Boundedness of Solutions of Matrix Nonlinear Volterra Difference Equations

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Nonlinear discrete-time Volterra equations in a Euclidean space are considered. Conditions for the boundedness of solutions are established by virtue of recent estimates for the norm of the resolvent of Volterra operators. The conditions are formulated in the terms of coefficients of considered equations. In addition, estimates for the c_0 - and l^2 -norms of solutions are derived.

Keywords: Volterra difference equations; Boundedness; Estimates for solutions; Matrix

INTRODUCTION

Volterra difference equations arise in the mathematical modeling of some real phenomena, and also in numerical schemes for solving differential and integral equations, cf. (Kolmanovskii and Myshkis, 1998; Kolmanovskii *et al.*, 2000) and references therein.

One of the basic methods in the theory of stability and boundedness of Volterra difference equations is the direct Lyapunov method (see (Crisci *et al.*, 1995; Elaydi, 1996; Elaydi and Murakami, 1996) and references therein). But finding of the Lyapunov functionals for Volterra difference equations is a difficult mathematical problem.

In this paper, to establish the boundedness conditions, we will interpret the Volterra equations with matrix kernels as operator equations in appropriate spaces. Such an approach for Volterra difference equations have been used by Kolmanovskii *et al.* (2000), Kolmanovskii and Myshkis (1998), Kwapisz (1992) and Medina (1996). Our main tool is the recent estimates for the norm of the resolvent of Volterra operators established in Gil' (1995). Under some restriction, these estimates allow us to generalize the main results from the papers of Kolmanovskii *et al.* (2000), Kolmanovskii and Myshkis (1998), Kwapisz (1992) and Medina (1996). Moreover, we derive inequalities for the c_0 - and l^2 -norms of solutions. Note that our boundedness conditions are formulated in the terms of coefficients of considered equations.

PRELIMINARIES

Let C^n be an n -dimensional complex Euclidean space with the Euclidean norm $|\cdot|_{C^n}$. For a positive r , put

$$\omega_r = \{h \in C^n : |\cdot|_{C^n} \leq r\}$$

As usual $c_0 = c_0(C^n)$ is the Banach space of sequences of vectors from C^n equipped with the norm

$$|h|_{c_0} = \sup_k |h_k|_{C^n} \quad (h = (h_k)_{k=1}^\infty \in c_0 \quad (h_k \in C^n;$$

$$k = 1, 2, \dots)$$

and $l^2 = l^2(C^n)$ is the Banach space of sequences of vectors from C^n equipped with the norm

$$|h|_{l^2} = \left[\sum_{k=1}^\infty |h_k|_{C^n}^2 \right]^{\frac{1}{2}}$$

$$(h = (h_k)_{k=1}^\infty \in l^2, h_k \in C^n, k = 1, 2, \dots).$$

Let a_{jk} ($j, k = 1, 2, \dots$) be $n \times n$ -matrices, and $f_j \in C^n$ ($j = 1, 2, \dots$). Consider the equation

$$x_j = f_j + \sum_{k=1}^{j-1} a_{jk}[x_k + G_k(x_k)], \quad (j = 1, 2, \dots) \quad (1)$$

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where the mappings $G_k : C^n \rightarrow C^n$ have the property

$$\begin{aligned} |G_k|(h)|_{C^n} &\leq q_k |h|_{C^n} \\ (h \in \omega_r; k = 1, 2, \dots; r \leq \infty). \end{aligned} \quad (2)$$

Rewrite Eq. (1) as

$$x = f + Vx + F(x), \quad (3)$$

where $f = (f_k)$, V and F are defined in c_0 by

$$[Vx]_j = \sum_{k=1}^{j-1} a_{jk} x_k$$

and

$$[Fx]_j = \sum_{k=1}^{j-1} a_{jk} G_k(x_k).$$

Here $[h]_j$ means the j -th coordinate of the element $h \in c_0$.

Let $N(A)$ be the Hilbert–Schmidt (Frobenius) norm of a matrix A . That is, $N^2(A) = \text{Trace}(AA^*)$. It is assumed that

$$N(V) \equiv \left[\sum_{j=1}^{\infty} \sum_{k=1}^{j-1} |a_{jk}|_{C^n}^2 \right]^{\frac{1}{2}} < \infty, \quad (4)$$

$$|f|_{l^2} \equiv \left[\sum_{k=1}^{\infty} |f_k|_{C^n}^2 \right]^{\frac{1}{2}} < \infty \quad (5)$$

and

$$N_q \equiv \left[\sum_{j=1}^{\infty} \sum_{k=1}^{j-1} |a_{jk}|_{C^n}^2 q_k^2 \right]^{\frac{1}{2}} < \infty. \quad (6)$$

Clearly, if $q_k \leq b_0 < \infty$ ($k = 1, 2, \dots$), then Eq. (4) implies $N_q \leq b_0 N(V)$.

