
ANDREAS RUFFING\textsuperscript{a,b,*}, PATRICK WINDPASSINGER\textsuperscript{b} and STEFAN PANIG\textsuperscript{b}

\textsuperscript{a}Center for Applied Mathematics and Theoretical Physics, University of Maribor, Koroška Ulica 2, SLO-2000 Maribor, Slovenia; \textsuperscript{b}Zentrum Mathematik, Technische Universität München, Arcisstrasse 21, D-80333 München, Germany

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We revise the interrelations between the classical Black Scholes equation, the diffusion equation and Burgers equation. Some of the algebraic properties the diffusion equation shows are elaborated and qualitatively presented. The related numerical elementary recipes are briefly elucidated in context of the diffusion equation. The quality of the approximations to the exact solutions is compared throughout the visualizations. The article mainly is based on the pedagogical style of the presentations to the Novacella Easter School 2000 on Financial Mathematics.

Keywords: Diffusion processes; Diffusion equations

1. INTRODUCTION

The Black Scholes equation has played and is playing a strong role in mathematical modelling of financial markets. It connects the understanding of option prizing with diffusion models and thus yields a fascinating interaction between phenomena in nature with phenomena in society. For a detailed description concerning the Black Scholes equation, see [10]. For general reference, see [1–4, 6–9, 11]. Discretizations of this equation have also been investigated in detail, and in principle it seems that there is nothing really new about this celebrated equation. Its symmetry properties however are still a remarkable mathematical fact, in detail with respect to its transformation properties into the classical heat or diffusion equation.

In this article, we prosecute the strategy of reducing solutions of the Black-Scholes equation to solutions of the classical diffusion equation as described in detail in [10, 11]. We give some explicit examples in which we compare the quality of exact algebraic solutions to the classical diffusion equation with solutions obtained by the standard numerical recipes as proposed in [10]. As a

*Corresponding author. e-mail: ruffing@appl-math.tu-muenchen.de, http://www-m6.mathematik.tu-muenchen.de/~ruffing/
consequence, we obtain the main result that has
been elucidated in great detail on the Novacella
Easter School 2000: The conventional recipes
proposed in [10, 11] yield a fine qualitative
accuracy. As for the standard error analysis, we
recommend the reader to refer to [5].

The Easter School in Novacella, Italy, was an
attempt to communicate questions in the area of
discrete dynamics in nature and society to students
in mathematical finance, mathematics, to students
in computer science and students in physics and to
encourage them to scientifically interact.

The homepage of the Novacella Easter School
2000 is www-m1.ma.tum.de/bilder/2000/ostersem_4_00

The organization of this article shall now be as
follows:

In the second chapter we revise the interrelation
between Black-Scholes equation, diffusion equa-
tion and Burgers equation.

In the third and the fourth chapter, algebraic
solutions and numerical approximations are devel-
oped. They are compared at the end of the fourth
chapter.

The authors would like to mention in particular
that the impressive graphic results were also
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strategies to ordinary differential equations during
the Novacella Easter School.

2. EXAMPLES OF EQUATION
TRANSFORMS

The connection between the Black Scholes
equation and the diffusion equation belongs to the
standard procedures being taught in computa-
tional finance. For the convenience of the reader,
we are going to state it right here at the very
beginning of the article.

The Black-Scholes equation is given by

$$\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0 \quad (1)$$

where $C$ refers to a call, $S$ denoting the underlying
asset, $t$ the time. $\sigma$ refers to the volatility of the
underlying asset, $r$ means the interest rate.

By setting

$$S = E e^x \quad t = T - \frac{\tau}{(1/2)\sigma^2} \quad C = Ev(x, \tau) \quad (2)$$

we remove the dimensions of the original
Black-Scholes PDE which now results in the
equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k - 1) \frac{\partial v}{\partial x} - kv \quad k = \frac{2r}{\sigma^2} \quad (3)$$

Putting now

$$v = e^{\alpha x + \beta \tau} u(x, \tau) \quad (4)$$

and setting

$$\alpha := -\frac{1}{2} (k - 1) \quad \beta := -\frac{1}{4} (k + 1)^2 \quad (5)$$

we end up with the diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad (6)$$

where $-\infty < x < \infty$, $\tau > 0$.

