Difference Equation for A Population Model

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(Received 11 August 1999)

In this paper we consider a difference equation of the form

$$u(t+2) = \alpha u(t+1) + \beta \frac{u(t+1) - \alpha u(t)}{\alpha u(t)}, \quad \text{for } t > -\infty,$$

seems to be a general statement of a relative socio-spatial dynamics. Indeed this equation is one of “population model”. We investigate behavior of solutions and expression of analytic general solutions of this model.

Keywords: Difference equation, Dynamics model, Analytic solution

1. INTRODUCTION

Recently many kinds of population models are formulated with difference equations [1,3,4]. Here we consider the following difference equation:

$$u(t+2) = \alpha u(t+1) + \beta \frac{u(t+1) - \alpha u(t)}{\alpha u(t)}, \quad (1.1)$$

where $\alpha = 1 + r$, in which r is the net (births minus death) endogenous population (stock) growth rate, and the second term is a function depicting net immigration at $t+1$, in turn a function of a “momentum” to grow from t to $t+1$. So that we assume that $\alpha > 0$ ($r > -1$) and $\beta > 0$ in (1.1).

The model (1.1) was proposed by Prof. Dimitrios Dendrinos [2]. We are interested in it because it is a simple three-time-period model. Let

$$u(t+2) = u_1(t+2) + u_2(t+2),$$

where

$$\begin{aligned} u_1(t+2) &= \alpha u(t+1), \\ u_2(t+2) &= \beta \frac{u(t+1) - \alpha u(t)}{\alpha u(t)}. \end{aligned}$$

In this model α and β are presumably independent of the time t . So we assume that α and β are constants. $u_1(t+2)$ is a term for endogenous population growth rate from $t+1$ to $t+2$, and $u_2(t+2)$ due to net in-migration rate. Indeed

$$\begin{aligned} u_1(t+2) &= \alpha u(t+1) = \alpha \{u_1(t+1) + u_2(t+1)\}, \\ u_2(t+2) &= \beta \frac{u(t+1) - \alpha u(t)}{\alpha u(t)} \\ &= \beta \frac{u(t+1) - u_1(t+1)}{u_1(t+1)} \\ &= \beta \frac{u_2(t+1)}{u_1(t+1)}, \end{aligned}$$

where $u_1(t+1)$ is the endogenous population growth rate from t to $t+1$, and $u_2(t+1)$ due to net in-migration rate.

In this paper we will study it, at first, from the viewpoint of (real) state-space analysis, especially of stability properties as $t \rightarrow \infty$, for positive solutions. Next we study it from the complex analytic view point.

We may write (1.1) as:

$$\begin{aligned} u(t+2) - \alpha u(t+1) \\ = \frac{c}{u(t)} \{u(t+1) - \alpha u(t)\}, \quad c = \frac{\beta}{\alpha}. \end{aligned} \quad (1.1')$$

When $\alpha \neq 1$, (1.1) admits the unique equilibrium value $c = \beta/\alpha$. When $\alpha = 1$, any value may be the equilibrium.

2. STABILITY PROPERTIES OF POSITIVE SOLUTIONS

In this section, we suppose that $u(t)$ is a solution of (1.1) which is positive for $t \geq t_0$, with the initial values $u(t_0)$ and $u(t_0+1)$.

We say $u(t)$ is *stable* if there are positive constants L_1, L_2 such that

$$0 < L_1 \leq u(t_0+n) \leq L_2 \quad \text{for } n \in \mathbb{N}.$$

$u(t)$ is *asymptotically stable* if there is b_0 such that, for any $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} |u(t_0+n) - b_0| &< \epsilon \quad \text{for } n \geq n_0, \\ \text{i.e. } u(t_0+n) &\rightarrow b_0 (n \rightarrow \infty). \end{aligned}$$

We write (1.1) as

$$\begin{aligned} u(t+2) - \alpha u(t+1) \\ = \frac{c}{u(t)} (u(t+1) - \alpha u(t)), \quad c = \frac{\beta}{\alpha}. \end{aligned} \quad (2.1)$$

It is easy to see that, if $\lim_{n \rightarrow \infty} u(t_0+n) = u_0$ for some t_0 and $\alpha \neq 1$, then we must have $u_0 = c$.

2.1. When $\alpha > 1$

Suppose $u(t_0+1) - \alpha u(t_0) \geq 0$ for some t_0 . Then we have $u(t_0+2) - \alpha u(t_0+1) \geq 0$ by (2.1). Hence

$$\begin{aligned} u(t+n) &\geq \alpha u(t+n-1) \geq \cdots \\ &\geq \alpha^{n-1} u(t+1) \geq \alpha^n u(t) > 0. \end{aligned}$$

Thus we have $u(t_0+n) \rightarrow \infty (n \rightarrow \infty)$ and the solution $u(t)$ is not stable. So, we assume that $u(t+1) - \alpha u(t) < 0$ for any $t \geq 0$.

