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Global Properties of Symmetric Competition Models with Riddling and Blowout Phenomena

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In this paper the problem of chaos synchronization, and the related phenomena of riddling, blowout and on–off intermittency, are considered for discrete time competition models with identical competitors. The global properties which determine the different effects of riddling and blowout bifurcations are studied by the method of critical curves, a tool for the study of the global dynamical properties of two-dimensional noninvertible maps. These techniques are applied to the study of a dynamic market-share competition model.

Keywords: Chaos synchronization; Noninvertible maps; Critical curves; Market share competition

1. INTRODUCTION

Dynamic models of strategic interaction between two competitors are often represented by a map of the plane into itself $T:(x_t, y_t) \to (x_{t+1}, y_{t+1})$, defined as

$$\begin{aligned} x_{t+1} &= T_1(x_t, y_t) \\ y_{t+1} &= T_2(x_t, y_t), \end{aligned} \tag{1}$$

where x_t and y_t represent the *state variables* which characterize, at time t, the behavior of the two competitors. The iteration of T gives the time evolution of the system: Given an initial condition $(x_0, y_0) \in \mathbb{R}^2$, the repeated application of the map T uniquely defines the *trajectory*

$$\tau(x_0, y_0) = \{(x_t, y_t) = T^t(x_0, y_0), t \in \mathbb{N}\} \tag{2}$$

Models of this form are often used for the description of biological, social or economic systems where two individuals, or two populations, compete in the environment where they operate. For example, in ecologic modeling x_t and y_t may represent the densities of two populations which live in the same territory and compete for

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resources; in economic modeling x_t and y_t may represent the choices (such as, productions or investments) of two firms which produce the same goods and compete in order to maximize their profits by acquiring greater portions of a given market. The long run (or asymptotic) evolution of such systems may be characterized by the coexistence of the two competitors, generally with some degree of dominance (or prevalence) of one of them, or by the complete elimination of a competitor, which may be seen as an extreme form of dominance. In a deterministic framework, the final outcome depends on the values of the parameters which characterize the competitors and, in some cases, on the initial condition (x_0, y_0), whose influence may be crucial when several coexisting attractors of the dynamical systems are present, each with its own basin of attraction. If the two competitors are very different, then one of them generally dominates in the long run, and the outcome of the competition may be independent of the initial point, *i.e.*, the stronger competitor may win even starting from a disadvantageous starting condition. More interesting situations may occur if the two competitors are similar or, at the limit, if they are absolutely identical. In this paper we investigate some particular properties of competition models with identical competitors.

In the case of identical competitors, the dynamical system must remain the same if the variables x and y are interchanged, *i.e.*, $T \circ S = S \circ T$, where $S: (x, y) \to (y, x)$ is the reflection through the diagonal

$$\Delta = \{(x, y) \in \mathbb{R}^2 | x = y\}. \tag{3}$$

This symmetry property implies that the diagonal is mapped into itself, *i.e.*, $T(\Delta) \subseteq \Delta$, which corresponds with the obvious statement that, in a deterministic framework, identical competitors, starting from identical initial conditions, behave identically for each time. The trajectories embedded into Δ, *i.e.*, characterized by $x_t = y_t$ for every t, are called *synchronized trajectories*, and they are governed by the one-dimensional map given by the restriction of T to the invariant submanifold Δ

$$x_{t+1} = f(x_t) \quad \text{with } f = T|_\Delta : \Delta \to \Delta. \tag{4}$$

In [8] the one-dimensional model (4) has been considered as the model of a *representative agent* whose dynamics summarize the common behavior of the two synchronized competitors.

A trajectory starting out of Δ, *i.e.*, with $x_0 \neq y_0$, is said to synchronize if $|x_t - y_t| \to 0$ as $t \to +\infty$. A question which naturally arises, in the case of symmetric competition models, is whether identical competitors starting from different initial conditions will synchronize, so that the asymptotic behavior is governed by the simpler one-dimensional model (4). This question can be reformulated as follows. Let A_s be an attractor of the one-dimensional map (4). Is it also an attractor for the two-dimensional map T? Of course, an attractor A_s of the restriction f is stable with respect to perturbations along Δ, so an answer to the question raised above can be given through a study of the stability of A_s with respect to perturbations transverse to Δ (*transverse stability*). If A_s is a cycle, then the study of the transverse stability is the usual one, based on the modulus of the eigenvalues of the cycle in the direction transverse to Δ, whereas the problem becomes more interesting when the dynamics restricted to the invariant submanifold are chaotic. In this case the phenomenon of *chaos synchronization* may occur (see *e.g.* [13, 29, 15]), *i.e.*, the time evolution of the two competitors synchronize in the long run even if each of them behaves chaotically.

Dynamical systems with chaotic trajectories embedded into an invariant submanifold of lower dimensionality than the total phase space have raised an increasing interest in the scientific community (see *e.g.* [5, 10]). *Milnor attractors* (see [22]) which are not stable in Lyapunov sense appear quite naturally in this context, together with phenomena like *on-off intermittency* and *riddled basins* (see *e.g.* [4, 28, 20]). In the recent

literature on chaos synchronization, stability statements are given in terms of the *transverse Lyapunov exponents*, by which the "average" local behavior of the trajectories in a neighborhood of the invariant set A_s can be understood, and new kinds of bifurcations can be detected, such as the *riddling bifurcation*, through which A_s is transformed from a Lyapunov attractor into an attractor in the weaker Milnor sense, or the *blowout bifurcation*, through which A_s is transformed from a Milnor attractor into a chaotic saddle.