Let us show that also another partial differential
equation can be reduced to the diffusion equation
by standard methods:

The partial differential equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \quad (7)$$

is nonlinear and usually referred to by the name
Burgers equation. It is remarkable that one can
linearize this equation in the following way.
Assume that $v(x, t)$ is a purely positive function
for which the expression

$$\frac{\partial}{\partial x} \ln(v(x, t)) \quad (8)$$
is well defined for any \((x,t) \in \mathbb{R}^2\). Defining the function

\[
u: \mathbb{R}^2 \to \mathbb{R},
\]

\[(x,t) \mapsto u(x,t) := -2 \frac{\partial}{\partial x} \ln(v(x,t)) \quad (9)\]

one obtains first by evaluating (7) that

\[
u(x,t) = \frac{2 \partial v}{v \partial x} \quad (10)\]

We now differentiate the function \(u\) with respect to \(t, x\) and obtain

\[rac{\partial u}{\partial t} = \frac{2}{v^2} \frac{\partial v}{\partial t} \frac{\partial v}{\partial x} - \frac{2}{v} \frac{\partial^2 v}{\partial t \partial x} \quad (11)\]

\[rac{\partial u}{\partial x} = \frac{2}{v^3} \left( \frac{\partial v}{\partial x} \right)^2 - \frac{2}{v} \frac{\partial v}{\partial x} \quad (12)\]

Now, taking the second derivative of \(u\) with respect to \(x\), we receive

\[rac{\partial^2 u}{\partial x^2} = -\frac{4}{v^3} \left( \frac{\partial v}{\partial x} \right)^3 + \frac{6}{v^2} \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2} - \frac{2}{v} \frac{\partial^3 v}{\partial x^3} \quad (13)\]

Combining these expressions in the sense of (7), one recognizes the following: The function \(u\) we have introduced by \(v\) yields a solution to Burgers equation if

\[rac{\partial v}{\partial x} \left( \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} \right) = v \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} \right) \quad (14)\]

Thus, if the solution \(v\) is a positive solution to the heat resp. diffusion Eq. (6) for any \((x,t) \in \mathbb{R}^2\), the function \(u\), defined by (9) provides a solution to Burgers equation.

We have now got some motivation to focus on comparing solutions to the diffusion equation which, as we have seen, is closely related to the Black-Scholes equation and to Burgers equation.

### 3. ALGEBRAIC SOLUTIONS TO THE DIFFUSION EQUATION

From a classification viewpoint, the diffusion resp. heat equation is a second order linear partial differential equation which is parabolic,

\[rac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (15)\]

In a conventional sense, linearity is understood as usual, i.e., along with the solutions \(u_1(x,t)\) and \(u_2(x,t)\) for the equation, all linear combinations \(c_1 u_1(x,t) + c_2 u_2(x,t)\) solve the equation as well. The solutions \(u(x,t)\) (\(u: \mathbb{R} \times \mathbb{R} \to \mathbb{R}\)) model the time dependent temperature in a long, fully insulated bar which is assumed to be one-dimensional.

One obtains the most simple solutions by combining polynomial functions in \(x\) and \(t\) as follows:

\[
u(x,t) = 1 \quad (c \in \mathbb{R}) \quad (16)\]

\[
u(x,t) = x \quad (17)\]

\[
u(x,t) = \frac{1}{2} x^2 + t \quad (18)\]

**GRAPH**

\[u(x,t) = 1.\]
Concerning the solution structure, "more difficult" solutions exist as well, for instance:

$$u(x, t) = c e^{nx + n^2 t} \quad (c, n \in \mathbb{R}) \quad (19)$$

as we can easily verify:

$$\frac{\partial u}{\partial t} = cn^2 e^{nx + n^2 t} = \frac{\partial}{\partial x} cn e^{nx + n^2 t} = \frac{\partial^2 u}{\partial x^2}$$