(i) Suppose $u(t_0) \geq c$. If $u(t_0+1) > u(t_0)$, then

$$\begin{aligned} 0 &\leq \alpha u(t_0+1) - u(t_0+2) \\ &= \frac{c}{u(t_0)} \{\alpha u(t_0) - u(t_0+1)\} \\ &\leq \alpha u(t_0) - u(t_0+1). \end{aligned}$$

So we have

$$0 < \alpha\{u(t_0+1) - u(t_0)\} \leq u(t_0+2) - u(t_0+1). \quad (2.2)$$

Since $u(t_0+1) > u(t_0) \geq c$, we have

$$\begin{aligned} u(t_0+n+1) - u(t_0+n) \\ \geq \alpha^n \{u(t_0+1) - u(t_0)\} > 0. \end{aligned}$$

Therefore we have $u(t_0+n) \rightarrow \infty$ and $u(t)$ is not stable. So we must have that $u(t+1) \leq u(t)$ whenever $u(t) \geq c$. If otherwise, i.e. if there is t_0 such that $u(t_0) \geq c$ and $u(t_0+1) > u(t_0)$, then $u(t)$ is not stable. If $u(t)$ is stable and $u(t) \geq c$ for any t , then c is an asymptotically stable equilib. point: $u(t) \downarrow c$.

(ii) Suppose $u(t_0) < c$ for some t_0 . If $u(t_0+1) < u(t_0)$, then

$$\begin{aligned} u(t_0+2) - u(t_0+1) &< \alpha(u(t_0+1) - u(t_0)) < 0, \\ u(t_0+2) &< u(t_0+1) < u(t_0) \leq c, \end{aligned}$$

and

$$\begin{aligned} u(t_0+n) \\ < (1 + \alpha + \cdots + \alpha^{n-1}) \{u(t_0+1) - u(t_0)\} < 0. \end{aligned}$$

Thus $u(t_0 + n)$ tend to $-\infty$, and $u(t)$ is not stable. Therefore, if $u(t)$ is stable, we must have

$$\begin{aligned} u(t+1) &\leq u(t) \quad \text{whenever } u(t) > c, \\ u(t+1) &\geq u(t) \quad \text{whenever } u(t) < c. \end{aligned} \quad (2.3)$$

Conversely, if (2.3) holds, then $u(t) \rightarrow c$ as $t \rightarrow \infty$. Therefore we see that $u(t)$ is stable if and only if $u(t)$ is asymptotically stable, and c is the only stable equilib. point.

In Section 4, we have really exist stable positive solutions when $\alpha > 1$.

2.2. When $\alpha < 1$

Suppose $u(t_0 + 1) - \alpha u(t_0) \leq 0$, for some t_0 . Then by (2.1) we have

$$\begin{aligned} 0 < u(t_0 + n) &\leq \alpha u(t_0 + n - 1) \leq \cdots \\ &\leq \alpha^{n-1} u(t_0 + 1) \leq \alpha^n u(t_0). \end{aligned}$$

Since $\alpha < 1$, we get $u(t_0 + n) \rightarrow 0$ and the population tends to death. So we may assume that

$$u(t+1) - \alpha u(t) > 0 \quad \text{for any } t.$$

(i) When $u(t_0) \leq c$ for some t_0 . We have

$$\begin{aligned} \alpha u(t_0 + 1) - u(t_0 + 2) &= \frac{c}{u(t_0)} \{ \alpha u(t_0) - u(t_0 + 1) \} \\ &\leq \alpha u(t_0) - u(t_0 + 1) < 0, \end{aligned}$$

hence

$$\alpha(u(t_0 + 1) - u(t_0)) \leq u(t_0 + 2) - u(t_0 + 1).$$

(i-a) Suppose $u(t_0 + 1) - u(t_0) \geq 0$. Then $u(t_0 + 2) - u(t_0 + 1) \geq 0$. Suppose there is k_0 such that $u(t_0 + k) \leq c$ ($0 \leq k < k_0$) and $u(t_0 + k_0) > c$. (If there is no such k_0 , we have $u(t_0 + k) \uparrow c$.) Then

$$\begin{aligned} u(t_0 + k_0 + 1) &= \alpha u(t_0 + k_0) + \frac{c}{u(t_0 + k_0 - 1)} u(t_0 + k_0) - \alpha c \\ &> \alpha u(t_0 + k_0) + \frac{c}{u(t_0 + k_0 - 1)} u(t_0 + k_0) \\ &\quad - \alpha u(t_0 + k_0) \\ &= \frac{c}{u(t_0 + k_0 - 1)} u(t_0 + k_0) \geq u(t_0 + k_0). \end{aligned}$$

