SPECTRAL ANALYSIS FOR A CLASS OF INTEGRAL-DIFFERENCE OPERATORS: KNOWN FACTS, NEW RESULTS, AND OPEN PROBLEMS

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We present state of the art, the new results, and discuss open problems in the field of spectral analysis for a class of integral-difference operators appearing in some nonequilibrium statistical physics models as collision operators. The author dedicates this work to the memory of Professor Ilya Prigogine, who initiated this activity in 1997 and whose interesting and most enlightening advices had gudided the author during all these years.

1. Introduction

This paper is devoted to the study of a class of integral-difference operators. The original idea of rigorous mathematical investigation of the properties of these operators is due to Professor Ilya Prigogine and goes back to 1997. The reason is that such operators appear in nonequilibrium statistical physics models [11] and became a subject of interest for physicists [10]. At the same time, these operators have interesting and delicate mathematical properties, so "intuitive" physical approach may not work [10].

The operators under consideration have the form

$$\mathcal{K}_\varphi : u(x) \longmapsto \int_{-\infty}^{\infty} \frac{u(x)\varphi(s) - u(s)\varphi(x)}{|x-s|} ds \tag{1.1}$$

acting in the Hilbert space $L_2(\mathbb{R}, dx)$. Here $\varphi(x)$ is the so-called equilibrium distribution function, having the following properties induced by its physical nature as a probability distribution:

$$\varphi(x) \geq 0; \qquad \int_{-\infty}^{\infty} \varphi(x) dx = 1. \tag{1.2}$$

As shown in our previous papers [5, 6, 7, 8, 9], spectral properties of $\mathcal{K}_\varphi$ depend essentially on the properties of equilibrium distribution function $\varphi(x)$. In particular, there is a very important distinction between the cases when $\varphi(x)$ has and does not have compact support. In the first case, it is also important to know if there exists such $\varepsilon > 0$ that $\varphi(x) \geq \varepsilon$ for all $x \in \operatorname{supp} \varphi$.

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The paper is organized as follows. In Section 2, we present some general properties of the family of operators $\mathcal{K}_\varphi$. In Section 3, we consider operators with equilibrium distribution function having compact support. We introduce an appropriate reference operator $\mathcal{K}_0$, develop complete spectral analysis for $\mathcal{K}_0$, and obtain results for $\mathcal{K}_\varphi$ using the resolvent comparison approach. Spectral estimations for the eigenvalues of $\mathcal{K}_\varphi$ are presented. We also discuss the contribution of the complement to the support of $\varphi(x)$ to the spectrum of $\mathcal{K}_\varphi$. In Section 4, we discuss one of the most physically interesting cases of Gaussian equilibrium distribution function $\varphi(x)$ and demonstrate that spectral properties of the corresponding collision operator $\mathcal{K}_\varphi$ differ drastically from the case of equilibrium distribution functions with a compact support. Section 5 contains the discussion of the results and of open problems.

2. Fourier transform, adjoint operator, and selfadjointness in weighted space

We note that one cannot represent $\mathcal{K}_\varphi$ as a difference of two operators corresponding to two terms in the nominator, as the two corresponding integrals will not converge. Another important point is that $\mathcal{K}_\varphi$ is not an integral operator as it is impossible to construct the corresponding integral kernel. However, it happens that under some simple conditions on $\varphi(x)$, Fourier transform of $\mathcal{K}_\varphi$ is an integral operator. We use Fourier representation in order to prove some important properties of operators $\mathcal{K}_\varphi$.

Lemma 2.1. *For any real-valued function $\varphi(x) \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$, the operator $\mathcal{K}_\varphi$ acting in the space $L_2(\mathbb{R})$ obeys the relation*

$$\mathcal{K}_\varphi \circ \varphi = \varphi \circ \mathcal{K}_\varphi^*, \tag{2.1}$$

where $\mathcal{K}_\varphi^*$ is the adjoint operator and φ stands for the operator of multiplication by the function $\varphi(x)$.

Proof. The proof [5, 6] is based on Fourier transform defined in a standard way:

$$Fu = \hat{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x)dx. \tag{2.2}$$

Consider the operator

$$\widehat{\mathcal{K}}_\varphi \stackrel{\text{def}}{=} F\mathcal{K}_\varphi F^{-1}. \tag{2.3}$$

It is an integral operator with the kernel

$$\begin{aligned}
\widehat{\mathcal{K}}_\varphi(k,k') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} (\mathcal{K}_\varphi e^{ik'x})(x)dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx \int_{-\infty}^{\infty} \frac{\varphi(s)e^{ik'x} - \varphi(x)e^{ik's}}{|x-s|} ds \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(k-k')x} dx \int_{-\infty}^{\infty} \frac{\varphi(x+t) - \varphi(x)e^{ik't}}{|t|} dt.
\end{aligned} \tag{2.4}$$

Above we used the change of variables $t = s - x$. As $\varphi(x) \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$, we can interchange the integrals and obtain

$$\begin{aligned}\widehat{\mathcal{K}}_\varphi(k,k') &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dt}{|t|}\left(\int_{-\infty}^{\infty} e^{-i(k-k')x}\varphi(x+t)dx - e^{ik't}\int_{-\infty}^{\infty} e^{-i(k-k')x}\varphi(x)dx\right)\\ &= \widehat{\varphi}(k-k')\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{i(k-k')t}-e^{ik't}}{|t|}dt,\end{aligned} \tag{2.5}$$

where $\widehat{\varphi}(k) = F\varphi$.

Now we calculate

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e^{i(k-k')t}-e^{ik't}}{|t|}dt &= \int_0^{\infty}\frac{e^{i(k-k')t}-e^{ik't}}{t}dt - \int_{-\infty}^{0}\frac{e^{i(k-k')t}-e^{ik't}}{t}dt\\ &= 2\operatorname{Re}\left(\int_0^{\infty}\frac{e^{i(k-k')t}-e^{ik't}}{t}dt\right)\\ &= 2\int_0^{\infty}\frac{\cos(k-k')t-\cos k't}{t}dt\\ &= 4\int_0^{\infty}\frac{\sin\left((2k'-k)/2\right)t\sin(k/2)t}{t}dt.\end{aligned} \tag{2.6}$$

The latter integral is known [4] to be

$$\begin{aligned}\int_0^{\infty}\frac{\sin\left((2k'-k)/2\right)t\sin(k/2)t}{t}dt &= \frac{1}{2}\operatorname{sign}\left(k(2k'-k)\right)\ln\left|\frac{k'}{k'-k}\right|\\ &= \frac{1}{2}\left|\ln\left|\frac{k'}{k'-k}\right|\right|.\end{aligned} \tag{2.7}$$