However, as noticed by many authors (see *e.g.* [5, 10, 15, 20]), even if the occurrence of riddling and blowout bifurcations is detected through the transverse Lyapunov exponents, *i.e.*, from a local analysis of the linear approximation of the map near Δ, their effects are determined by the global properties of the map. In fact, the effect of these bifurcations is related to the fate of the trajectories which are locally repelled away from a neighborhood of the Milnor attractor A_s, since they may reach another attractor or they may be folded back toward A_s by the action of the nonlinearities acting far from Δ. In the models with one-dimensional chaos synchronization the map (1) is often a noninvertible map of the plane, because its one-dimensional restriction f must be a noninvertible map in order to have chaotic motion along the invariant subspace Δ. In this case, the global dynamical properties of the map T can be described by the method of *critical curves* (see [14, 24, 3]) and, in particular, the reinjection of the locally repelled trajectories can be described in terms of their folding action (see *e.g.* [24] or [25] for a description of the geometric properties of a noninvertible map related to the folding, or foliation, of its phase space). This idea has been recently proposed in [9] for the study of symmetric maps arising in game theory, and in [8] for the study of the effects of small asymmetries due to parameters mismatches. In these two papers, the critical curves have been used to obtain the boundary of a compact trapping region, called *absorbing area* following [24], inside which intermittency and blowout phenomena are confined. These methods have been recently introduced in the physical literature, for the study of a system of coupled chaotic oscillators, in [21] and [6]. In particular, in [6] the concept of *minimal invariant absorbing area* is defined in order to give a global characterization of the different dynamical scenarios related to riddling and blowout bifurcations.

The main purposes of this paper are to explain the relations between the problems related to chaos synchronization and the properties of critical curves, and to illustrate their application to the study of symmetric competition models. These relations may be important in practical problems because they can be used to define compact regions of the phase plane that acts as trapping bounded vessels inside which the trajectories starting near Δ are confined, thus giving an upper bound for the oscillations (bursts) which characterize both the transient dynamics of the trajectories which eventually synchronize, and the persistent oscillations (on – off intermittency) which characterize the dynamics just after a blowout bifurcation. Moreover, contacts between the portions of critical curves bounding the minimal absorbing area surrounding a Milnor attractor and the basin boundaries may mark the transition between local and global riddling phenomena (see [21, 6]).

The paper is organized as follows. In Section 2 we recall some definitions and results related to the study of transverse stability, and related local bifurcations, revealed by the study of transverse Lyapunov exponents. In Section 3 we present some properties of noninvertible maps of the plane, and in particular we describe a procedure to obtain the boundary of an invariant absorbing area. In Section 4 the results described in Sections 2 and 3 are applied to the study of a brand competition model for market share. In Section 5 we conclude with a brief outline of some possible extensions to higher dimensional models and to models where the symmetry is broken by small parameters' mismatches.

2. CHAOTIC SYNCHRONIZATION AND RELATED LOCAL BIFURCATIONS

In this section we recall some definitions and results related to the problem of chaos synchronization, see [10] for a more complete treatment. Let T be a map of the plane, Δ a one-dimensional trapping subspace and A_s a chaotic attractor (with absolutely continuous invariant measure on it) of the restriction (4) of T to Δ. The key property for the study of the transverse stability of A_s is that it includes infinitely many periodic orbits which are unstable in the direction along Δ. For any of these cycles it is easy to compute the associated eigenvalues. In fact, due to the symmetry of the map, the Jacobian matrix of T computed at any point of Δ, say $DT(x, x) = \{T_{ij}(x)\}$, is such that $T_{11} = T_{22}$ and $T_{12} = T_{21}$. The two orthogonal eigenvectors of such a symmetric matrix are one parallel to Δ, say $\mathbf{v}_{\|} = (1, 1)$, and one perpendicular to it, say $\mathbf{v}_{\perp} = (1, -1)$, with related eigenvalues given by

$$\lambda_{\|}(x) = T_{11}(x) + T_{12}(x) \quad \text{and}$$
$$\lambda_{\perp}(x) = T_{11}(x) - T_{12}(x)$$

respectively. Of course, $\lambda_{\|}(x) = f'(x)$. Since the product of matrices with the structure of $DT(x, x)$ has the same structure as well, a k-cycle $\{x_1, \ldots, x_k\}$ embedded into Δ has eigenvalues $\lambda_{\|}^k = \prod_{i=1}^k \lambda_{\|}(x_i)$ and $\lambda_{\perp}^k = \prod_{i=1}^k \lambda_{\perp}(x_i)$, with eigenvectors $\mathbf{v}_{\|}$ and $\mathbf{v}_{\perp}$ respectively. So, for a chaotic set $A_s \subset \Delta$, infinitely many transverse Lyapunov exponents can be defined as

$$\Lambda_{\perp} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N} \ln |\lambda_{\perp}(x_i)| \tag{5}$$

