\quad < x_2^*\ \ln(x_2^*/x_1^*A_2)\ln(x_3^*/x_1^*A_3) < 1.\end{aligned} \tag{51}$$

The equation of the flip boundary $\mathrm{Tr}\, J^* = -(\Delta^* + 1)$ becomes:

$$\begin{aligned}&1 - x_2^*\ \ln(x_2^*/x_1^*A_2) - x_3^*\ \ln(x_3^*/x_1^*A_3)\\ &\quad + x_2^*\ \ln(x_2^*/x_1^*A_2)\ln(x_3^*/x_1^*A_3) = 0.\end{aligned} \tag{52}$$

The equation of the flutter boundary $\Delta^* = 1$ becomes:

$$x_2^*\ \ln(x_2^*/x_1^*A_2)\ln(x_1^*/x_1^*A_3) = 1, \tag{53}$$

and the equation of the divergence boundary $\operatorname{Tr} J^* = \Delta^* + 1$ becomes:

$$1 + x_2^* \ln(x_2^*/x_1^* A_2) + x_3^* \ln(x_3^*/x_1^* A_3) + x_2^* \ln(x_2^*/x_1^* A_2) \ln(x_3^*/x_1^* A_3) = 0. \quad (54)$$

Furthermore, the positions of resonances on a flutter boundary are the solutions of the following system of equations:

$$\begin{aligned} -x_2^* \ln(x_2^*/x_1^* A_2) - x_3^* \ln(x_3^*/x_1^* A_3) &= 2\cos 2\pi\Omega, \quad 0 \le \Omega \le 1, \\ x_2^* \ln(x_2^*/x_1^* A_2) \ln(x_3^*/x_1^* A_3) &= 1, \\ x_1^* + x_2^* + x_3^* &= 1. \end{aligned} \quad (55)$$

Thus, the images of domain of stability and flip, flutter and divergence boundaries clarify the qualitative description of the local features of dynamics within the vicinity of fixed points, and also the global features of dynamics connected with the existence of periodic doubling, resonance invariant curves, strange attractors and other types of chai.

6. *Example of linear bifurcation analysis* Consider a set of models of the type (37) with the constant parameters $A_2 = A_3 = 1$ and the changeable external parameters μ_{23}, μ_{31}. The inequalities (51) define the same domain of stability of equilibria for all these models (see Fig. 3 where this domain is shadowed). Eqs. (52)–(53) correspond

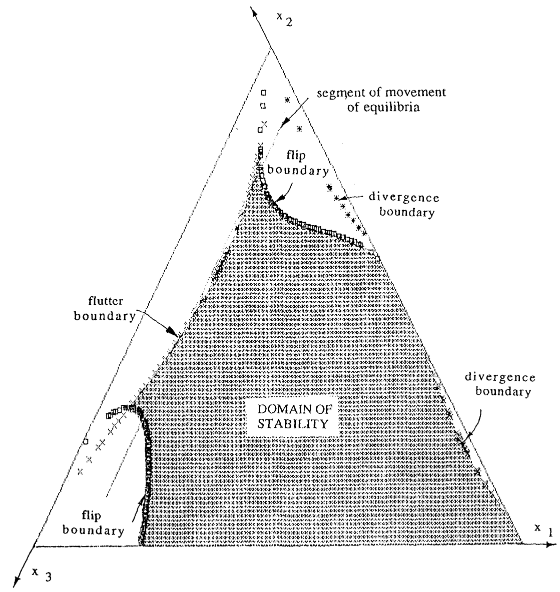

FIGURE 3 Domain of stability and flip, flutter and divergence boundaries of one population/three location relative dynamics.

to the flip, flutter and divergence boundaries. The travel of equilibria along the straight line between the points (0.1, 0.1, 0.8) and (0.1, 0.85, 0.05) is chosen with the purpose to show the transfer from stability to flutter and to flip bifurcations (see Fig. 3). Figure 4 presents the usual bifurcation diagram for the first coordinates $x_1(t)$ of the orbits. This diagram shows the following sequence of qualitative phenomena: stable two-periodic cycle, stable attractor, series of resonances

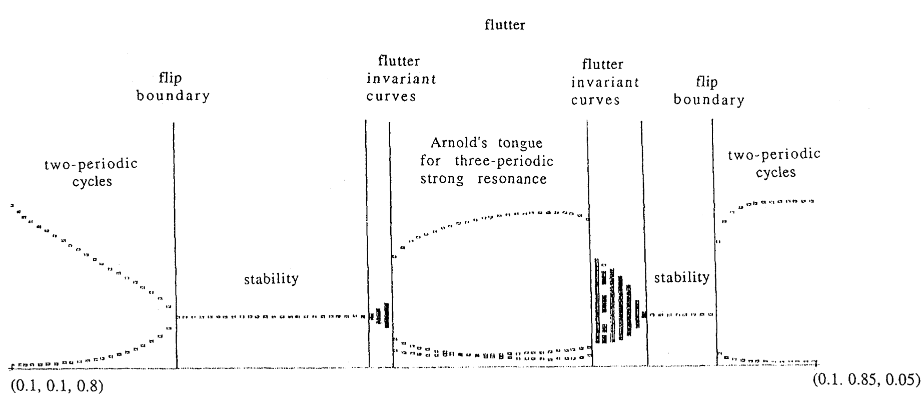

FIGURE 4 Bifurcation diagram for the population share x_1.

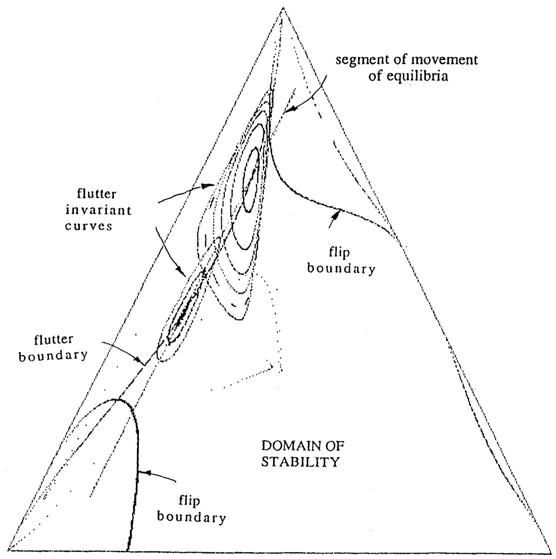

FIGURE 5 Planar bifurcation diagram: one population/three location relative dynamics.

including the Arnold tongue for three-period strong resonance, stable attractor and the mode-locking tongue for two-periodic cycle starting within the domain of stability. The corresponding planar bifurcation diagram is presented in Fig. 5 where the two locuses of invariant curves are clearly visible.

The reader can find other examples of the applications of linear bifurcation analysis to the labor-capital core-periphery relative discrete dynamics and to the analysis of new bifurcation phenomena in the classical Henon map in Sonis (1993; 1994; 1996).

7 CONCLUSIONS

This study presents three-tier vision of the recent developments in the discrete non-linear dynamics: the level of new mathematical models of the discrete non-linear dynamics recently developed in different – social and natural – fields of inquiry; the level of unified conceptual framework of the information minimizing or entropy maximizing principles for discrete non-linear dynamics and the level of linear bifurcation analysis defining the domains of structural stability and boundaries of structural changes in the qualitative properties of orbits. The development of the specific "calculus of bifurcations" obtains at present the theoretical and practical importance especially in connection with the new emerging interest to the analysis of the sustainability properties of economic, social and societal dynamics.

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