Therefore, we have

$$\widehat{\mathcal{K}}_\varphi(k,k') = \sqrt{\frac{2}{\pi}}\widehat{\varphi}(k-k')\left|\ln\left|\frac{k'}{k'-k}\right|\right|. \tag{2.8}$$

The kernel of the adjoint operator $\widehat{\mathcal{K}}_\varphi^*$ can be obtained from the latter expression by the interchange of the arguments k, k' and complex conjugation:

$$\begin{aligned}\widehat{\mathcal{K}}_\varphi^*(k,k') &= \sqrt{\frac{2}{\pi}}\widehat{\varphi}^*(k'-k)\left|\ln\left|\frac{k}{k-k'}\right|\right|\\ &= \sqrt{\frac{2}{\pi}}\widehat{\varphi}(k-k')\left|\ln\left|\frac{k}{k-k'}\right|\right|.\end{aligned} \tag{2.9}$$

Here we have used the condition $\varphi(x) \in \mathbb{R}$, hence $\widehat{\varphi}^*(k'-k) = \widehat{\varphi}(k-k')$.

The kernel of the operator $F(\mathcal{K}_\varphi \circ \varphi)$ is given by the convolution

$$\begin{aligned} F(\mathcal{K}_\varphi \circ \varphi)(k,k') &= \int_{-\infty}^{\infty} \widehat{\mathcal{K}}_\varphi(k,p)\widehat{\varphi}(p-k')dp \\ &= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \widehat{\varphi}(k-p)\left|\ln\left|\frac{p}{p-k}\right|\right|\widehat{\varphi}(p-k')dp \\ &= \int_{-\infty}^{\infty} \widehat{\varphi}(k-p)\mathcal{K}_\varphi^*(p,k')dp \\ &= F(\varphi \circ \mathcal{K}_\varphi^*)(k,k'). \end{aligned} \tag{2.10}$$

Therefore, $F(\mathcal{K}_\varphi \circ \varphi) = F(\varphi \circ \mathcal{K}_\varphi^*)$ and, consequently, $\mathcal{K}_\varphi \circ \varphi = \varphi \circ \mathcal{K}_\varphi^*$. The lemma is proved. □

Note that $\mathcal{K}_\varphi$ is not an integral operator, but its Fourier transform $\widehat{\mathcal{K}}_\varphi$ is an integral operator with a weakly singular kernel.

From Lemma 2.1, we obviously have the following corollary.

Corollary 2.2. *For any real-valued function $\varphi(x) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, the operator $\mathcal{K}_\varphi$ is selfadjoint in the weighted space $L_2(\mathbb{R}, dx/\varphi(x))$.*

3. Equilibrium distribution functions with compact support

In this section, we consider an important case $\operatorname{supp}\varphi \subseteq [a,b] \subset \mathbb{R}$. Without losing generality, one can assume that $a = -1$, $b = 1$. Indeed, it is enough to make linear change of variables $x \to x' = (2x-b-a)/(b-a)$ to get equivalent problem on the renormalized real line such that $[a,b] \to [-1,1]$.

3.1. Reference operator $\mathcal{K}_0$: introduction. In this subsection, we introduce reference operator $\mathcal{K}_0$ as a prototype for operators $\mathcal{K}_\varphi$ corresponding to $\varphi(x)$ with compact support $\operatorname{supp}\varphi \subseteq [a,b]$. Operator $\mathcal{K}_0$ corresponds to the equilibrium distribution function

$$\chi_{[-1,1]}(x) = \begin{cases} 1, & x \in [-1,1], \\ 0, & \text{otherwise}. \end{cases} \tag{3.1}$$

Namely,

$$(\mathcal{K}_0 u)(x) = \int_{-\infty}^{\infty} \frac{u(x)\chi_{[-1,1]}(s) - u(s)\chi_{[-1,1]}(x)}{|x-s|} ds. \tag{3.2}$$

Note that in order to satisfy the normalization condition $\int_{-\infty}^{\infty} \varphi(x)dx = 1$, one should take function $(1/2)\chi_{[-1,1]}$, but for technical convenience, we will omit the coefficient 1/2.

Operator $\mathcal{K}_0$ allows complete spectral analysis [5, 6], which is based on the decomposition of the Hilbert space $L_2(\mathbb{R})$ into the orthogonal sum of the subspaces

$$L_2(\mathbb{R}) = L_2(-\infty,-1) \oplus L_2[-1,1] \oplus L_2(1,\infty). \tag{3.3}$$

It corresponds to the representation of an arbitrary function $u \in L_2(\mathbb{R})$ in the form

$$u = \begin{pmatrix} u_- \\ u_0 \\ u_+ \end{pmatrix}, \tag{3.4}$$

where $u_0 = P_0 u$; $u_\pm = P_\pm u$; P_-, P_0, and P_+ stand for the projection operators on the subspaces $L_2(-\infty,-1)$, $L_2[-1,1]$, and $L_2(1,\infty)$, correspondingly (operators of multiplication by the indicators of the corresponding intervals). In this representation, operator $\mathcal{K}_0$ can be written in the form

$$\mathcal{K}_0 = \begin{pmatrix} P_-\mathcal{K}_0P_+ & P_-\mathcal{K}_0P_0 & P_-\mathcal{K}_0P_+ \\ P_0\mathcal{K}_0P_+ & P_0\mathcal{K}_0P_0 & P_0\mathcal{K}_0P_+ \\ P_+\mathcal{K}_0P_+ & P_+\mathcal{K}_0P_0 & P_+\mathcal{K}_0P_+ \end{pmatrix}. \tag{3.5}$$

By straightforward calculations [5, 6], one can check that the matrix elements of the above operator-valued matrix are as follows.