where $\{x_i = f^i(x_0),\ i \geq 0\}$ is a trajectory embedded in A_s. If x_0 belongs to a k-cycle then $\Lambda_{\perp} = \ln|\lambda_{\perp}^k|$, so that the cycle is transversely stable if $\Lambda_{\perp} < 0$, whereas if x_0 belongs to a generic aperiodic trajectory embedded inside the chaotic set A_s then $\Lambda_{\perp}$ is the *natural transverse Lyapunov exponent* $\Lambda_{\perp}^{\text{nat}}$, where by the term "natural" we mean the Lyapunov exponent associated to the natural, or SBR (Sinai-Bowen-Ruelle), measure, *i.e.*, computed for a typical trajectory taken in the chaotic attractor A_s. Since infinitely many cycles, all unstable along Δ, are embedded inside a chaotic attractor A_s, a spectrum of transverse Lyapunov exponents can be defined, see [10]

$$\Lambda_{\perp}^{\min} \leq \cdots \leq \Lambda_{\perp}^{\text{nat}} \leq \cdots \leq \Lambda_{\perp}^{\max} \tag{6}$$

The meaning of the inequalities in (6) can be intuitively understood on the basis of the property that $\Lambda_{\perp}^{\text{nat}}$ expresses a sort of "weighted balance" between the transversely repelling and transversely attracting cycles (see [27]). If $\Lambda_{\perp}^{\max} < 0$, *i.e.*, all the cycles embedded in A_s are transversely stable, then A_s is asymptotically stable, in the usual Lyapunov sense, for the two-dimensional map T. However, it may occur that some cycles embedded in the chaotic set A_s become transversely unstable, *i.e.*, $\Lambda_{\perp}^{\max} > 0$, while $\Lambda_{\perp}^{\text{nat}} < 0$. In this case, A_s is no longer Lyapunov stable, but it continues to be a *Milnor attractor* [22] *i.e.*, it attracts a positive (Lebesgue) measure set of points of the two-dimensional phase space. The transition from asymptotic stability to attractivity only in Milnor sense, marked by a change of sign of $\Lambda_{\perp}^{\max}$ from negative to positive, is denoted as the *riddling bifurcation* in [18] (or *bubbling bifurcation* in [32]).

Even if the occurrence of such bifurcations is detected through the study of the transverse Lyapunov exponents, their effects depend on the action of the nonlinearities far from Δ, that is, on the global properties of the dynamical system. In fact, after the riddling bifurcation two possible scenarios can be observed according to the fate of the trajectories that are locally repelled along (or near) the local unstable manifolds of the transversely repelling cycles:

(**L**) they can be reinjected towards Δ, so that the dynamics of such trajectories are characterized by some bursts far from Δ before synchronizing on it (a very long sequence of such bursts, which can be observed when $\Lambda_{\perp}$ is close to zero, has been called *on-off intermittency* in [28]);

(**G**) they may belong to the basin of another attractor, in which case the phenomenon of *riddled basins* ([4]) is obtained.

Some authors call *local riddling* the situation (L) and, by contrast, *global riddling* the situation (G) (see [5, 19, 21]). When also $\Lambda_{\perp}^{\text{nat}}$ becomes positive, due to the fact that the transversely unstable periodic orbits embedded into A_s have a greater weight as compared with the stable ones, a *blowout bifurcation* occurs, after which A_s is no longer a Milnor attractor, because it attracts a set of points of zero measure, and becomes a *chaotic saddle*, see [10]. In particular, for $\lambda_{\perp}^{\min} > 0$ all the cycles embedded into Δ are transversely repelling, and A_s is called *normally repelling chaotic saddle* see [10]. Also the macroscopic effect of a blowout bifurcation is strongly influenced by the behavior of the dynamical system far from the invariant submanifold Δ: The trajectories starting close to the chaotic saddle may be attracted by some attracting set far from Δ or remain inside a two-dimensional compact set located around the chaotic saddle A_s, thus giving on-off intermittency. The study of transverse Lyapunov exponents says nothing about the fate of the locally repelled trajectories, and the occurrence of the different scenarios described above is determined by the global properties of the map. When T is a noninvertible map, these global properties can be described by the method of *critical curves*, which may be used to obtain the *minimal invariant absorbing area* inside which intermittency phenomena are confined.

3. GLOBAL PROPERTIES OF NONINVERTIBLE MAPS AND ABSORBING AREAS

Noninvertible map means "many-to-one", that is, distinct points $p_1 \neq p_2$ may have the same image, *i.e.*, $T(p_1) = T(p_2) = p$. Geometrically, the action of a noninvertible map of the plane can be expressed by saying that it "folds and pleats" the plane, so that the two distinct points p_1 and p_2 are mapped into the same point p. This is formally expressed by saying that p has several distinct rank-1 preimages, *i.e.*, several inverses are defined in p, and that these inverses "unfold" the plane.