Furthermore, denote

$$m(V) = \sum_{k=0}^{\infty} \frac{N^k(V)}{\sqrt{k!}}.$$

By the Schwarz inequality it is not hard to show that

$$m(V) \leq \sqrt{2} \exp[N^2(V)].$$

THE MAIN RESULT

Lemma 1. *Let condition (2) hold with $r = \infty$. Then under the conditions (4–6), and*

$$m(V)N_q < 1 \quad (7)$$

a solution $x = (x_k)$ of Eq. (1) is in l^2 . Moreover, it satisfies the inequality

$$|x|_{l^2} \leq m(V)(1 - m(V)N_q)^{-1} |f|_{l^2}. \quad (8)$$

Proof. By Theorem 2.4.1 from Gil' (1995),

$$\|(I - V)^{-1}\|_{l^2} \leq m(V). \quad (9)$$

Moreover, due to Eq. (2) and the Schwarz inequality

$$\begin{aligned} |F(x)|_{l^2}^2 &= \sum_{j=1}^{\infty} \left| \sum_{k=1}^{j-1} a_{jk} G_k(x_k) \right|_{C^n}^2 \\ &\leq \sum_{j=1}^{\infty} \left[\sum_{k=1}^{j-1} |a_{jk}|_{C^n} q_k |x_k|_{C^n} \right]^2 \\ &\leq \sum_{j=1}^{\infty} \sum_{k=1}^{j-1} |a_{jk}|_{C^n}^2 q_k^2 \sum_{k=1}^{j-1} |x_k|_{C^n}^2. \end{aligned}$$

Hence,

$$|F(x)|_{l^2} \leq N_q |x|_{l^2}. \quad (10)$$

Rewrite Eq. (3) as

$$x = (I - V)^{-1}(f + F(x)). \quad (11)$$

From Eqs. (9) and (10), it follows that

$$|x|_{l^2} \leq m(V)[|f|_{l^2} + N_q |x|_{l^2}].$$

Thus, by Eq. (7) we get Eq. (8), concluding the proof. $\square$

Note that the comparison theorems for Volterra difference equations of the type Eq. (1) were established in Medina (1999).

Put

$$M_1(V, q) = \max_j \left[\sum_{k=1}^{j-1} |(1 + q_k) a_{jk}|_{C^n}^2 \right]^{\frac{1}{2}}$$

and

$$c(f) = |f|_{c_0} + |f|_{l^2} M_1(V, q) m(V) (1 - m(V)N_q)^{-1}.$$

Lemma 2. *Under the hypotheses of Lemma 1, the estimate*

$$|x|_{c_0} \leq c(f) \quad (12)$$

is valid.

Proof. By Eqs. (1) and (2),

$$|x|_{c_0} \leq |f|_{c_0} + \max_j \sum_{k=1}^{j-1} |a_{jk}|_{C^n} (1 + q_k) |x_k|_{C^n}.$$

The Schwarz inequality yields

$$\begin{aligned} |x|_{c_0} &\leq |f|_{c_0} + \max_j \left[\sum_{k=1}^{j-1} |(1+q_k)a_{jk}|_{C^n}^2 \sum_{k=1}^{j-1} |x_k|^2 \right]^{\frac{1}{2}} \\ &\leq |f|_{c_0} + M_1(V, q)|x|_{l^2}. \end{aligned}$$

From Lemma 1 we infer that

$$|x|_{c_0} \leq |f|_{c_0} + m(V)(1 - m(V)N_q)^{-1}|f|_{l^2}M_1(V, q) = c(f)$$

concluding the proof. $\square$

By a simple application of the Uryshon lemma and Lemma 2 we get the main result of the paper

Theorem 1. *Under conditions (2) and (4–7), let*

$$c(f) \leq r. \quad (13)$$

Then, a solution $x = (x_k)$ of Eq. (1) is bounded. Moreover, the inequality Eq. (12) is valid.

PARTICULAR CASES

Let us consider the linear system with disturbances

$$x_j = f_j + \sum_{k=1}^{j-1} (a_{jk} + b_{jk})x_k; \quad j, k = 1, 2, \dots \quad (14)$$

where (a_{jk}) and (b_{jk}) are $n \times n$ -matrices, and $f_j \in C^n$ ($j, k = 1, 2, \dots$). Rewrite Eq. (14) as

$$x = f + Vx + Wx, \quad (15)$$

where $f = (f_k)$, V and W are defined in c_0 by

$$[Vx]_j = \sum_{k=1}^{j-1} a_{jk}x_k,$$

and

$$[Wx]_j = \sum_{k=1}^{j-1} b_{jk}x_k.$$

It is assumed that

$$N(W) = \left[\sum_{j=1}^{\infty} \sum_{k=1}^{j-1} |b_{jk}|_{C^n}^2 \right]^{\frac{1}{2}} < \infty. \quad (16)$$