From (2.1) we have, using

$$\begin{aligned} \frac{c}{u(t_0 + k_0)} - 1 &\leq 0, \\ \frac{1}{\alpha} \{ u(t_0 + k_0 + 2) - u(t_0 + k_0 + 1) \} &= \left\{ 1 + \frac{1}{\alpha} \left(\frac{c}{u(t_0 + k_0)} - 1 \right) \right\} u(t_0 + k_0 + 1) - c \\ &= \frac{u(t_0 + k_0 + 1)}{\alpha} \left(\frac{c}{u(t_0 + k_0)} - 1 \right) \\ &\quad + u(t_0 + k_0 + 1) - c \\ &\leq u(t_0 + k_0) \left(\frac{c}{u(t_0 + k_0)} - 1 \right) \\ &\quad + u(t_0 + k_0 + 1) - c \\ &= u(t_0 + k_0 + 1) - u(t_0 + k_0). \end{aligned}$$

So we have

$$\begin{aligned} u(t_0 + k_0 + 2) - u(t_0 + k_0 + 1) &\leq \alpha \{ u(t_0 + k_0 + 1) - u(t_0 + k_0) \}. \end{aligned}$$

If $u(t_0 + k_0 + 2) - u(t_0 + k_0 + 1) \geq 0$, we may repeat the same argument. If we assume $u(t_0 + k_0 + n + 1) - u(t_0 + k_0 + n) \geq 0$ for any $n \geq 1$, we would have

$$\begin{aligned} u(t_0 + k_0 + n) &< (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha + 1) \{ u(t_0 + k_0 + 1) \\ &\quad - u(t_0 + k_0) \} + u(t_0 + k_0) \\ &= \frac{1 - \alpha^n}{1 - \alpha} (u(t_0 + k_0 + 1) - u(t_0 + k_0)) \\ &\quad + u(t_0 + k_0), \end{aligned}$$

and

$$\begin{aligned} u(t_0 + k_0 + n) \uparrow u_0 &\leq \frac{1}{1 - \alpha} \{ u(t_0 + k_0 + 1) - \alpha u(t_0 + k_0) \} < \infty. \end{aligned}$$

Obviously $u_0 \geq u(t_0 + k_0) > c$, which is a contradiction since $u(t_0 + n) \rightarrow u_0$ and we must have $u_0 = c$. Hence there is n_0 such that $u(t_0 + k_0 + n) \leq u(t_0 + k_0 + n + 1)$ ($n < n_0$) and $u(t_0 + k_0 + n_0) > u(t_0 + k_0 + n_0 + 1)$.

(i-b) Suppose $u(t_0 + 1) < u(t_0) \leq c$. If $u(t_0 + n) \leq u(t_0 + n - 1)$ ($n \geq 1$), then $u(t_0 + n) \downarrow u_0$ and u_0 cannot be positive. Hence $u(t_0 + n) \downarrow 0$.

If there is k_0 such that $u(t_0 + k) \leq u(t_0 + k - 1)$ ($k < k_0$) and $u(t_0 + k_0) > u(t_0 + k_0 - 1)$, we have from (2.1), using

$$\begin{aligned} & \frac{c}{u(t_0 + k_0 - 1)} - 1 > 0, \\ & \frac{1}{\alpha} \{u(t_0 + k_0 + 1) - u(t_0 + k_0)\} \\ &= \frac{u(t_0 + k_0)}{\alpha} \left(\frac{c}{u(t_0 + k_0 - 1)} - 1 \right) \\ &+ u(t_0 + k_0) - c. \end{aligned}$$

Since $u(t_0 + k_0) - \alpha u(t_0 + k_0 - 1) > 0$, we have

$$\begin{aligned} & \frac{1}{\alpha} \{u(t_0 + k_0 + 1) - u(t_0 + k_0)\} \\ &> u(t_0 + k_0 - 1) \left(\frac{c}{u(t_0 + k_0 - 1)} - 1 \right) \\ &+ u(t_0 + k_0) - c \\ &= u(t_0 + k_0) - u(t_0 + k_0 - 1). \end{aligned}$$

If $u(t_0 + k_0) \leq c$, we may repeat the argument. Suppose there is h_0 such that $u(t_0 + k_0 + h) \leq c$ ($h < h_0$) and $u(t_0 + k_0 + h_0) > c$. (If there is no such h_0 , then $u(t_0 + n) \uparrow c$.) By the above argument, we have $u(t_0 + k_0 + h_0 + 1) > u(t_0 + k_0 + h_0) > c$. Now we proceed to the following case (ii):