More formally, a two-dimensional map $T: (x, y) \to (x', y')$, defined by (1), is said to be noninvertible if the rank-1 preimages $(x, y) = T^{-1}(x', y')$, obtained by solving the system (1) with respect to x and y, may be more than one. In this case, the plane can be subdivided into regions Z_k, $k \geq 0$, whose points have k distinct rank-1 preimages. Generally, as the point (x', y') varies in the plane $\mathbb{R}^2$, pairs of preimages appear or disappear as it crosses the boundaries separating different regions, hence such boundaries are characterized by the presence of at least two coincident (merging) preimages. This leads to the definition of the critical curves, one of the distinguishing features of noninvertible maps. Following the notations of [14, 24, 3], the *critical set LC* (from the French "Ligne Critique") is defined as the locus of points having two, or more, coincident rank-1 preimages, located on a set (*set of merging preimages*) called LC_{-1}. LC is the two-dimensional generalization of the notion of critical value (when it is a local minimum or maximum value) of a one-dimensional map,[1] LC_{-1} is the generalization of the notion of critical point (when it is a local extremum point). Arcs of LC separate the regions of the phase plane characterized by a different number of real rank-1 preimages (see [14, 24, 3]).

Points of LC_{-1} in which the map is differentiable are necessarily points where the Jacobian determinant vanishes: in fact in any neighborhood of a point of LC_{-1} there are at least two distinct points which are mapped by T in the same point, hence the map is not locally invertible in points of LC_{-1}. This implies, for a differentiable map, that

$$LC_{-1} \subseteq J_0 = \{(x, y) \in \mathbb{R}^2 | \det DT(x, y) = 0\} \quad (7)$$

[1] This terminology, and notation, originates from the notion of critical points as it is used in the classical works of Julia and Fatou.

More generally, since T is locally an orientation preserving map near points (x, y) such that det $DT(x, y) > 0$ and orientation reversing if det $DT(x, y) < 0$, then for a continuous map T the fold LC_{-1} is included in the set where det $DT(x, y)$ changes sign.

The critical sets of rank k are the images of rank k of LC_{-1} denoted by $LC_{k-1} = T^k(LC_{-1}) = T^{k-1}(LC)$, LC_0 being LC. Segments of critical curves of rank-k, $k = 0, 1, \ldots$, can be used in order to define trapping regions of the phase plane. An *absorbing area* $\mathcal{A}$ is a bounded region of the plane whose boundary is given by critical curve segments (segments of the critical curve LC and its images) such that a neighborhood $U \supset \mathcal{A}$ exists whose points enter $\mathcal{A}$ after a finite number of iterations and then never escape it, *i.e.*, $T(\mathcal{A}) \subseteq \mathcal{A}$ (see [24], Chapter 4, or [6], for more details).

Following [24] or [3] a practical procedure can be outlined in order to obtain the boundary of an absorbing area (although it is difficult to give a general method). Starting from a portion of LC_{-1}, approximately taken in the region occupied by the area of interest, its images of increasing rank are computed until a closed region is obtained. When such a region is mapped into itself, then it is an absorbing area $\mathcal{A}$. The length of the initial segment is to be taken, in general, by a trial and error method, although several suggestions are given in the books referenced above. Once an absorbing area $\mathcal{A}$ is found, in order to see if it is invariant (or strictly mapped into itself) the same procedure must be repeated by taking only the portion

$$\gamma = \mathcal{A} \cap LC_{-1} \tag{8}$$

as the starting segment. Then one of the following two cases occurs:

(i) the union of m iterates of γ (for a suitable m) covers the whole boundary of $\mathcal{A}$; in which case $\mathcal{A}$ is an invariant absorbing area, and

$$\partial\mathcal{A} \subset \bigcup_{k=1}^{m} T^k(\gamma) \tag{9}$$

(ii) no natural m exists such that $\bigcup_{i=1}^{m} T^i(\gamma)$ covers the whole boundary of $\mathcal{A}$, in which case $\mathcal{A}$ is not invariant but strictly mapped into itself. An invariant absorbing area is obtained by $\cap_{n>0} T^n(\mathcal{A})$ (and may be obtained by a finite number of images of $\mathcal{A}$).

The *minimal invariant absorbing area* is the smallest absorbing area that includes the Milnor attractor on which the synchronized dynamics occur. Indeed, boundaries of trapping regions can also be obtained by the union of segments of critical curves and portions of unstable sets of saddle cycles, and in this case we have a so called *absorbing areas of mixed type* (see [24]). We don't enter here in such details, as in the examples given in this paper only standard absorbing areas (*i.e.*, completely bounded by critical arcs) are used. However, the arguments given in the following remain substantially unchanged if absorbing areas of mixed type are met.

4. A COMPETITION MODEL FOR MARKET SHARE

As an example, we consider a dynamic *brand competition model* proposed in [5]. This model describes a market where a population of consumers can choose between two brands of homogeneous goods, produced by two competing firms. Let x, y represent the marketing efforts of two firms (advertising, R&D, *etc.*) and B the total sales potential of the market (in terms of customer market expenditures). If firm 1 spends x dollars of effort and firm 2 spends y dollars, then the share of the market (sales revenue) accruing to firm 1 and to firm 2 is Bs_1 and $Bs_2 = B - Bs_1$, respectively, where

$$s_1 = \frac{ax^{\beta_1}}{ax^{\beta_1} + by^{\beta_2}}, \quad s_2 = \frac{by^{\beta_2}}{ax^{\beta_1} + by^{\beta_2}}. \tag{10}$$

