Controlling Hopf Bifurcations: Discrete-Time Systems

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(Received 20 July 1999)

Bifurcation control has attracted increasing attention in recent years. A simple and unified state-feedback methodology is developed in this paper for Hopf bifurcation control for discrete-time systems. The control task can be either shifting an existing Hopf bifurcation or creating a new Hopf bifurcation. Some computer simulations are included to illustrate the methodology and to verify the theoretical results.

Keywords: Feedback control, Hopf bifurcation, Limit cycle, Period-doubling bifurcation, Stability

1. INTRODUCTION

Bifurcation control means to design a controller that can modify the bifurcative properties of a given nonlinear system, so as to obtain some desired dynamical behaviors. Typical examples include delaying the onset of an inherent bifurcation, relocating an existing bifurcation point, modifying the shape or type of a bifurcation chain, introducing a new bifurcation at a preferable parameter value, stabilizing a bifurcated periodic trajectory, changing the multiplicity, amplitude, and/or frequency of some limit cycles emerging from bifurcation, optimizing the system performance near a bifurcation point, or a certain combination of some of these [1,2,4–6,9,10,14,17]. Bifurcation control is important not only in its own right, but also in providing

an effective strategy for chaos control. In particular, period-doubling bifurcation is a typical route to chaos in many nonlinear discrete-time dynamical systems. Bifurcation control is useful in many engineering applications, as discussed in [10].

System bifurcations can be controlled by using different methods, such as linear delayed state-feedback [7,8] or nonlinear state-feedback [2], using a wash-out filter [16], employing harmonic balance approximation [4,5,11,14,15], and applying the quadratic invariants in the normal form [13]. In [10], a unified linear as well as simple nonlinear state-feedback technique was developed for Hopf bifurcation control for continuous-time systems. In this paper, this new methodology is further extended to discrete-time systems. Discrete-time systems differ from the continuous ones in many aspects. For instance,

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the former has the typical period-doubling bifurcation but the latter generally does not. In the investigation of this paper, both problems of shifting and creating a Hopf bifurcation are discussed. Computer simulations are included to illustrate the methodology and to verify the theoretical results.

This paper is organized as follows. Section 2 briefly summarizes the classical Hopf bifurcation theory for discrete-time systems. Sections 3 and 4 study the state-feedback control problem for Hopf bifurcations, including computer simulation results. Section 5 concludes the investigation with some discussion.

2. HOPF BIFURCATIONS: PRELIMINARIES

In this section, the classical criterion for discrete Hopf bifurcations is briefly reviewed.

Consider a general two-dimensional parametrized system:

$$\begin{aligned} x_{k+1} &= f(x_k, y_k; \mu) \\ y_{k+1} &= g(x_k, y_k; \mu), \end{aligned} \quad (1)$$

with a real variable parameter $\mu \in R$ and an equilibrium point (x^*, y^*) , satisfying $x^* = f(x^*, y^*; \mu)$ and $y^* = g(x^*, y^*; \mu)$ simultaneously for all μ . Let $J(\mu)$ be its Jacobian at this equilibrium, with eigenvalues $\lambda_{1,2}(\mu)$ satisfying $\lambda_2(\mu) = \bar{\lambda}_1(\mu)$. If, despite some minor details,

$$|\lambda_1(\mu^*)| = 1 \quad \text{and} \quad \left. \frac{\partial |\lambda_1(\mu)|}{\partial \mu} \right|_{\mu=\mu^*} > 0 \quad (2)$$

then the system undergoes a *Hopf bifurcation* at the bifurcation point (x^*, y^*, μ^*) [3,12,14]. More precisely, in any small left-neighborhood of μ^* (i.e., $\mu < \mu^*$), (x^*, y^*) is a stable focus; and in any small right-neighborhood of μ^* (i.e., $\mu > \mu^*$), this focus changes to be unstable, usually surrounded by a limit cycle.

Here, the second condition in (2) refers to as the *transversality* condition for the crossing of the

eigenlocus at the unit circle, namely, the eigenlocus is not tangent to the circle. Moreover, both supercritical and subcritical bifurcations can be further distinguished, via however a rather complicated series coordinate transformations (see Theorem 9.7 of [12]).

3. CONTROLLING THE HOPF BIFURCATION

Conceptually, Hopf bifurcations can be relatively easily created for a given two-dimensional system, in a way similar to that for the continuous-time case studied in [10]. Both calculation and simulation have confirmed this observation. Therefore, the current interest is twisted to finding out if Hopf bifurcations can also be created in a one-dimensional system in some way, which is not intuitively obvious. For this purpose, consider a parametrized one-dimensional system in a general form

$$x_{k+1} = f(x_k; \mu) \quad (3)$$

with a real parameter $\mu \in R$ and an equilibrium point x^* . Clearly, this one-dimensional system cannot have a classical Hopf bifurcation due to the dimensional deficit. However, one may consider the following bifurcation control problem instead: Design a state-feedback controller, $u_k = u_k(x_k; \mu)$, to be added to the right-hand side of the given system (3), such that the controlled system displays a Hopf bifurcation in the extended phase plane associated with system (3), in a sense to be further described below.