Lemma 3.1. (1) $P_-\mathcal{K}_0P_0 = P_-\mathcal{K}_0P_0 = P_+\mathcal{K}_0P_+ = P_+\mathcal{K}_0P_- = 0$;

(2) $P_-\mathcal{K}_0P_-$ *and* $P_+\mathcal{K}_0P_+$ *are the operators of multiplication by the functions*

$$q_\mp(x) = \left| \ln \left| \frac{x-1}{x+1} \right| \right|, \tag{3.6}$$

where the sign $\mp$ *means that the variable* x *varies in the intervals* $x < -1$ *and* $x > 1$, *respectively;*

(3) $P_0\mathcal{K}_0P_\mp = -K_\mp$, *where*

$$\begin{aligned} (K_-u_-)(x) &= \int_{-\infty}^{-1} \frac{u_-(s)}{|x-s|}\,ds, \quad x \in [-1,1], \\ (K_+u_+)(x) &= \int_{1}^{\infty} \frac{u_+(s)}{|x-s|}\,ds, \quad x \in [-1,1]; \end{aligned} \tag{3.7}$$

(4) *the restricted operator* $K_0 := P_0\mathcal{K}_0P_0$ *acts in the space* $L_2[-1,1]$ *as follows:*

$$(K_0u_0)(x) = \int_{-1}^{1} \frac{u_0(x)-u_0(s)}{|x-s|}\,ds. \tag{3.8}$$

Therefore, one can represent operator $\mathcal{K}_0$ as follows:

$$\mathcal{K}_0 = \begin{pmatrix} q_- & 0 & 0 \\ -K_- & K_0 & -K_+ \\ 0 & 0 & q_+ \end{pmatrix}. \tag{3.9}$$

This representation makes important study of the restricted operator K_0 in the space $L_2[-1,1]$.

3.2. Study of the restricted operator K_0.

THEOREM 3.2. *The operator K_0 in the Hilbert space $L_2[-1,1]$ is selfadjoint, $K_0 = K_0^*$, and its spectrum $\sigma(K_0)$ is discrete and equal to the set of simple eigenvalues*

$$\sigma(K_0) = \{\mu_n\}_{n=0}^{\infty}, \tag{3.10}$$

where

$$\mu_0 = 0; \qquad \mu_n = 2\sum_{j=1}^{n}\frac{1}{j}, \quad n \geq 1. \tag{3.11}$$

The correspondent eigenfunctions are Legendre polynomials:

$$p_0(x) = \frac{1}{2}; \qquad p_n(x) = \sqrt{\frac{2n+1}{2}}\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n \geq 1. \tag{3.12}$$

Proof. Selfadjointness of the operator K_0 follows from Lemma 2.1. Indeed, in our case, $\varphi(x) = \chi_{[-1,1]}(x)$, and due to Lemma 2.1, we have $\mathcal{K}_0 \circ \chi_{[-1,1]} = \chi_{[-1,1]} \circ \mathcal{K}_0^*$. The operator of multiplication by the indicator $\chi_{[-1,1]}(x)$ is the projection operator P_0, therefore, $\mathcal{K}P_0 = P_0\mathcal{K}^*$, which implies

$$K_0 = P_0\mathcal{K}_0P_0 = P_0\mathcal{K}_0^*P_0 = K_0^*. \tag{3.13}$$

Now we consider a polynomial of the order n:

$$p_n(x) = \sum_{k=0}^{n} b_k^{(n)} x^n \tag{3.14}$$

and demonstrate that we can uniquely choose the coefficients $b_k^{(n)}$ such that $b_n^{(n)} \neq 0$ and $K_0 p_n = \mu_n p_n$. One can represent

$$x^k - s^k = (x-s)\sum_{j=1}^{k} x^{k-j}s^{j-1}. \tag{3.15}$$

Therefore,

$$\begin{aligned}(K_0p_n)(x) &= \sum_{k=1}^{n} b_k^{(n)}\sum_{j=1}^{k} x^{k-j}\left(\int_{-1}^{x} s^{j-1}ds - \int_{x}^{1} s^{j-1}ds\right)\\ &= \sum_{k=1}^{n} b_k^{(n)}\left(2x^k\sum_{j=1}^{k}\frac{1}{j} - \sum_{j=1}^{k}\frac{1+(-1)^j}{j}x^{k-j}\right).\end{aligned} \tag{3.16}$$

We introduce the notation

$$\alpha_j \overset{\text{def}}{=} \frac{1+(-1)^j}{j} \tag{3.17}$$

and rewrite the latter equality as follows:

$$\begin{aligned}(K_0 p_n)(x) &= \sum_{k=1}^{n} b_k^{(n)} \left(\mu_k x^k - \sum_{j=1}^{k} \alpha_j x^{k-j} \right) \\ &= \sum_{k=1}^{n} b_k^{(n)} \mu_k x^k - \sum_{k=0}^{n-1} x^k \sum_{j=k+1}^{n} b_j^{(n)} \alpha_{j-k} \\ &= b_n^{(n)} \mu_n x^n + \sum_{k=0}^{n-1} \left(b_k^{(n)} \mu_k - \sum_{j=k+1}^{n} \alpha_{j-k} b_j^{(n)} \right) x^k.\end{aligned} \tag{3.18}$$

In order to satisfy the equation $K_0 p_n = \mu p_n$, we have to make equal the coefficients at all the powers of x:

$$\begin{gathered}(\mu - \mu_n) b_n^{(n)} = 0; \\ (\mu - \mu_k) b_k^{(n)} = - \sum_{j=k+1}^{n} \alpha_{j-k} b_j^{(n)}, \quad 0 \le k \le n-1.\end{gathered} \tag{3.19}$$

This $(n+1)$-dimensional system is equivalent to the vector equation

$$(A + M - \mu I) \vec{b}^{(n)} = 0, \tag{3.20}$$

where $M = \operatorname{diag}\{\mu_0, \mu_1, \mu_2, \ldots, \mu_n\}$ and A is an upper-triangular matrix with zero main diagonal (a nilpotent matrix) with the matrix elements $A_{kj} = \alpha_{j-k}$, $j \ge k+1$:

$$A = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ 0 & 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & 0 & \alpha_1 & \cdots & \alpha_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.21}$$

One can notice that due to the definition of the coefficients α_j, they vanish for all odd indices j. The determinant D_μ of the matrix $(A + M - \mu I)$ is equal to

$$D_\mu = -\mu \prod_{k=1}^{n} (\mu_k - \mu), \tag{3.22}$$

hence the eigenvalues are $\mu = 0, \mu_1, \mu_2, \ldots, \mu_n$. The only eigenvector with $b_n^{(n)} \neq 0$ (which implies that the polynomial $p_n(x)$ is of the order n) corresponds to $\mu = \mu_n$. Due to the triangular form of (3.20), all the coefficients $b_j^{(n)}$ can be constructed by a recurrent procedure and they determine the polynomial $p_n(x) \in L_2[-1,1]$ satisfying the equation $K_a p_n(x) = \mu_n p_n(x)$. Therefore, the set $\{\mu_n\}_{n=0}^{\infty}$ is in the discrete spectrum of the operator K_0 and the correspondent eigenfunctions are the polynomials $p_n(x)$. As $K_0 = K_0^*$, polynomials $p_n(x)$ are Legendre polynomials and $\sigma(K_0) = \{\mu_n\}_{n=0}^{\infty}$. □