The terms $A_1 = ax^{\beta_1}$ and $A_2 = by^{\beta_2}$ represent the recruitment of customers by firm 1 and 2, respectively, given x and y units of effort, and

the parameters a and b denote the relative effectiveness of the effort made by the firms. Since $(dA_1/dx)(x/A_1)=\beta_1$ and $(dA_2/dx)(x/A_2)=\beta_2$ the parameters β_1 and β_2 denote the elasticities of the attraction of firm (or brand) i with regard to the effort of firm i. Market share attraction models have been used frequently in empirical work; (see, *e.g.* [11, 26]). The dynamic model given in [7] is obtained by assuming that the two competitors change their marketing efforts adaptively, in response to the profits achieved in the previous period:

$$\begin{aligned} x_{t+1} &= x_t + \lambda_1(Bs_{1t} - x_t)x_t \\ y_{t+1} &= y_t + \lambda_2(Bs_{2t} - y_t)y_t \end{aligned} \tag{11}$$

The decision rule the firms use is an adaptive adjustment (a type of anchoring attitude) widely used in decision theory (see [31, 30]). The parameters $\lambda_i > 0$, $i=1,2$, measure the rate of adjustment. If we insert the market shares from Eq. (10), the competition model becomes

$$T: \begin{cases} x_{t+1} = x_t + \lambda_1 x_t\left(B\frac{x_t^{\beta_1}}{x_t^{\beta_1}+ky_t^{\beta_2}} - x_t\right) \\ y_{t+1} = y_t + \lambda_2 y_t\left(B\frac{ky_t^{\beta_2}}{x_t^{\beta_1}+ky_t^{\beta_2}} - y_t\right) \end{cases} \tag{12}$$

where $k := b/a$. A general study of the dynamic properties of the map (12) is given in [7].

The map (12) is a noninvertible map of $Z_4-Z_2-Z_0$ type. The set of points for which det $DT(x, y)=0$ is given by the union of two branches, denoted by $LC_{-1}^{(a)}$ and $LC_{-1}^{(b)}$ in Figure 1a. Also LC is the union of two branches, denoted by $LC^{(a)} = T(LC_{-1}^{(a)})$ and $LC^{(b)} = T(LC_{-1}^{(b)})$, see Figure 1b. The branch $LC^{(b)}$ separates the region Z_0, whose points have no preimages, from the region Z_2, whose points have two distinct rank-1 preimages. $LC^{(a)}$ separates the region Z_2 from Z_4, where the points in Z_4 have four distinct preimages.

Here we consider the symmetric case of identical firms, obtained for

$$\lambda_1 = \lambda_2 = \lambda; \quad \beta_1 = \beta_2 = \beta; \quad k = 1$$

This case has already been considered by Kopel *et al.* [16], where it is argued that the parameter β measures the degree of competition between the firms. We now use this example to show some applications of the methods described in Section 3.

The restriction (4) of the symmetric map to the invariant diagonal is given by

$$f(x) = \left(1 + \frac{1}{2}\lambda B\right)x - \lambda x^2. \tag{13}$$

which is conjugate to the standard logistic map $z=\mu z(1-z)$, with

$$\mu = 1 + \frac{1}{2}\lambda B \tag{14}$$

by the linear transformation $x=z(1+\lambda B/2)/\lambda$. For the symmetric map, the Jacobian matrix, computed at a point of the diagonal Δ, is

$$\begin{aligned} &DT(x, x; \lambda, B, \beta) \\ &= \begin{bmatrix} 1 - 2\lambda x + \frac{\lambda B(\beta+2)}{4} & -\frac{\lambda B\beta}{4} \\ -\frac{\lambda B\beta}{4} & 1 - 2\lambda + \frac{\lambda B(\beta+2)}{4} \end{bmatrix}. \end{aligned} \tag{15}$$

Hence, the eigenvalues are $\lambda_{\|} = 1+(1/2)\lambda B-2\lambda x$ and $\lambda_{\perp} = 1+(1/2)\lambda B(1+\beta)-2\lambda x$, and the transverse Lyapunov exponents are readily obtained:

$$\Lambda_{\perp} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}\ln\left|1 + \frac{1}{2}\lambda B(1+\beta) - 2\lambda x_n\right|.$$

It is important to note that the parameter β only appears in the transverse eigenvalue $\lambda_{\perp}$, *i.e.*, β is a *normal parameter*: it has no influence on the dynamics along the invariant submanifold Δ, and only influences the transverse stability. This allows us to consider fixed values of the parameters λ and B, such that a chaotic attractor $A_s \subset \Delta$ of the map (13) exists, with an absolutely continuous invariant measure on it. So, we can study the transverse stability of A_s as the degree of