(ii) When $u(t_0) > c$. Then

$$\begin{aligned} 0 &\geq \alpha u(t_0 + 1) - u(t_0 + 2) \\ &= \frac{c}{u(t_0)} \{ \alpha u(t_0) - u(t_0 + 1) \} \\ &> \alpha u(t_0) - u(t_0 + 1), \end{aligned}$$

so we get

$$\alpha \{u(t_0 + 1) - u(t_0)\} > u(t_0 + 2) - u(t_0 + 1). \quad (2.4)$$

(ii-a) Suppose $u(t_0 + 1) > u(t_0) > 0$. If $u(t_0 + n) > u(t_0 + n - 1)$ ($n \geq 1$), we get

$$\begin{aligned} & \alpha \{u(t_0 + n - 1) - u(t_0 + n - 2)\} \\ &> u(t_0 + n) - u(t_0 + n - 1), \\ & u(t_0 + n) - u(t_0 + n - 1) < \alpha^{n-1} \{u(t_0 + 1) - u(t_0)\} \\ & u(t_0 + n) < u(t_0) + \frac{1 - \alpha^n}{1 - \alpha} \{u(t_0 + 1) - u(t_0)\} \\ &< \frac{u(t_0 + 1) - \alpha u(t_0)}{1 - \alpha}, \end{aligned}$$

hence

$$c < u(t_0 + n) \uparrow u_0 < \frac{u(t_0 + 1)}{1 - \alpha},$$

which is a contradiction. Thus there is n_0 such that

$$\begin{aligned} & u(t_0 + n) > u(t_0 + n - 1) (n < n_0) \quad \text{and} \\ & u(t_0 + n) > u(t_0 + n - 1). \end{aligned}$$

Thus we proceed to the following case (ii-b).

(ii-b) Suppose $u(t_0 + 1) < u(t_0)$. Then $u(t_0 + 2) < u(t_0 + 1)$ by (2.4). If $u(t_0 + n) \geq c$, then we get $u(t_0 + n) \downarrow c$. If $u(t_0 + n) \geq c$ ($n < n_0$) and $u(t_0 + n_0) < c$, then we come back to the case (i).

Thus we see that, when $0 < \alpha < 1$, we have either $u(t + n) \rightarrow 0$ or $u(t)$ is stable.

2.3. When $\alpha = 1$

We have in this case

$$u(t + 2) - u(t + 1) = \frac{c}{u(t)} (u(t + 1) - u(t))$$

Note that,

$$\text{if } \lim_{n \rightarrow \infty} u(t_0 + n) = u_0 \neq \pm\infty,$$

$$\text{then } \lim_{n \rightarrow \infty} \frac{u(t_0 + n + 1)}{u(t_0 + n)} = 1.$$

(i) When $u(t_0 + 1) > u(t_0)$. Then $u(t_0 + n) > u(t_0 + n - 1)$ ($n \in \mathbb{N}$). If $u(t_0) \geq c$, then

$$b = \frac{c}{u(t_0 + 1)} < 1.$$

Hence

$$\begin{aligned}
& u(t_0 + n) - u(t_0 + n - 1) \\
&= \frac{c}{u(t_0 + n - 2)} \cdots \frac{c}{u(t_0 + 1)} \frac{c}{u(t_0)} \\
&\quad \times \{u(t_0 + 1) - u(t_0)\} \\
&< b^{n-2} \{u(t_0 + 1) - u(t_0)\}, \\
&u(t_0 + n) < \frac{1 - b^{n-1}}{1 - b} \{u(t_0 + 1) - u(t_0)\} \\
&\quad + u(t_0 + 1), \\
&u(t_0 + n) \uparrow u_0 \leq \frac{(2 - b)u(t_0 + 1) - u(t_0)}{1 - b}, \quad (u_0 > c).
\end{aligned}$$

If $u(t_0) < c$, then $u(t_0 + 2) - u(t_0 + 1) = c/u(t_0) \times \{u(t_0 + 1) - u(t_0)\} > u(t_0 + 1) - u(t_0) > 0$ ($n \in \mathbb{N}$), hence there is n_0 such that $u(t_0 + n_0) > c$. Hereafter $u(t_0 + n)$ behaves as above and we get $u(t_0 + n) \uparrow u_0 < \infty$, ($u_0 > c$).

(ii) When $u(t_0 + 1) < u(t_0)$. Then $u(t_0 + n + 1) - u(t_0 + n) < 0$ for $u(t_0 + j) > 0$ for $0 \leq j \leq n - 1$.

If $u(t_0) \leq c$, then $u(t_0 - 1 + n) < c$, hence

$$\begin{aligned}
& u(t_0 + n + 2) - u(t_0 + n + 1) \\
&= \frac{c}{u(t_0 + n)} \{u(t_0 + n + 1) - u(t_0 + n)\} \\
&\leq u(t_0 + n + 1) - u(t_0 + n) < 0,
\end{aligned}$$

if $u(t_0 + n) > 0$. Thus there is n_0 such that $u(t_0 + n) > 0$ for $n < n_0$ and $u(t_0 + n_0) \leq 0$. Then $u(t_0 + n_0 + 2) - u(t_0 + n_0 + 1) > 0$ since $u(t_0 + n_0 + 1) - u(t_0 + n_0) < 0$, and $u(t_0 + n_0 + 3) - u(t_0 + n_0 + 2) < 0$, since $u(t_0 + n_0 + 1) < 0$. Thus $u(t_0 + n)$ oscillate.