We make a small remark [7]. Theorem 3.2 implies that operator K_0 commutes with the operator $L = (x^2-1)(d^2/dx^2)+2x(d/dx)$ generating the Legendre polynomials, $(LK_0 - K_0L) = 0$. This can be also checked by a straightforward independent calculation for every monomial x^n. The eigenvalues of the operator L are $n(n+1)$. Using the formula [4]

$$\begin{gathered}\sum_{j=1}^{n}\frac{1}{j} = C+\ln n+\frac{1}{2n}-\sum_{k=2}^{\infty}\frac{A_k}{n(n+1)\cdots(n+k-1)},\\ A_k = \frac{1}{k}\int_0^1 x(1-x)(2-x)\cdots(k-1-x)dx,\end{gathered} \tag{3.23}$$

one can obtain the following relation between the operators K_0 and L:

$$K_0 = 2CI+2\ln G+G^{-1}-2\sum_{k=2}^{\infty}A_kG^{-1}(G+I)^{-1}\cdots\left(G+(k-1)I\right)^{-1}, \tag{3.24}$$

where $G := (1/2)(I+4L)^{1/2}-I$.

3.3. Spectrum of the reference operator $\mathcal{K}_0$. Now we are ready to describe [5, 6] the spectrum of the reference operator $\mathcal{K}_0$. Using representation (3.9), one can see that the spectral problem

$$\mathcal{K}_0u^{\mu} = \mu u^{\mu} \tag{3.25}$$

is equivalent to the following system of equations:

$$\begin{gathered}q_-(x)u_-^{\mu}(x) = \mu u_-^{\mu}(x), \quad x<-1;\\ (K_0-\mu)u_0^{\mu}(x) = (K_-u_-^{\mu})(x)+(K_+u_+^{\mu})(x), \quad x\in[-1,1];\\ q_+(x)u_+^{\mu}(x) = \mu u_+^{\mu}(x), \quad x>1.\end{gathered} \tag{3.26}$$

Together with Theorem 3.2, it allows [5, 6] to get the following result.

Theorem 3.3. *Spectrum $\sigma(\mathcal{K}_0)$ of the operator $\mathcal{K}_0$ in the space $L_2(\mathbb{R})$ fills the positive semi-axis $\mathbb{R}_+$. The spectrum $\sigma(K_0) = \{\mu_n\}_{n=0}^{\infty}$ of the restricted operator K_0 is the discrete spectrum of the operator $\mathcal{K}_0$ and the correspondent eigenfunctions are $\chi_{[-1,1]}(x)p_n(x)$. If $\mu\in\mathbb{R}_+$ and $\mu\notin\sigma(K_0)$, then point μ has double multiplicity and the correspondent generalized eigenfunctions have the form*

$$u_{\mu}^{\pm}(x) = \chi_{[-1,1]}(x)u_0^{\mu\pm}(x)+\delta(x-\lambda_{\mu}^{\pm}), \tag{3.27}$$

where

$$u_0^{\mu\pm}(x) = (K_0-\mu)^{-1}\left(|x-\lambda_{\mu}^{\pm}|^{-1}\right) = -\sum_{n\geq 0}p_n(\lambda_{\mu}^{\pm})p_n(x)\in L_2[-1,1], \tag{3.28}$$

polynomials $p_n(x)$ are the normalized eigenfunctions of the operator K_0 (Legendre polynomials), and

$$\lambda_\mu^{\pm} = \mp\frac{1+e^{\mu}}{1-e^{\mu}}. \tag{3.29}$$

Proof. First, we notice that the spectrum $\sigma(K_0)$ of the restricted operator K_a is a subset of the spectrum $\sigma(\mathcal{K})$ of the operator $\mathcal{K}$. Indeed, one can assume $u_- \equiv 0$, $u_+ \equiv 0$; then system (3.26) turns into the equation $K_0 u_0 = \mu u_0$ and an eigenfunction $u_0^{\mu}(x)$ of the operator K_0 generates an eigenfunction $\chi_{[-1,1]}(x)u_0^{\mu}(x)$ of the operator $\mathcal{K}_0$.

Now, we assume that a positive value $\mu \notin \sigma(K_0)$. Then at least one of the functions u_-, u_+ should not be identical to the zero. The first and the third equations of system (3.26) imply that μ should belong to the image of the function $q_-(x)$ and/or to the image of the function $q_+(x)$. One can check that the positive semiaxis $\mathbb{R}_+$ is the image of both functions q_- and q_+. We assume that $u_-(x) \equiv 0$; then $u_+^{\mu}(x)$ is a delta function with the support in the point λ_μ^+ such that $q_+(\lambda_\mu^+) = \mu > 0$. One can check that

$$\lambda_\mu^{+} = -\frac{1+e^{\mu}}{1-e^{\mu}}, \tag{3.30}$$

hence

$$u_+^{\mu}(x) = \delta(x-\lambda_\mu^+). \tag{3.31}$$

Using statement (3) of Lemma 3.1 and (3.31), we get

$$(K_+u_+^{\mu})(x) = |x-\lambda_\mu^+|^{-1} \in L_2[-1,1]. \tag{3.32}$$

Under the assumption that $\mu \notin \sigma(K_0)$, the operator $(K_0-\mu)$ is invertible and using (3.26) and (3.32), we have

$$u_0^{\mu}(x) = (K_0-\mu)^{-1}\left(|x-\lambda_\mu|^{-1}\right) \in L_2[-1,1]. \tag{3.33}$$

The similar result is true if we assume that $u_+ \equiv 0$. Then for the function $u_-^{\mu}(x)$, we have

$$u_-^{\mu}(x) = \delta(x-\lambda_\mu^-). \tag{3.34}$$

As proved in Theorem 3.2, the operator K_0 is selfadjoint, therefore, its resolvent can be represented as follows:

$$(K_0-\mu)^{-1} = \sum_{n\geq 0}\frac{p_n\langle\cdot,p_n\rangle_{L_2[-1,1]}}{\mu_n-\mu}. \tag{3.35}$$