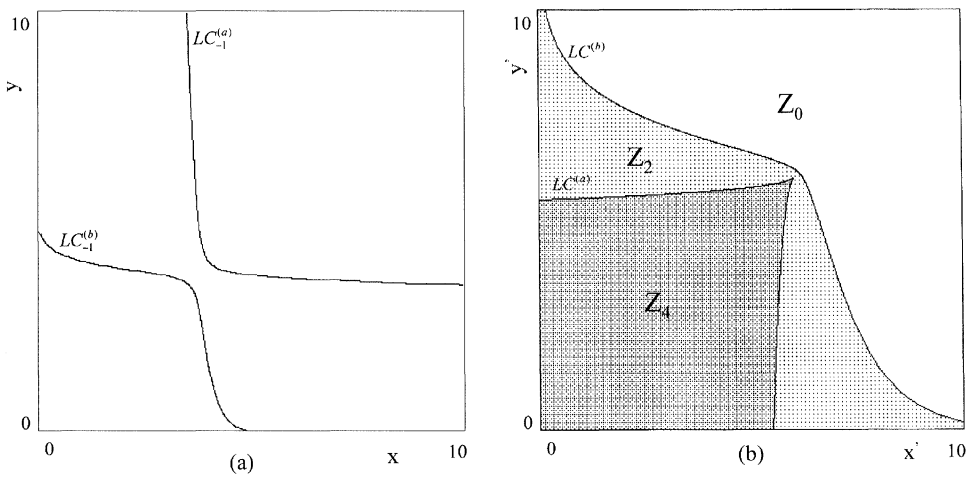

FIGURE 1 (a) Critical curves of rank-0, LC_{-1}, for the map (12), numerically obtained as the locus of points where the Jacobian $\det(DT(x,y))=0$, for $\lambda_1=\lambda_2=0.5$, $\beta_1=\beta_2=0.2$, $B=10$ and $k=1$. (b) Critical curves of rank-1, $LC=T(LC_{-1})$. These curves separate the plane into the regions Z_4, Z_2 and Z_0, whose points have four, two, and no rank-1 preimages respectively.

competition between the two firms, measured by the parameter β, varies. Suitable values of the aggregate parameter λB, at which chaotic intervals for the restriction (13) exist, are obtained from the well known properties of the logistic map (see *e.g.* [12, 23]). For example, at the parameter value $\bar{\mu}_2 = 3.5748049387592\cdots$ the period-4 cycle of the logistic map undergoes the homoclinic bifurcation, at which four cyclic chaotic intervals are obtained by the merging of 8 cyclic chaotic intervals. By using $\lambda B = 2(\bar{\mu}_2 - 1)$ we get a four-band chaotic set A_s along the diagonal Δ, as shown in Figure 2a. With the parameters used in Figure 2, *i.e.*, $B=10$ and $\beta=0.16$, we have $\Lambda_\perp^{\max} > 0$ and $\Lambda_\perp^{\text{nat}} = -2.6\times 10^{-2} < 0$. Hence, A_s is a Milnor attractor and local riddling occurs. The generic trajectory starting from initial conditions taken in the white region of Figure 2a leads to synchronization, and the points of the dark region generate interrupted trajectories, involving negative values of the state variables. The Milnor attractor A_s is included inside a minimal invariant absorbing area whose boundary can be easily obtained by five iterations of an arc of LC_{-1}, as shown in Figure 2b. This absorbing area constitutes a trapping region inside which the bursts observed during the transient are contained. This is clearly seen in Figure 2c, where the points of the transient part of a typical trajectories which synchronizes are represented. During the transient, the time evolution of the system is characterized by several bursts away from Δ before synchronization occurs, as shown in Figure 2d, where the difference $x_t - y_t$, computed along the trajectory of Figure 2c, is represented *versus* time.

In such a situation, a method to obtain trajectories which never synchronize, so that the bursts never stop and the iterated points fill up the whole minimal absorbing area, consists in the introduction of a small parameters' mismatch (see *e.g.* [6]), such as λ_1 slightly different from λ_2 or β_1 slightly different from β_2, so that the symmetry is broken. This implies that the invariance of Δ is lost, and consequently the one-dimensional Milnor attractor embedded in no longer exists. The study of the effects of small parameters' mismatches may be important in economic dynamic modelling, as stressed in [8, 16].

A similar effect is obtained even in the symmetric case, if the value of β is increased so that $\Lambda_\perp^{\text{nat}}$ increases until it changes sign, *i.e.*, a blowout bifurcation occurs. After this bifurcation the bursts which characterize the first part of the trajectory of Figures 2c and 2d, never stop, *i.e.*,