As $$(\partial u/\partial t) = (\partial^2 u/\partial x^2)$$ is a linear differential equation, we obtain other solutions by constructing linear combinations of the particular solutions of type (19):

$$u(x, t) = \sum_{n=1}^{k} c_n e^{nx + n^2 t} \quad (c_n \in \mathbb{R}) \quad (20)$$

This can easily be checked by using induction methods: The case $$k = 1$$ is obviously clear, therefore we address $$k - 1 \rightarrow k$$:

$$u_k(x, t) = \sum_{n=1}^{k} c_n e^{nx + n^2 t} = \left( \sum_{n=1}^{k-1} c_n e^{nx + n^2 t} \right) + c_k e^{kx + k^2 t}.$$ 

As the partial derivative of a function is linear, $$\sum_{n=1}^{k} c_n e^{nx + n^2 t}$$ gives a solution because $$\sum_{n=1}^{k-1} c_n e^{nx + n^2 t}$$ satisfies the initial condition. As for $$c_k e^{kx + k^2 t}$$ compare (19). Starting the summation at a lower negative index doesn’t cause any problems.

We can construct a different solution fulfilling the following boundary conditions:

$$u(x, 0) = f(x) \quad 0 < x < \pi$$
$$u(0, t) = 0 \quad t > 0$$
$$u(\pi, t) = 0 \quad t > 0$$
Obviously further solutions are obtained by changing\( \sin \) into \( \cos, \sinh \) or \( \cosh \).

Due to the linearity of the heat resp. diffusion equation, the sum again gives a solution:

\[
 u(x, t) = \sum_{n=1}^{k} c_n \sin (nx) e^{-n^2 t} \quad (c_n \in \mathbb{R}) \quad (22)
\]
Note that it is very similar to the probability density of the normal distribution \( \phi(S)_{\sigma,\mu} = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}((x-\mu)/\sigma)^2} \).

Function (23) yields a solution of the heat equation:

\[
\frac{\partial u}{\partial t} = -\frac{1}{2} \frac{c}{\sqrt{t}} e^{-(1/4)(x^2/t)} + \frac{c}{\sqrt{t}} e^{-(1/4)(x^2/t)} 
\]

Looking at the probability density \( \phi(s)_{\sigma,\mu} = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}((x-\mu)/\sigma)^2} \) and choosing \( \mu = 0 \), \( \phi(s) \) becomes symmetric with respect to the y-axis.

The proof is similar to the one stated above and one concludes that the sum (22) fulfills the given boundary conditions as well.

From the viewpoint of stochastics, a quite interesting solution is given as follows:

\[
u(x, t) = \frac{c}{\sqrt{t}} e^{-(1/4)(x^2/t)} \quad (t > 0, c \in \mathbb{R}) \tag{23}
\]
SOLUTIONS OF DIFFUSION EQUATIONS

If additionally $\sigma^2$ is substituted by $2t$, we exactly get the same result as in (23) with $c = (1/2\sqrt{\pi})$.

The factor $c$ is necessary to ensure that the integral $\Phi = \int_{-\infty}^{+\infty} \phi(x)dx$ equals 1 which reflects the fact that the probability is conserved.

By direct calculation one can verify that the density function with parameter $\mu$ is already a solution of the heat equation when substituting $\sigma^2$ by $2t$:

$$\frac{\partial}{\partial t} \phi(x, \sqrt{\sigma^2 t}) =$$
$$= \frac{\partial}{\partial t} \frac{1}{\sqrt{2\pi \sigma^2 t}} e^{-(1/2)((x-\mu)/\sigma)^2} =$$
$$= -\frac{1}{4\sqrt{\pi \sigma^2 t}} e^{-(1/4)((x-\mu)^2/t)} +$$
$$+ \left( \frac{1}{2\sqrt{\pi t}} e^{-(1/4)((x-\mu)^2/t)} \right) \left( \frac{1}{4} \frac{(x-\mu)^2}{t^2} \right) =$$
$$= \left( \frac{1}{8\sqrt{\pi \sigma^2 t}} \right)^{3/2} \cdot e^{-(1/4)((x-\mu)^2/t)}.$$