Therefore,

$$u_0^{\mu\pm}(x) = (K_0-\mu)^{-1}\left(|x-\lambda_\mu^{\pm}|^{-1}\right) = \sum_{n\geq 0}\frac{p_n(x)}{\mu_n-\mu}\int_{-1}^{1}\frac{p_n(s)}{|s-\lambda_\mu^{\pm}|}ds. \tag{3.36}$$

The legendre polynomials $p_n(x)$ are the eigenfunctions of the restricted operator K_0, therefore, they satisfy the equality

$$\int_{-1}^{1} \frac{p_n(x) - p_n(s)}{|x-s|} ds = \mu_n p_n(x) \tag{3.37}$$

for all $x \in \mathbb{R}$, including $x \notin [-1,1]$. We take $x = \lambda_\mu^\pm$ in the latter equality and get

$$p_n(\lambda_\mu^\pm) \int_{-1}^{1} \frac{ds}{|\lambda_\mu^\pm - s|} - \int_{-1}^{1} \frac{p_n(s)}{|\lambda_\mu^\pm - s|} ds = \mu_n p_n(\lambda_\mu^\pm). \tag{3.38}$$

Therefore,

$$\begin{aligned} \int_{-1}^{1} \frac{p_n(s)}{|s-\lambda_\mu^\pm|} ds &= p_n(\lambda_\mu^\pm) \left(\int_{-1}^{1} \frac{ds}{|\lambda_\mu^\pm - s|} - \mu_n \right) \\ &= p_n(\lambda_\mu^\pm)(q_\pm(\lambda_\mu^\pm) - \mu_n). \end{aligned} \tag{3.39}$$

Using (3.36) and (3.39), we obtain

$$u_0^{\mu\pm}(x) = -\sum_{n\geq 0} p_n(\lambda_\mu^\pm) p_n(x). \tag{3.40}$$

We have shown that both the spectrum $\sigma(K_0)$ of the restricted operator K_0 and its complement $\mathbb{R}_+ \backslash \sigma(K_0)$ are subsets of the spectrum $\sigma(\mathcal{K}_0)$, which means $\mathbb{R}_+ \subseteq \sigma(\mathcal{K}_0)$. On the other hand, the spectrum of the operator K_0 is nonnegative. Therefore, if $\mu \notin \mathbb{R}_+$ and $\mu \in \sigma(\mathcal{K}_0)$, either the first or the third equations of the system (3.26) should be satisfied with nontrivial u_- or u_+. However, that is impossible, because the images of the functions $q_\mp$ are positive. Therefore, $\sigma(\mathcal{K}_0) \subseteq \mathbb{R}_+$ and finally we have $\sigma(\mathcal{K}_0) = \mathbb{R}_+$. □

3.4. Spectral analysis of the operators $\mathcal{K}_\varphi$ corresponding to equilibrium distribution functions $\varphi(x)$ with compact support. Let $\operatorname{supp}\varphi(x) = [a,b]$. We denote by P_{ab} the operator of multiplication by the indicator of the interval $[a,b]$, and by P_- and P_+ the operators of multiplication by the indicators of the intervals $(-\infty,a)$ and (b,∞), correspondingly. The main result can be formulated as follows [7].

Theorem 3.4. *Let the equilibrium distribution function $\varphi(x)$ satisfy the following conditions:*

(i) *$\varphi(x)$ has a compact support:* $\operatorname{supp}\varphi(x) \subseteq [a,b]$;
(ii) *$\varphi(x)$ is bounded, positive, and separated from zero on $[a,b]$: there exist ε, A such that $0 < \varepsilon \leq \varphi(x) \leq A$ for all $x \in [a,b]$;*
(iii) *$\varphi(x) \in \operatorname{Lip}(\alpha)$ for some $\alpha > 0$, that is, there exist $\alpha, C > 0$ such that $|\varphi(x) - \varphi(s)| \leq C|x-s|^\alpha$ for all $x,s \in [a,b]$.*

Then the spectrum $\sigma(\mathcal{K}_\varphi)$ of the corresponding collision operator $\mathcal{K}_\varphi$ given by (1.1) fills the positive semiaxis $\mathbb{R}_+$. Additionally, the operator $\mathcal{K}_\varphi$ has a discrete real spectrum $\sigma_d(\mathcal{K}_\varphi) = \{\tau_n\}$, $\tau_n \to \infty$ when $n \to \infty$, which coincides with the discrete spectrum $\sigma(K_\varphi)$ of the restricted operator $K_\varphi = P_{ab}\mathcal{K}_\varphi P_{ab}$. If $\lambda \in \mathbb{R}_+ \backslash \sigma(K_\varphi)$, then $\lambda \in \sigma(\mathcal{K}_\varphi)$ and has a double multiplicity.

The corresponding generalized eigenfunctions are

$$u_\lambda^\pm(x) = \delta(x - y_\lambda^\pm) + \chi_{ab}(x)(K_\varphi - \lambda)^{-1}\frac{\varphi(x)}{|x - y_\lambda^\pm|}, \tag{3.41}$$

where $y_\lambda^\pm$ *are the inverse images of the function* $q_\varphi(x) = \int_a^b(\varphi(s)/|x-s|)ds$ *in the point* λ : $q_\varphi(y_\lambda^\pm) = \lambda$, $y_\lambda^+ > b$, $y_\lambda^- < a$.

Condition (iii) is not very restrictive from a physical point of view. Indeed, smoothness of the equilibrium distribution function is a natural physical property. As the absolute values of the parameters a and b can be arbitrary large, one could think that condition (i) is also not physically restrictive. However, as we will show later, condition (ii) and related condition (i) are crucial for the spectral properties of the corresponding operators.