If $u(t_0) > c$ and $u(t_0 + n_1) \leq c$ for some n_1 , then from above discussion, $u(t_0 + n)$ oscillate.

If $u(t_0) > c$ and $u(t_0 + n) > c$ for all n , then there is a u_0 such that $u(t_0 + n) \downarrow u_0 \geq c$. Indeed if we put $u(t_0 + n) = c(1 + (1/n))$, then $u(t_0 + n) \downarrow u_0 = c$. Therefore, when $\alpha = 1$, we get that

$$\begin{aligned}
& \text{either } u(t_0 + n_0) \leq 0 \quad (\exists n_0), \\
& \text{or } u(t_0 + n) \downarrow u_0 \geq c, \\
& \text{or } u(t_0 + n) \uparrow u_0 > c.
\end{aligned}$$

2.4. Summary of the Section

Behaviors of $u(t)$ vary greatly according to the initial values $u(t_0)$ and $u(t_0 + 1)$, which might be seen to be a default of this model. But the model is very interesting because it includes immigration rate β , in addition to the net growth rate α . May be we better consider β to depend time t and population $u(t)$, and α would be influenced by β .

Next we consider complex analytic treatment of the Eq. (1.1), since it admits interesting general solution which would give some prospects in dealing population models.

3. ANALYTIC SOLUTION

3.1. When $\alpha > 1$

We have showed that, if $u(t_0 + n)$ is stable, then $u(t_0 + n) \rightarrow c$ as $n \rightarrow \infty$. Then $v(t) = u(t) - c$ satisfies $v(t_0 + n) \rightarrow 0$ and

$$\begin{aligned}
v(t + 2) &= \alpha v(t + 1) + \beta \frac{v(t + 1) - v(t)}{\alpha v(t) + \beta} \\
&= \alpha v(t + 1) + \frac{v(t + 1) - v(t)}{1 + (v(t)/c)} \\
&= (\alpha + 1)v(t + 1) - v(t) \\
&\quad + F(v(t), v(t + 1)),
\end{aligned} \tag{3.1}$$

where $F(v_1, v_2)$ is the sum of terms of higher degree with respect to v_1, v_2 . Thus, if t is large and $v(t)$ is small, then $v(t)$ approximately satisfies

$$v(t + 2) - (\alpha + 1)v(t + 1) + v(t) = 0. \tag{3.2}$$

Therefore

$$u(t_0 + n) = c + C_1 \lambda_1^n + C_2 \lambda_2^n + w(t_0 + n), \tag{3.2'}$$

in which λ_1, λ_2 , are roots of the characteristic equation

$$\begin{aligned}
& \lambda^2 - (\alpha + 1)\lambda + 1 = 0, \\
& \lambda_1 = \frac{1}{2} \left(\alpha + 1 + \sqrt{(\alpha + 1)^2 - 4} \right), \quad \lambda_2 = \frac{1}{\lambda_1}.
\end{aligned} \tag{3.3}$$

Since $\lambda_1 > 1$, we must have $C_1 = 0$ if we suppose $u(t+n)$ is stable when $n \rightarrow \infty$. Therefore we consider a particular solution of (1.1) in the form

$$u(t_0 + t) = c + \sum_{k=1}^{\infty} a_k \lambda_2^{kt}, \quad (3.4)$$

where

$$0 < \lambda = \lambda_2 = \frac{1}{2} \left(\alpha + 1 - \sqrt{(\alpha + 1)^2 - 4} \right) < 1.$$

If we suppose $u(t+n)$ is stable when $n \rightarrow -\infty$, then we also consider solutions in the form

$$u(t_0 + t) = c + \sum_{k=1}^{\infty} a_{-k} \lambda_2^{-kt}. \quad (3.5)$$

Furthermore, if $u(t_0 - n) \rightarrow 0$ ($n \rightarrow \infty$), then we have

$$\lim_{n \rightarrow -\infty} \frac{u(t+n+2)}{u(t+n+1)} = \alpha.$$

So that we put a formal solution to (1.1) in the form

$$u(t) = \sum_{k=1}^{\infty} a_k \alpha^{kt}.$$

Then a_1 is arbitrary and $a_k = 0$ ($k \geq 2$). Suppose $x(t) = \phi(t) \alpha^t$ be a solution of (1.1) such that $x(t-n) \rightarrow 0$ as $n \rightarrow +\infty$, and ϕ satisfies $\lim_{n \rightarrow \infty} |\phi(t-n) - a_1| < M$ uniformly on any compact set for some constant M . Then we have

$$\lim_{n \rightarrow \infty} (\phi(t-n+1) - \phi(t-n)) = 0.$$

3.2. When $\alpha < 1$

Suppose $u(t_0 + n)$ is stable. Suppose $u(t_0 + n) \rightarrow c$. Then $v(t) = u(t) - c$ satisfy (3.2), hence $u(t_0 + n)$ is written in the form (3.2'), where $|\lambda_j| = 1$, $j = 1, 2$, which shows that $u(t_0 + n)$ cannot tend to c . Hence $u(t_0 + n)$ is stable but not asymptotically stable.