$$\frac{\partial^2}{\partial x^2} \phi(x, \sqrt{\sigma^2 t}) =$$
$$= \frac{\partial}{\partial x} \frac{1}{2\sqrt{\pi t}} e^{-(1/4)((x-\mu)^2/t)} \cdot \left( \frac{1}{4} \cdot 2(x-\mu) \right) =$$
$$= \frac{1}{2\sqrt{\pi t}} e^{-(1/4)((x-\mu)^2/t)} \left( \frac{1}{2t} (x-\mu) \right) \cdot \left( \frac{1}{2t} (x-\mu) \right) +$$
$$+ \frac{1}{2\sqrt{\pi t}} e^{-(1/4)((x-\mu)^2/t)} \left( \frac{1}{2t} (x-\mu) \right) =$$
$$= \left( \frac{1}{8\sqrt{\pi \sigma^2 t}} \right)^{3/2} \cdot e^{-(1/4)((x-\mu)^2/t)}.$$

4. NUMERICAL APPROXIMATIONS TO THE DIFFUSION EQUATION

4.1. Stating the Difference Scheme

As usual, we replace the partial derivatives by difference quotients where we introduce first order difference quotients and second order difference quotients separately:
4.1.1. Derivatives of First Order

\[
\frac{\partial u}{\partial \tau}(x, \tau) = \lim_{\delta \tau \to 0} \frac{u(x, \tau + \delta \tau) - u(x, \tau)}{\delta \tau}
\]

Backward Difference

\[
\frac{\partial u}{\partial \tau}(x, \tau) = \lim_{\delta \tau \to 0} \frac{u(x, \tau) - u(x, \tau - \delta \tau)}{\delta \tau}
\]

Forward Difference

\[
\frac{\partial u}{\partial \tau}(x, \tau) = \lim_{\delta \tau \to 0} \frac{u(x, \tau + \delta \tau) - u(x, \tau - \delta \tau)}{2 \delta \tau}
\]

Central Difference

The approximation is performed as follows:

\[
\frac{\partial u}{\partial \tau}(x, \tau) \approx \frac{u(x, \tau + \delta \tau) - u(x, \tau)}{\delta \tau} \approx \frac{u(x, \tau) - u(x, \tau - \delta \tau)}{\delta \tau} \approx \frac{u(x, \tau + \delta \tau) - u(x, \tau - \delta \tau)}{2 \delta \tau}
\]

\(\delta \tau\) is assumed to be sufficiently small – The evaluation for \(\frac{\partial u}{\partial x}\) \((x, \tau)\) goes in an analogous way.

4.1.2. Derivatives of Second Order

Reducing the second order derivatives to the given derivatives of first order yields

\[
\frac{\partial^2 u}{\partial \tau^2}(x, \tau) = \lim_{\delta \tau \to 0} \frac{u(x, \tau + \delta \tau) - 2u(x, \tau) + u(x, \tau - \delta \tau)}{\delta \tau^2}
\]

Forward Difference

\[
\frac{\partial^2 u}{\partial \tau^2}(x, \tau) = \lim_{\delta \tau \to 0} \frac{u(x, \tau + \delta \tau) - 2u(x, \tau) + u(x, \tau - \delta \tau)}{\delta \tau^2}
\]

Backward Difference

\[
\frac{\partial^2 u}{\partial \tau^2}(x, \tau) = \lim_{\delta \tau \to 0} \frac{u(x + \delta x, \tau) - 2u(x, \tau) + u(x - \delta x, \tau)}{(\delta x)^2}
\]

Central Difference

There are three given methods for approximating the first derivative.

\(\Rightarrow\) In total we obtain 27 possibilities of approximating the second derivatives in terms of finite differences to the \(u\)-functions.