Proof of Theorem 3.4. Using the technique used above for the analysis of the operator $\mathcal{K}_0$, we decompose the Hilbert space $L_2(\mathbb{R})$ in the orthogonal sum of the subspaces:

$$L_2(\mathbb{R}) = L_2(-\infty, a) \oplus L_2[a,b] \oplus L_2(b,\infty), \tag{3.42}$$

which corresponds to the representation of an arbitrary function $u \in L_2(\mathbb{R})$ in the form

$$u = \begin{pmatrix} u_- \\ u_{ab} \\ u_+ \end{pmatrix}, \tag{3.43}$$

where $u_\mp = P_\mp u$, $u_{ab} = P_{ab}u$. In this representation, the operator $\mathcal{K}_\varphi$ can be written in the form

$$\mathcal{K}_\varphi = \begin{pmatrix} P_-\mathcal{K}_\varphi P_- & P_-\mathcal{K}_\varphi P_{ab} & P_-\mathcal{K}_\varphi P_+ \\ P_{ab}\mathcal{K}_\varphi P_- & P_{ab}\mathcal{K}_\varphi P_{ab} & P_{ab}\mathcal{K}_\varphi P_+ \\ P_+\mathcal{K}_\varphi P_- & P_-\mathcal{K}_\varphi P_{ab} & P_+\mathcal{K}_\varphi P_+ \end{pmatrix}. \tag{3.44}$$

By straightforward calculations similar to the ones in Lemma 3.1 using condition (i) of our theorem, one can check the following:

(1) $P_-\mathcal{K}_\varphi P_{ab} = P_-\mathcal{K}_\varphi P_+ = P_+\mathcal{K}_\varphi P_{ab} = P_+\mathcal{K}_\varphi P_- = 0$;

(2) $P_-\mathcal{K}_\varphi P_-$ and $P_+\mathcal{K}_\varphi P_+$ are the operators of multiplication by the functions $\chi_-(x)q_\varphi(x)$ and $\chi_+(x)q_\varphi(x)$, respectively, where the function $q_\varphi(x)$ is as follows:

$$q_\varphi(x) = \int_a^b \frac{\varphi(s)}{|x-s|}ds; \tag{3.45}$$

(3) $P_{ab}\mathcal{K}_\varphi P_\mp = -K_\varphi^\mp$, where

$$\begin{aligned} (K_\varphi^- u_-)(x) &= \varphi(x)\int_{-\infty}^a \frac{u_-(s)}{|x-s|}ds, \\ (K_\varphi^+ u_+)(x) &= \varphi(x)\int_b^\infty \frac{u_+(s)}{|x-s|}ds; \end{aligned} \tag{3.46}$$

(4) the restricted operator $P_{ab}\mathcal{K}_\varphi P_{ab} = K_\varphi$ acts in the space $L_2[a,b]$ as follows:

$$(K_\varphi u_{ab})(x) = \int_a^b \frac{u_{ab}(x)\varphi(s) - u_{ab}(s)\varphi(x)}{|x-s|}ds. \tag{3.47}$$

Therefore, one can write the operator $\mathcal{K}_\varphi$ in the form similar to (3.9):

$$\mathcal{K}_\varphi = \begin{pmatrix} q_\varphi & 0 & 0 \\ -K_\varphi^- & K_\varphi & -K_\varphi^+ \\ 0 & 0 & q_\varphi \end{pmatrix}, \tag{3.48}$$

where q_φ stands for the operator of multiplication by $q_\varphi(x)$. Hence, the spectral problem

$$\mathcal{K}_\varphi u = \lambda u \tag{3.49}$$

becomes equivalent to the system of equations similar to (3.26):

$$\begin{gathered} q_\varphi(x)u_-(x) = \lambda u_-(x), \quad x < a; \\ (K_\varphi - \lambda)u_{ab}(x) = (K_\varphi^- u_-)(x) + (K_\varphi^+ u_+)(x), \quad x \in [a,b]; \\ q_\varphi(x)u_+(x) = \lambda u_+(x), \quad x > b. \end{gathered} \tag{3.50}$$

There always exist trivial solutions of the first and the third equations of system (3.50), $u_-(x) \equiv 0$, $u_+(x) \equiv 0$. In this case, the second equation of this system turns into $K_\varphi u_{ab} = \lambda u_{ab}$, and every eigenfunction $u_{ab}^\lambda(x)$ of the restricted operator K_φ generates an eigenfunction $u^\lambda(x) = \chi_{ab}(x)u_{ab}^\lambda(x)$ of the operator $\mathcal{K}_\varphi$. Therefore, the spectrum of the operator K_φ is a subset of the spectrum of the operator $\mathcal{K}_\varphi$.

Similar to Theorem 3.3, one can show that every point $\lambda \in \mathbb{R}_+$ belongs to the spectrum of the operator $\mathcal{K}_\varphi$, and if $\lambda \in \mathbb{R}_+ \backslash \sigma(K_\varphi)$, it has a double multiplicity. The corresponding generalized eigenfunctions can be also calculated similar to Theorem 3.3.

In order to accomplish the proof of our theorem, we have to study the spectrum $\sigma(K_\varphi)$ of the operator K_φ. As shown above, $\sigma(K_\varphi) \subset \sigma(\mathcal{K}_\varphi)$.

Due to Lemma 2.1, under our conditions, the operator $\mathcal{K}_\varphi$ obeys the relation $\mathcal{K}_\varphi \circ \varphi = \varphi \circ \mathcal{K}_\varphi^*$, where $\mathcal{K}_\varphi^*$ is the adjoint operator and φ stands for the operator of multiplication by the function $\varphi(x)$. Therefore, due to condition (i), we have $K_\varphi \circ \varphi = P_{ab}\mathcal{K}_\varphi P_{ab} \circ \varphi = P_{ab}\mathcal{K}_\varphi \circ \varphi P_{ab} = P_{ab}\varphi \circ \mathcal{K}_\varphi^* P_{ab} = \varphi \circ P_{ab}\mathcal{K}_\varphi^* P_{ab} = \varphi \circ K_\varphi^*$. Hence, due to condition (ii), the operator K_φ is selfadjoint in the space $L_2([a,b], dx/\varphi(x))$. Consequently, its spectrum is real.

As mentioned above, by linear change of variables, without losing generality, we can assume that $a = -1$, $b = 1$. Hereafter in this proof, we accept this assumption.

By straightforward calculation, one can get

$$K_\varphi = \varphi \circ K_0 - (K_0\varphi), \tag{3.51}$$

where $(K_0\varphi)$ is the operator of multiplication by the function $(K_0\varphi)(x)$. Indeed,

$$\begin{aligned}(K_\varphi)(x) &= \int_{-1}^{1} \frac{u(x)\varphi(s) - u(s)\varphi(x)}{|x-s|} ds \\ &= \int_{-1}^{1} \frac{u(x)\varphi(s) - u(s)\varphi(x) + u(x)\varphi(x) - u(x)\varphi(x)}{|x-s|} ds \\ &= \varphi(x) \int_{-1}^{1} \frac{u(x) - u(s)}{|x-s|} ds - u(x) \int_{-1}^{1} \frac{\varphi(x) - \varphi(s)}{|x-s|} ds \\ &= \varphi(x)(K_0 u)(x) - u(x)(K_0\varphi)(x).\end{aligned} \tag{3.52}$$

We denote by $R_\varphi(z) = (K_\varphi - z)^{-1}$ the resolvent of the operator K_φ. One can check that the following relation is valid:

$$R_\varphi(z) = R_0(z) \circ \frac{1}{\varphi} - R_\varphi(z) \circ [z(\varphi - I) - (K_0\varphi)] \circ R_0(z) \circ \frac{1}{\varphi}, \tag{3.53}$$

therefore,

$$R_\varphi(z) = R_0(z) \circ \frac{1}{\varphi} \circ \left[I + [z(\varphi - I) - (K_0\varphi)] \circ R_0(z) \circ \frac{1}{\varphi} \right]^{-1} \tag{3.54}$$

if the inverse operator in the right-hand side of (3.54) exists.