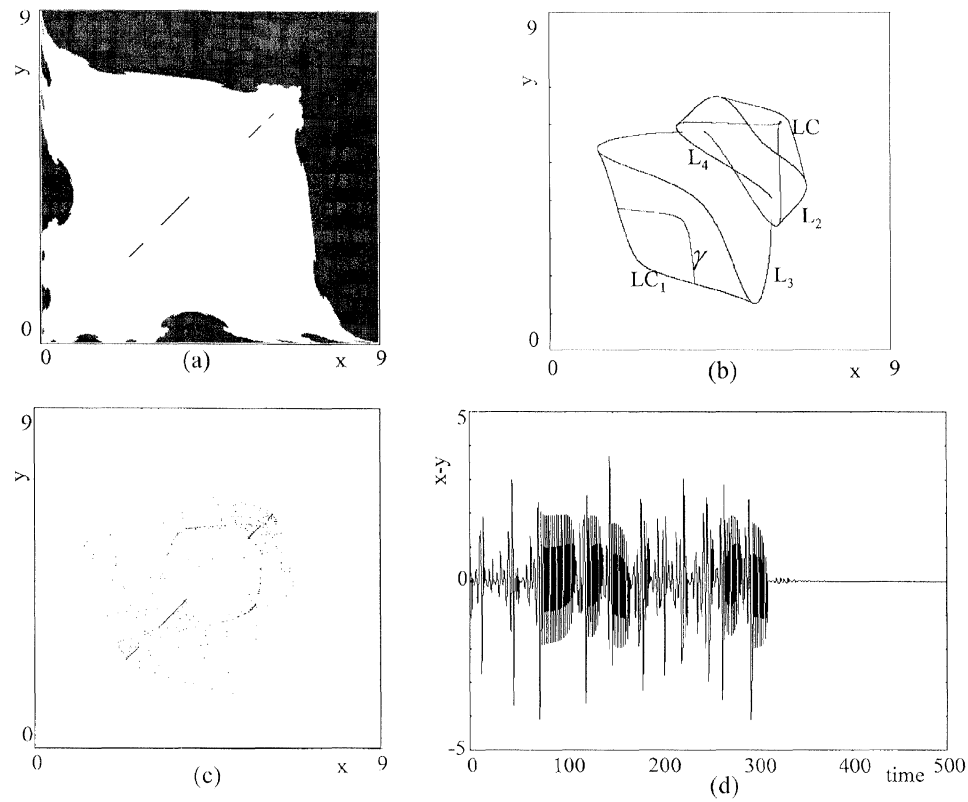

FIGURE 2 Numerical explorations of the dynamic behaviors of the symmetric brand competition model (12) with parameters $\beta_1 = \beta_2 = 0.16$, $B = 10$, $k = 1$ and $\lambda_1 = \lambda_2 = 0.51496098\ldots = 2(\bar{\mu}_2 - 1)/B$. (a) The white region represents the set of points that generate trajectories which synchronize on the 4-cyclic chaotic set A_s; the points of the grey region generate interrupted trajectories, involving negative values of the state variables. (b) Boundary of the absorbing area around the Milnor attractor A_s, obtained by the segments of critical curves $LC = T(\gamma)$, $LC_1 = T(LC), \ldots, LC_k = T(LC_{k-1})$, $k = 2, \ldots, 4$, images of the arc $\gamma \in LC_{-1}$. (c) Points of the transient part of a typical trajectories which synchronizes. (d) The difference $x_t - y_t$, computed along the trajectory of Figure 2c, is represented *versus* time during the first 500 periods.

the firms never synchronize. A_s is now a chaotic saddle, and on-off intermittency is observed. This is what happens in the situation shown in Figure 3, obtained for $\beta = 0.19$, at which $\Lambda_\perp^{\text{nat}} = 1.6 \times 10^{-2} > 0$. Now the point of a generic trajectory starting from the white region fill the whole absorbing area, still bounded by segments of critical arcs.

Another situation, obtained with $\beta = 0.09$, is shown in Figure 4. In Figure 4a the 4-cyclic chaotic set A_s, which is a Milnor attractor with $\Lambda_\perp^{\max} > 0$ and $\Lambda_\perp^{\text{nat}} = -0.15 < 0$, coexists with an attracting cycle C_2 of period 2, with periodic points out of Δ: the white region represents the set of points generating trajectories that synchronize to $\mathbf{A}_s$, the light grey regions represent the basin of the cycle C_2. The locally repelled trajectories cannot reach C_2 because of the presence of an absorbing area surrounding A_s. This is shown in Figure 4a. The locally repelled trajectories are folded back by the boundaries of the absorbing area, and after some bursts away from Δ they synchronize. An increase of β causes a contact between the absorbing area and the basin of C_2 which leads to the destruction of the absorbing area, so that some of the trajectories that are repelled from A_s can converge to C_2, and the basin of A_s becomes riddled (Fig. 4b). This example shows a transition from a locally riddled to a globally riddled dynamics caused by a contact

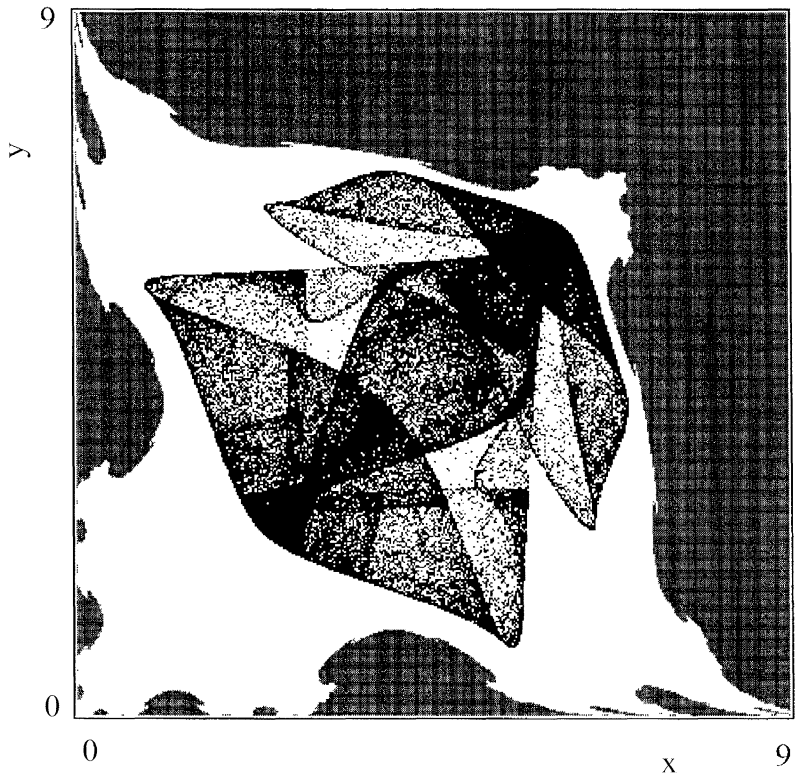

FIGURE 3 With the same parameters λ_i, B and k as in Figure 2, and $\beta_1 = \beta_2 = 0.19$, $\Lambda_\perp^{nat} > 0$, hence a generic trajectory starting from the white region fills the whole absorbing area, bounded by segments of critical arcs.