Furthermore, if $u(t_0 + n) \rightarrow 0$ ($n \rightarrow +\infty$), then we have

$$\lim_{n \rightarrow +\infty} \frac{u(t+n+2)}{u(t+n+1)} = \alpha.$$

So that we put a formal solution to (1.1) in the form

$$u(t) = \sum_{k=1}^{\infty} a_k \alpha^{kt}.$$

Then a_1 is arbitrary and $a_k = 0$ ($k \geq 2$). Suppose $x(t) = \phi(t) \alpha^t$ be a solution of (1.1) such that $x(t+n) \rightarrow 0$ as $n \rightarrow +\infty$, and ϕ satisfies $\lim_{n \rightarrow \infty} |\phi(t+n) - a_1| < M$ uniformly on any compact set for some constant M . Then we have

$$\lim_{n \rightarrow \infty} (\phi(t+n+1) - \phi(t+n)) = 0.$$

3.3. When $\alpha = 1$

Suppose $u(t_0 + n) \uparrow u_0$ or $u(t_0 + n) \downarrow u_0$ with $u_0 > c$, then

$$\frac{u(t_0 + n + 2) - u(t_0 + n + 1)}{u(t_0 + n + 1) - u(t_0 + n)} \rightarrow \frac{c}{u_0} < 1.$$

Hence we get

$$u(t_0 + t + 1) - u(t_0 + t) = \sum_{k=1}^{\infty} d_k \left(\frac{c}{u_0} \right)^{kt}$$

and $u(t_0 + t)$ will be obtained.

4. EXISTENCE OF ANALYTIC STABLE SOLUTIONS IN THE CASE $\alpha > 1$

In this section, we write $u(t_0 + t)$ simply as $u(t)$. Time t is of course a real variable. But in this section, we consider t to be a complex variable, and we will prove existence of analytic solutions.

First we shall show existence of a particular solution of (3.1).

When $\alpha > 1$, we can determine a formal solution to (3.1) in the form (4.1)

$$v(t) = \sum_{n=1}^{\infty} a^n \lambda^{nt}, \quad (4.1)$$

where $0 < \lambda = \lambda_2 < 1$.

$$\begin{aligned} & a_n \beta \{ \lambda^{2n} - (1 + \alpha) \lambda^n + 1 \} \\ &= \alpha \sum_{k=1}^{n-1} a_k a_{n-k} \lambda^{n-k} (\alpha - \lambda^{n-k}), (a_1 : \text{arbitrary}) \end{aligned}$$

4.1. Make a Map T and Existence of a Fixed Point of T

We rewrite (3.1) as

$$v(t) = \beta \frac{(1 + \alpha)v(t+1) - v(t+2)}{\alpha v(t+2) - \alpha^2 v(t+1) + \beta}$$

which is written as

$$v(t) = f(v(t+1), v(t+2)) \quad (4.2)$$

where

$$f(v_1, v_2) = \beta \frac{(1 + \alpha)v_1 - v_2}{\alpha v_2 - \alpha^2 v_1 + \beta}.$$

Let N be a positive integer. Put $P_N(t) = \sum_{n=1}^N a_n \lambda^{nt}$ and $y(t) = v(t) - P_N(t)$. Then by (4.2) we obtain

$$\begin{aligned} y(t) &= f(y(t+1) + P_N(t+1), y(t+2) + P_N(t+2)) \\ &\quad - P_N(t) \\ &= g_1(t, y(t+1), y(t+2)) + g_2(t) \\ &= g(t, y(t+1), y(t+2)), \end{aligned} \quad (4.3)$$

in which

$$\begin{aligned} & g_1(t, y(t+1), y(t+2)) \\ &= f(y(t+1) + P_N(t+1), y(t+2) + P_N(t+2)) \\ &\quad - f(P_N(t+1), P_N(t+2)), \\ & g_2(t) = f(P_N(t+1), P_N(t+2)) - P_N(t). \end{aligned}$$

Put

$$\begin{aligned} S(\rho) &= \{t \in \mathbb{C} : |\lambda^t| \leq \rho\} \\ H(A, \rho) &= \{y : y(t) \text{ is holomorphic and} \\ &\quad |y(t)| \leq A|\lambda^t|^{N+1} \text{ for } t \in S(\rho)\}. \end{aligned}$$

Take $A > 0$ and $0 < \rho < 1$, which will be determined later. For $y(t) \in H(A, \rho)$, put

$$\begin{aligned} T[y](t) &= g(t, y(t+1), y(t+2)) \\ &= g_1(t, y(t+1), y(t+2)) + g_2(t). \end{aligned} \quad (4.4)$$

We will show that, constants A and ρ may be chosen so that T maps $H(A, \rho)$ into $H(A, \rho)$.