4.2. Considering the Errors

As a consequence of the respective Taylor expansion and the additional assumption that \(u(x, \tau)\) allows to calculate its lowest derivatives, we can derive the following equations that give a first insight into the error behavior:

On the one hand, we obtain

\[
u(x, \tau + \delta \tau) = u(x, \tau) + \frac{\partial u}{\partial \tau}(x, \tau)(\delta \tau) + R_2 \frac{(\delta \tau)^2}{2!}
\]

(25)

with

\[
|R_2| \leq \max_{\tau \leq \xi \leq \tau + \delta \tau} \left| \frac{\partial^2 u}{\partial \tau^2}(x, \xi) \right|
\]

and on the other hand

\[
\frac{\partial u}{\partial \tau}(x, \tau) = \frac{u(x, \tau + \delta \tau) - u(x, \tau)}{\delta \tau} + O(\delta \tau)
\]
respectively

\[ \frac{\partial u}{\partial \tau}(x, \tau) = \frac{u(x, \tau) - u(x, \tau - \delta \tau)}{\delta \tau} + O(\delta \tau) \]

with

\[ O(\delta \tau) = \frac{1}{2} R_2 \delta \tau \]

This means that the error term of the forward resp. backward difference goes to zero in a linear way as \( \delta \tau \to 0 \).

Adding the forward and backward difference, we can derive the following approximations from (24):

\[ \frac{\partial u}{\partial x}(x, \tau) = \frac{u(x + \delta x, \tau) - u(x - \delta x, \tau)}{2 \delta x} + O((\delta x)^2) \]

respectively

\[ \frac{\partial^2 u}{\partial x^2}(x, \tau) = \frac{u(x + \delta x, \tau) - 2u(x, \tau) + u(x - \delta x, \tau)}{(\delta x)^2} + O((\delta x)^2) \]

Thus the given central differences yield a quadratic convergence in the stated sense. The stated facts open the possibility of determining exact error bounds.

4.3. Evaluating the Difference Scheme

The discretization is performed by choosing an equidistant lattice with mesh lengths \( \delta x \) resp. \( \delta \tau \) and lattice coordinates \((n \delta x, m \delta \tau)\):

\[ u^m_n = u(n \delta x, m \delta \tau) \]
Graph 17 \[ u(x, t) = \frac{1}{2}x^2 + t. \]

Graph 18 \[ u(x, t) = \frac{1}{1000} \sum_{n=-10}^{10} e^{(1/10)x + (1/100)t^2}. \]

Graph 19 \[ u(x, t) = e^{(1/2)x + (1/4)t}. \]
GRAPH 20 \[ u(x, t) = \sin \left( \frac{1}{2}x \right) e^{-\left(\frac{1}{4}\right)t} \].

GRAPH 21 \[ u(x, t) = \sum_{n=0}^{100} \sin (nx) e^{-\omega t} \].

GRAPH 22 \[ u(x, t) = \left( \sqrt{\frac{1}{4}} \right) e^{-\left(\frac{1}{4}\right)x^2/t} \].
4.3.1. The Explicit Approach

Inserting the forward and symmetric backward central difference into the heat equation now yields

\[
\frac{u(x_0 + \delta x, \tau) - u(x_0, \tau)}{\delta \tau} + O(\delta \tau) = \frac{u(x_0 + 2\delta x, \tau) - 2u(x_0, \tau) + u(x_0 - \delta x, \tau)}{(\delta x)^2} + O((\delta x)^2),
\]

i.e.,

\[
\frac{u^{m+1}_n - u^m_n}{\delta \tau} = \frac{u^{m+1}_{n+1} - 2u^m_n + u^m_{n-1}}{(\delta x)^2} + O((\delta x)^2) \quad (26)
\]

Ignoring the error terms we receive the following approximation formula

\[
\nu^{m+1}_n = \nu^m_n + \alpha(\nu^m_{n+1} + \nu^m_{n-1} - 2\nu^m_n) \quad \text{mit } \alpha = \frac{\delta \tau}{(\delta x)^2}
\]

The way it operates is visualized in Graph 16.

4.3.2. The Implicit Approach

If we insert the backward difference scheme into Eq. (26) instead of the forward difference scheme, we obtain an implicit difference method.

In the case of the implicit approach, one now makes the following observation: To determine all approximations along the finite lattice, the data along one of its bars are required.

Let us now compare the exact solutions and their approximations.

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