Now, we consider the resolvent $R_0(z) = (K_0 - z)^{-1}$ of the operator K_0. Obviously, this resolvent has a discrete spectrum with the eigenvalues $\gamma_n(z) = 1/(\mu_n - z)$. For every $z \notin \sigma(K_0)$, the set $\{\gamma_n(z)\}$ is bounded from above, $|\gamma_n(z)| \le 1/\min_n |\mu_n - z|$, and has one accumulation point $\gamma_\infty = 0$, that is, $\gamma_n \to 0$ when $n \to \infty$. Therefore, except for the discrete countable set $z = \mu_n$, the resolvent $R_0(z)$ is a compact operator.

Due to condition (ii) of our theorem, the operators of multiplication by the functions $\varphi(x)$ and $1/\varphi(x)$ are bounded. Due to condition (iii), the same is true for the operator of multiplication by the function $(K_0\varphi)(x)$. Indeed,

$$\begin{aligned}\|(K_0\varphi)\|^2 &\le \max_{x\in[-1,1]} |(K_0\varphi)(x)|^2 = \max_x \left| \int_{-1}^{1} \frac{\varphi(x) - \varphi(s)}{|x-s|} ds \right|^2 \\ &\le C^2 \max_x \left| \int_{-1}^{1} |x-s|^{\alpha-1} ds \right|^2 < \infty\end{aligned} \tag{3.55}$$

because $\alpha > 0$.

Therefore, as the resolvent $R_0(z)$ is a compact operator except for a countable discrete set of z, the operator $[z(\varphi - I) - (K_0\varphi)] \circ R_0(z) \circ (1/\varphi)$ is compact outside of this set because it is a product of a compact and a bounded operator. Hence, his spectrum cannot have an accumulation point at $\lambda = -1$. Outside of the mentioned set, it is an analytic operator-valued function of z. Consequently, the point $\lambda = -1$ can be an eigenvalue of this operator only for a countable discrete set of z. Therefore, outside of this set, there exists a bounded operator $[I + [z(\varphi - I) - (K_0\varphi)] \circ R_0(z) \circ (1/\varphi)]^{-1}$. Then in the right-hand

side of (3.54), we have a product of a bounded operator and a compact operator $R_0(z)$. Therefore, the resolvent $R_\varphi(z)$ is a compact operator except for a countable discrete set of z. This implies that the operator K_φ can have only a discrete spectrum and its eigenvalues $\tau_n \to \infty$ when $n \to \infty$. These eigenvalues form the discrete spectrum of the operator $\mathcal{K}_\varphi$. □

3.5. Spectral estimation for the discrete spectrum. As shown above, in the case $\operatorname{supp}\varphi = [a,b]$, the spectrum of the restricted operator $K_\varphi = P_{ab}\mathcal{K}_\varphi P_{ab}$ generates discrete spectrum of the operator $\mathcal{K}_\varphi$. Without losing generality, we can assume that $a = -1$ and $b = 1$. We will use representation (3.51) in order to obtain spectral estimations for operator K_φ. We assume that $\varphi(x)$ satisfies the conditions of Theorem 3.4.

As shown above, operator K_φ is selfadjoint in Hilbert space $L_2([-1,1], dx/\varphi(x))$. We denote the scalar product in this space as follows:

$$\langle u,v\rangle := \int_{-1}^{1} u(x)\bar{v}(x)\frac{dx}{\varphi(x)} \tag{3.56}$$

and the norm as $\mathbf{I}u\mathbf{I}^2 := \langle u,u\rangle$. Operator K_0 is selfadjoint in Hilbert space $L_2([-1,1], dx)$; we denote the scalar product in this space as follows:

$$(u,v) := \int_{-1}^{1} u(x)\bar{v}(x)dx \tag{3.57}$$

and the norm as $\|u\|^2 := (u,u)$.

We consider the spectral problem

$$K_\varphi u = \lambda u. \tag{3.58}$$

As shown above, under the conditions of Theorem 3.4, operator K_φ is semibounded from below, has purely discrete spectrum, and selfadjoint in Hilbert space $L_2([-1,1], dx/\varphi(x))$. Due to the maximinimal principle [1], the nth eigenvalue of operator K_φ can be calculated as follows:

$$\lambda_n = \max_{\Phi_n \subset D_\varphi} \inf_{u\in\Phi_n, \mathbf{I}u\mathbf{I}=1} \langle K_\varphi u, u\rangle, \tag{3.59}$$

where D_φ is the domain of operator K_φ and Φ_n is an arbitrary linear set satisfying condition $\dim D_\varphi \backslash \Phi_n \le n$. Using representation (3.51), we get from (3.59)

$$\begin{aligned}\lambda_n &= \max_{\Phi_n \subset D_\varphi} \inf_{u\in\Phi_n, \mathbf{I}u\mathbf{I}=1} \left[\langle \varphi K_0 u, u\rangle - \langle (K_0\varphi)u, u\rangle\right] \\ &= \max_{\Phi_n \subset D_\varphi} \inf_{u\in\Phi_n, \mathbf{I}u\mathbf{I}=1} \left[(K_0 u, u) - \langle (K_0\varphi)u, u\rangle\right].\end{aligned} \tag{3.60}$$

Obviously,

$$\begin{aligned}
|\langle (K_0\varphi)u,u\rangle| &= \left| \int_{-1}^{1} (K_0\varphi)(x)\,|u(x)|^2 \frac{dx}{\varphi(x)} \right| \\
&\le \max_{x\in[-1,1]} |(K_0\varphi)(x)| \int_{-1}^{1} |u(x)|^2 \frac{dx}{\varphi(x)} \\
&= \max_{x\in[-1,1]} |(K_0\varphi)(x)|\,\mathbf{I}u\mathbf{I}^2,
\end{aligned} \tag{3.61}$$

therefore,

$$\begin{aligned}
&\max_{\Phi_n\subset D_\varphi} \inf_{u\in\Phi_n,\mathbf{I}u\mathbf{I}=1} - \max_{x\in[-1,1]} |(K_0\varphi)(x)| \\
&\qquad \le \lambda_n \le \max_{\Phi_n\subset D_\varphi} \inf_{u\in\Phi_n,\mathbf{I}u\mathbf{I}=1} + \max_{x\in[-1,1]} |(K_0\varphi)(x)|.
\end{aligned} \tag{3.62}$$