We note that, if $|v_1|, |v_2| < \beta/(4\alpha^2)$, then

$$\left| \frac{\partial f}{\partial v_1} \right|, \left| \frac{\partial f}{\partial v_2} \right| \leq 8(1 + \alpha). \quad (4.5)$$

Since

$$\begin{aligned} & g_1(t, y(t+1), y(t+2)) \\ &= \int_0^1 \frac{d}{ds} f(sy(t+1) + P_N(t+1), sy(t+2) \\ &\quad + P_N(t+2)) ds \\ &= \int_0^1 \left\{ y(t+1) \frac{\partial f}{\partial v_1} (*) + y(t+2) \frac{\partial f}{\partial v_2} (*) \right\}, \end{aligned} \quad (4.6)$$

where $(*) = sy(t+1) + P_N(t+1), sy(t+2) + P_N(t+2)$, if we suppose

$$\begin{aligned} & |sy(t+1) + P_N(t+1)|, |sy(t+2) + P_N(t+2)| \\ &< \frac{\beta}{4\alpha^2}, \end{aligned} \quad (4.7)$$

then we have

$$\begin{aligned} & |g_1(t, y(t+1), y(t+2))| \\ &\leq \int_0^1 (|y(t+1)| + |y(t+2)|) \cdot 8(1 + \alpha) ds \\ &\leq 16(1 + \alpha) \{ A|\lambda^{t+1}|^{N+1} \cdot |\lambda|^{N+1} \}. \end{aligned} \quad (4.8)$$

Suppose N is so large such that

$$16(1 + \alpha)|\lambda|^{N+1} \leq \frac{1}{4}, \quad (4.9)$$

then we have, by (4.8)

$$|g_1(t, y(t+1), y(t+2))| \leq \frac{1}{4} A |\lambda^t|^{N+1}, \quad (4.10)$$

if (4.7) holds.

We take sufficiently small δ such that

$$|y(t+1)|, |y(t+2)|, |P_N(t+1)|, |P_N(t+2)| < \frac{\beta}{8\alpha^2}.$$

Then we have (4.7), hence obtain (4.10).

Since (4.1) is a formal solution of (4.2), we have

$$|g_2(t)| \leq K_2 |\lambda^t|^{N+1}. \quad (4.11)$$

with a constant K_2 which depends on N . Suppose N is so large that (4.9) holds, and A is so large that

$$\frac{1}{4} A + K_2 < \frac{1}{2} A,$$

i.e.,

$$A > 2K_2$$

and ρ is so small that (4.7) holds, then we obtain that T in (4.4) maps $H(A, \rho)$ into itself.

The map T is obviously continuous if $H(A, \rho)$ is endowed with the topology of uniform convergence $S(\rho)$.

$H(A, \rho)$ is clearly convex, and is compact by the theorem of Montel [5].

Thus by Schauder's fixed point theorem in [5], we obtain the existence of a fixed point $y(t) = y_N(t) \in H(A, \rho)$ of T .

4.2. Uniqueness of the Fixed Point

Suppose there is another fixed point $y^*(t) = y_N^*(t) \in H(A^*, \rho^*)$. Put

$$A_0 = \max(A, A^*), \quad \rho \leq \min(\rho, \rho^*),$$

$$v(t) = y_N(t) + P_N(t),$$

$$v^*(t) = y_N^*(t) + P_N(t),$$

and

$$w(t) = y_N(t) - y_N^*(t).$$

From (4.3), we have

$$\begin{aligned} w(t) &= \{f(y_N^*(t+1) + P_N(t+1), y_N^*(t+2) + P_N(t+2)) \\ &\quad - P_N(t)\} \\ &\quad - \{f(y_N(t+1) + P_N(t+1), y_N(t+2) + P_N(t+2)) \\ &\quad - P_N(t)\} \\ &= f(w(t+1) + v_N(t+1), w(t+2) + v_N(t+2)) \\ &\quad - f(v_N(t+1), v_N(t+2)) \\ &= \int_0^1 \{w(t+1) \frac{\partial f}{\partial v_1}(**) + w(t+2) \frac{\partial f}{\partial v_2}(**)\} ds, \end{aligned}$$

where $(**) = (sw(t+1) + v_N(t+1), sw(t+2) + v_N(t+2))$.