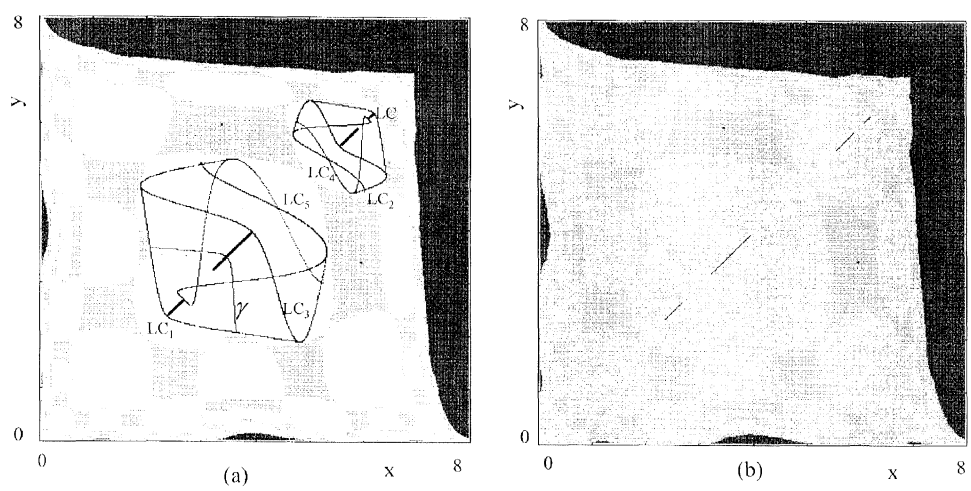

FIGURE 4 (a) With the same parameters λ_i, B and k as in Figure 2, and $\beta_1 = \beta_2 = 0.09$, the 4-cyclic chaotic set A_s, which is a Milnor attractor with $\Lambda_\perp^{max} > 0$ and $\Lambda_\perp^{nat} = -0.15 < 0$, coexists with an attracting cycle C_2 of period 2, with periodic points out of Δ. The points of the white region generate trajectories which synchronize on A_s, whereas points in the light-grey region converge to C_2. The boundaries of the minimal absorbing area including A_s are also shown. (b) For $\beta_1 = \beta_2 = 0.1$ the basin of A_s is riddled with the basin of C_2.

between the boundary of a minimal invariant absorbing area and the boundary of its basin of attraction.

5. CONCLUSIONS

We have analyzed a particular type of two-dimensional discrete dynamical systems that one can meet in the modeling of competition between two economic agents. Due to the assumed symmetry between the agents, the phenomenon of chaos synchronization can be observed.

The critical curves, a tool for the study of the global properties of noninvertible maps of the plane, have been used to obtain the boundary of compact trapping regions, called *absorbing areas*, inside which intermittency and blowout phenomena are confined.

Many of these concepts can be generalized to the case of more than two competitors. In general, a linear transformation M is called a *symmetry* of a map T if $M \circ T = T \circ M$. In this case, the set of fixed points of M is an invariant subspace for the dynamical system. For example, with three identical competitors, whose time evolution is modeled by the iteration of the map $T: (x_t, y_t, z_t) \to (x_{t+1}, y_{t+1}, z_{t+1})$, an evident group of symmetries is represented by the set of permutations of the coordinates. This implies, for example, that the planes Π_1, Π_2 and Π_3, of equations $y = z$, $x = z$; and $x = y$ respectively, are invariant, and the trajectories embedded inside them, are governed by the two-dimensional restrictions of T to Π_i, represent *partial synchronization*, since they are characterized by the fact that two of the three competitors behave in a synchronized way. The intersection of the three invariant planes is the invariant line $S = \{(x, x, x) \in \mathbb{R}^3\}$ where *total synchronization* occurs. Examples of models describing the competition among three competitors are the Cournot triopoly games recently studied in [1, 2]. For higher dimensional models, such as oligopoly models with n identical competitors, it may be interesting to study the formation of *clusters* of $k < n$ synchronized competitors, coexisting with other clusters or with non-synchronized competitors. The extension of the global methods described in this paper involves critical surfaces (or hypersurfaces) in order to bound compact regions where intermittency phenomena are confined.

The delimitation of minimal absorbing regions, which include the Milnor attractors embedded in invariant subspaces of lower dimensionality, also allows one to understand the effects of asymmetries causes by small parameters mismatches, that is, the consequences of the presence of quasi-identical economic agents. In this case, as shown in [6, 9] for the two-dimensional case, on-off intermittency phenomena are observed, and the boundaries of the minimal absorbing area behave as a vessel inside which the intermittent busts are confined (see also [8, 16] for a discussion on the economic implications).

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