To facilitate quantitative analysis and calculation, a specific form of controller, namely, the popular delayed state-feedback controller [7,8]

$$u_k = u_k(x_k - x_{k-1}; \mu), \quad (4)$$

is chosen in the following discussion. It is easily seen that, in principle, the methodology can be applied to other forms of controllers.

By introducing a new state variable, $y_k = x_k - x_{k-1}$, the controlled system can be written

in the following extended form:

$$\begin{aligned} x_{k+1} &= f(x_k; \mu) + u_k(y_k; \mu), \\ y_{k+1} &= x_{k+1} - x_k. \end{aligned} \quad (5)$$

If the controller (4) is designed to satisfy $u_k(0; \mu) = 0$, then it will not change the original equilibrium point, x^* , of the given system (3). The controlled system has the Jacobian at $(x^0, y^0) = (x^*, 0)$ as

$$J(\mu) = \begin{bmatrix} f_x & u_y \\ f_x - 1 & u_y \end{bmatrix}_{x=x^*, y=y^0=0}, \quad (6)$$

where $f_x = \partial f / \partial x_k$ and $u_y = \partial u_k / \partial y_k$, with eigenvalues

$$\lambda_{1,2}(\mu) = \frac{1}{2}(f_x + u_y) \pm \frac{1}{2}\sqrt{(f_x + u_y)^2 - 4u_y}. \quad (7)$$

Conditions for the controller to satisfy $\lambda_1(\mu) = \bar{\lambda}_2(\mu)$ and (2) are

$$\begin{aligned} (f_x + u_y)^2 &\leq 4u_y, \quad |\lambda_{1,2}(\mu^*)| = 1 \quad \text{and} \\ \frac{\partial |\lambda_{1,2}(\mu^*)|}{\partial \mu} &> 0. \end{aligned} \quad (8)$$

4. SIMULATION RESULTS

In this section, some simple but illustrative numerical examples are presented.

To show an example, the familiar Logistic map,

$$x_{k+1} = \mu x_k(1 - x_k), \quad (9)$$

is used, which has equilibrium points $x^* = 0$ and $x^* = (\mu - 1)/\mu$. This one-dimensional system does not have Hopf bifurcation, but rather, has the well-known period-doubling bifurcation leading to chaos as μ varies from 1.0 to 4.0 (see Fig. 1). With the controller $u_k(y_k; \mu) = e^\mu y_k$ being added to it, the extended controlled system

$$\begin{aligned} x_{k+1} &= \mu x_k(1 - x_k) + u_k(y_k; \mu), \\ y_{k+1} &= x_{k+1} - x_k, \end{aligned} \quad (10)$$

satisfies all the stated conditions, and, as expected, has a Hopf bifurcation at $\mu^0 = 0$. The corresponding simulation results are shown in Figs. 2–4, where in Fig. 2: $\mu = -0.01 < 0.0 = \mu^0$, the controlled system has a stable focus, $(x^*, y^*) = (0, 0)$; in Fig. 3: $\mu = 0.0 = \mu^0$, the stable focus bifurcates to a limit cycle of period six,

$$(x^*, y^*) = \{(0.5, 0.5), (0.5, 0.0), (0.0, -0.5), (-0.5, -0.5), (-0.5, 0.0), (0.0, 0.5)\}$$

when the initial point is $(x_0, y_0) = (0.5, 0.5)$; in Fig. 4: $\mu = 0.01 > 0.0 = \mu_0$, the focus $(x^*, y^*) = (0, 0)$ becomes unstable.

This bifurcation control strategy has been tested on other systems; for instance, on the following

FIGURE 1 Period-doubling bifurcation of the Logistic map ($1 < \mu \leq 4$).

FIGURE 2 Orbit of the controlled Logistic map ($\mu = -0.01 < 0.0 = \mu^0$).

FIGURE 3 Orbit of the controlled Logistic map ($\mu = 0.0 = \mu^0$).

system:

$$x_{k+1} = -\mu x_k - (1 + \mu)x_k^2. \quad (11)$$

This system also has a period-doubling bifurcations rather than a Hopf bifurcation. Yet, a new Hopf bifurcation can be created at $(x^0, y^0, \mu^0) = (0, 0, 1)$, by various feedback controllers. For example, either one of the following controllers works quite well:

$$u_k = \mu y_k, \quad u_k = \mu^2 y_k$$

FIGURE 4 Orbit of the controlled Logistic map ($\mu = 0.01 > 0.0 = \mu^0$).

which are linear with respect to y_k (and x_k), and

$$u_k = (2\mu x_k + 2\mu + \mu^2)y_k$$

which is nonlinear with respect to (x_k, y_k) .

5. CONCLUSIONS

In this paper, a simple and unified state-feedback control methodology has been developed for Hopf bifurcations for discrete-time systems, for both problems of shifting and creating a Hopf bifurcation point in the controlled system. Although the discussion here has been restricted to two-dimensional systems, the basic idea and the proposed approach can be extended to higher-dimensional dynamical systems. It is anticipated that some real applications of the new control method can be found in the near future.

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