If ρ_0 is sufficiently small, then

$$\left| \frac{\partial f}{\partial v_1}(**) \right|, \left| \frac{\partial f}{\partial v_2}(**) \right| < 8(1 + \alpha),$$

and

$$32(1 + \alpha) |\lambda|^{N+1} < \frac{1}{2}.$$

Thus we have

$$\begin{aligned} |w(t)| &< 8(1 + \alpha)(|w(t+1)| + |w(t+2)|) \\ &\leq 8(1 + \alpha)(2A_0 |\lambda^{t+1}|^{N+1} + 2A_0 |\lambda^{t+2}|^{N+1}) \\ &< 32(1 + \alpha) |\lambda|^{N+1} \cdot A_0 |\lambda^t|^{N+1} \\ &< \frac{1}{2} A_0 |\lambda^t|^{N+1}. \end{aligned}$$

Then

$$\begin{aligned} |v(t) - v^*(t)| &\leq \frac{1}{2} A_0 |\lambda^t|^{N+1} = \frac{1}{4} (2A_0) |\lambda^t|^{N+1}, \\ &\text{for } t \in S(\rho_0). \end{aligned}$$

Repeating this procedure k times we obtain

$$|v(t) - v^*(t)| < \left(\frac{1}{4}\right)^k (2A_0) |\lambda^t|^{N+1}, \\ k = 1, 2, \dots$$

Letting $k \rightarrow \infty$, we have

$$v(t) = v^*(t), \quad t \in S(\rho_0).$$

Thus $v(t)$ and $v^*(t)$ are holomorphic in $|\lambda^t| \leq \min(\rho, \rho^*)$ and $v(t) \equiv v^*(t)$ there.

4.3. Arbitrariness of N (Independence of N)

Finally we will show that the solution $y_N(t)$ does not depend on N , which shows that (4.1) is actually a solution.

Let $y_N(t) \in H(A_N, \rho_N)$ and $y_{N+1}(t) \in H(A_{N+1}, \rho_{N+1})$ be fixed points, and

$$\begin{aligned} v_{N+1}(t) &= y_{N+1}(t) + P_{N+1}(t) \\ &= y_{N+1}(t) + a_{N+1} \lambda^{(N+1)t} + P_N(t) \\ &= \tilde{y}_N(t) + P_N(t). \end{aligned}$$

Obviously

$$\begin{aligned} \tilde{y}_N(t) &= |y_{N+1}(t) + a_{N+1} \lambda^{(N+1)t}| \\ &\leq A_{N+1} \lambda^{N+2} |\lambda^t|^{N+2} + |a_{N+1}| \cdot |\lambda^t|^{N+1} \\ &= A_N^* |\lambda^t|^{N+1}. \end{aligned}$$

By uniqueness of fixed point, $\tilde{y}_N(t) = y_N(t)$ for $t \in S(\rho_N) \cap S(\rho_{N+1})$. Thus

$$v_N(t) = v_{N+1}(t) \quad \text{in } S(\rho_N) \cap S(\rho_{N+1}).$$

By analytic prolongation, both of $v_N(t)$ and $v_{N+1}(t)$ are holomorphic in $S(\rho_N) \cap S(\rho_{N+1})$ and coincide there.

Thus we have proved that a solution $v(t)$ is defined and holomorphic in $S(\rho)$ for a $\rho > 0$, which has the expansion (4.1). By analytic prolongation,

the solution $v(t)$ is defined on the whole plane by using the Eq. (4.12)

$$v(t-2) = \frac{\alpha\beta\{(1+\alpha)v(t-1) - v(t)\}}{\alpha v(t) - \alpha^2 v(t-1) + \beta}. \quad (4.12)$$

4.4. Analytic General Solutions

Analytic solutions of some difference equations are investigated in [6–9]. In this section, we shall investigate analytic general solutions of (1.1).

From [7], we have the following theorem:

THEOREM 1 *Let $v(s)$ be the solution of (4.2) obtained in Section 3. Suppose $x(t)$ be an analytic solution of (4.2) such that $x(t-n) \rightarrow 0$ as $n \rightarrow +\infty$, uniformly on any compact set. Then there is a periodic entire function $\pi(t)$, ($\pi(t+1) = \pi(t)$), such that*

$$x(t) = v\left(t + \frac{\log \pi(t)}{\log \lambda}\right).$$

Conversely if we put

$$x(t) = v\left(t + \frac{\log \pi(t)}{\log \lambda}\right),$$

where π is a periodic function whose period is one, then $x(t)$ is a solution of (3.1).

Now we have a general solution of (3.1) such that

$$x(t) = v\left(t + \frac{\log \pi(t)}{\log \lambda}\right),$$

where π is an arbitrarily periodic function whose period is one. And we have sought general solutions of the population model which is given by the Eq. (1.1) such that

$$u(t) = \frac{\beta}{\alpha} + v\left(t + \frac{\log \pi(t)}{\log \lambda}\right),$$

where $\pi(t)$ is an arbitrarily periodic function whose period is one.

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