Lemma 1 implies

Corollary 1. *Under conditions (4), (5) and (16), let*

$$m(V)N(W) < 1, \quad (17)$$

Then a solution $x = (x_k)$ of Eq. (14) is in l^2 . Moreover, it

satisfies the inequality

$$|x|_{l^2} \leq m(V)(1 - m(V)N(W))^{-1}|f|_{l^2}. \quad (18)$$

Furthermore, put

$$M_2(V) = \max_j \left[\sum_{k=1}^{j-1} |a_{jk}|_{C^n}^2 \right]^{\frac{1}{2}},$$

$$M_3(W) = \max_j \left[\sum_{k=1}^{j-1} |b_{jk}|_{C^n}^2 \right]^{\frac{1}{2}},$$

and

$$\begin{aligned} c_1(f) &= |f|_{c_0} + m(V)(M_2(V) + M_3(W))(1 \\ &\quad - m(V)N(W))^{-1}|f|_{l^2}. \end{aligned}$$

Lemma 2 implies

Corollary 2. *Under the hypotheses of Corollary 1, for a solution of Eq. (14) the estimate*

$$|x|_{c_0} \leq c_1(f) \quad (19)$$

is valid.

Now let us consider the following equation:

$$x_j = U_j(x_j) + \sum_{k=1}^{j-1} a_{jk}[x_k + G_k(x_k)] \quad (j = 1, 2, \dots) \quad (20)$$

where the mappings $G_k : C^n \rightarrow C^n$ have the property Eq. (2) and the mappings $U_j : C^n \rightarrow C^n$ have the property

$$|U_j(x_j)|_{C^n} \leq \sum_{k=1}^{j-1} \alpha_{jk}|x_k|_{C^n}, \quad j = 1, 2, \dots \quad (21)$$

and $\alpha_{jk} \geq 0$ are nonnegative numbers.

Rewrite Eq. (20) as

$$x = U(x) + Vx + F(x),$$

where $U = (U_k)$, V , and F are defined in $c_0(C^n)$ by

$$[Vx]_j = \sum_{k=1}^{j-1} a_{jk}x_k \quad \text{and} \quad [F(x)]_j = \sum_{k=1}^{j-1} a_{jk}G_k(x_k).$$

Suppose that

$$N_\alpha \equiv \left[\sum_{j=1}^{\infty} \sum_{k=1}^{j-1} |\alpha_{jk}|^2 \right]^{\frac{1}{2}} < \infty. \quad (23)$$

The following result is a particular extension of Lemma 1.

Theorem 2. *Let conditions (2) and (21) hold, with $r = \infty$. Then, under the conditions (4), (6), (23) and*

$$m(V)(N_\alpha + N_q) < 1, \quad (24)$$

a solution $x = (x_k)$ of Eq. (20) is in l^2 . Moreover, it satisfies the inequality

$$|x|_{l^2} \leq m(V)[1 - m(V)(N_\alpha + N_q)]^{-1}.$$

Proof. We have

$$\begin{aligned} |U_j(x_j)|_{C^n}^2 &\leq \left(\sum_{k=1}^{j-1} \alpha_{jk} |x_k|_{C^n} \right)^2 \\ &\leq \left(\sum_{k=1}^{j-1} |\alpha_{jk}|^2 \right) \left(\sum_{k=1}^{j-1} |x_k|_{C^n}^2 \right). \end{aligned}$$

This yields,

$$\begin{aligned} |U|_{l^2}^2 &= \sum_{j=1}^{\infty} |U_j(x_j)|_{C^n}^2 \leq \sum_{j=1}^{\infty} \left[\sum_{k=1}^{j-1} |\alpha_{jk}|^2 \sum_{k=1}^{j-1} |x_k|_{C^n}^2 \right] \\ &\leq \left(\sum_{j=1}^{\infty} \sum_{k=1}^{j-1} |\alpha_{jk}|^2 \right) \sum_{k=1}^{\infty} |x_k|_{C^n}^2 = N_\alpha^2 |x|_{l^2}^2. \end{aligned}$$

On the other hand,

$$x = (I - V)^{-1}(U(x) + F(x)).$$

Thus,

$$\begin{aligned} |x|_{l^2} &\leq m(V)[|U|_{l^2} + |F|_{l^2}] \\ &\leq m(V)[N_\alpha |x|_{l^2} + N_q |x|_{l^2} + 1]. \end{aligned}$$

By Eq. (24), it follows that

$$|x|_{l^2} \leq m(V)[1 - m(V)(N_\alpha + N_q)]^{-1}.$$

As claimed. $\square$

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