As $\varphi(x) > 0$, one can easily check that

$$\frac{1}{\max_{x\in[-1,1]}\varphi(x)}\|u\|^2 \le \mathbf{I}u\mathbf{I}^2 \le \frac{1}{\min_{x\in[-1,1]}\varphi(x)}\|u\|^2, \tag{3.63}$$

therefore,

$$\begin{aligned}
\min_{x\in[-1,1]} \varphi(x) \inf_{u\in\Phi_n} \frac{(K_0u,u)}{\|u\|^2} &\le \inf_{u\in\Phi_n,\mathbf{I}u\mathbf{I}^2} (K_0u,u) \\
&\le \max_{x\in[-1,1]} \varphi(x) \inf_{u\in\Phi_n} \frac{(K_0u,u)}{\|u\|^2}.
\end{aligned} \tag{3.64}$$

This implies

$$\begin{aligned}
\min_{x\in[-1,1]} \varphi(x) \inf_{u\in\Phi_n,\|u\|=1} (K_0u,u) &\le \inf_{u\in\Phi_n,\mathbf{I}u\mathbf{I}=1} (K_0u,u) \\
&\le \max_{x\in[-1,1]} \varphi(x) \inf_{u\in\Phi_n,\|u\|=1} (K_0u,u).
\end{aligned} \tag{3.65}$$

Operator K_0 is selfadjoint semibounded from below, has purely discrete spectrum in Hilbert space $L_2([-1,1],dx)$. Using the maximinimal principle [1], we get

$$\max_{\Phi_n\in D_0} \inf_{u\in\Phi_n,\|u\|=1} (K_0u,u) = \mu_n, \tag{3.66}$$

where μ_n, defined in Theorem 3.2, are the eigenvalues of operator K_0. Under the conditions of Theorem 3.4, the domains D_φ and D_0 of the operators K_φ and K_0 coincide: $D_\varphi = D_0$. Therefore, combining (3.62), (3.65), and (3.66), we obtain the following estimation

for the eigenvalues of operator K_φ:

$$\mu_n \min_{x\in[-1,1]} \varphi(x) - \max_{x\in[-1,1]} \left|(K_0\varphi)(x)\right| \le \lambda_n \le \mu_n \max_{x\in[-1,1]} \varphi(x) + \max_{x\in[-1,1]} \left|(K_0\varphi)(x)\right|. \tag{3.67}$$

In particular, this estimation can be applied to the case when the equilibrium distribution function $\varphi(x)$ is similar to the homogeneous equilibrium distribution function

$$\varphi(x) = \frac{1}{2} + \gamma h(x), \tag{3.68}$$

where $\gamma \in \mathbb{R}$, $\int_{-1}^{1} h(x)dx = 0$, and the perturbation function $h(x)$ satisfies condition (iii) of Theorem 3.4 and $\max_{x\in[-1,1]} |h(x)| \le C$. Condition (iii) of Theorem 3.4 means that there exist constants $A, \alpha > 0$ such that

$$\left|h(x) - h(s)\right| \le A|x-s|^\alpha, \quad \forall x, s \in [-1,1]. \tag{3.69}$$

In this case, one can easily estimate

$$\max_{x\in[-1,1]} \left|(K_0\varphi)(x)\right| = \max_{x\in[-1,1]} \left|\int_{-1}^{1} \frac{h(x)-h(s)}{|x-s|} ds\right| \le \frac{2A}{\alpha}. \tag{3.70}$$

In this case, (3.68) can be written as follows:

$$\frac{1}{2}\mu_n - \gamma\left(\mu_n C + \frac{2A}{\alpha}\right) \le \lambda_n \le \frac{1}{2}\mu_n + \gamma\left(\mu_n C + \frac{2A}{\alpha}\right). \tag{3.71}$$

We say that $\varphi(x)$ is similar to the homogeneous equilibrium distribution function if it has the form (3.68) and $\gamma \ll \alpha/2A$, $\gamma \ll 1/C$. In this case, it is easy to construct approximations of the eigenvalues and eigenfunctions of the operator K_φ in terms of the powers of the small parameter γ. We will write down here the first-order approximation. From (3.51), (3.58), and (3.68), we have

$$\frac{1}{2}K_0 u + \gamma(hK_0 u - (K_0 h)u) = \lambda u. \tag{3.72}$$

Decomposing the eigenfunction u in a series of normalized Legendre polynomials $u = \sum_{m\ge 0} u^{(m)} p_m(x)$ and calculating the scalar product of the both hand sides of (3.72) with $p_l(x)$, we get

$$\left(\lambda - \frac{1}{2}\mu_l\right)u^{(l)} = \gamma \sum_{m\ge 0} u^{(m)}((\mu_m - K_0)h, p_l p_m). \tag{3.73}$$

From this equation, we get the following first-order approximations for nth eigenvalue and eigenfunction of operator K_0:

$$\begin{aligned} \lambda_n &= \frac{1}{2}\mu_n + \gamma\big((\mu_n - K_0)h, p_n^2\big) + o(\gamma); \\ u_n(x) &= p_n(x) + 2\gamma \sum_{m\neq n} \frac{\big((\mu_n - K_0)h, p_n p_m\big)}{\mu_n - \mu_m} p_m(x) + o(\gamma), \end{aligned} \tag{3.74}$$

valid if $|((\mu_n - K_0)h, p_n p_m)| \ll |\mu_n - \mu_m|$, $\gamma \ll 1/C\mu_n$, and $\gamma \ll \alpha/2A$.

3.6. Continuous spectrum generated by the complement to the support of the equilibrium distribution function. As one can see from the previous discussion, the support of the equilibrium distribution function $\varphi(x)$ is "responsible" for the discrete spectrum of the operator $\mathcal{K}_\varphi$ (coinciding with the spectrum of the restricted operator $K_\varphi = P_{\operatorname{supp}\varphi}\mathcal{K}_\varphi P_{\operatorname{supp}\varphi}$), while the complement to the support "generates" branches of the continuous spectrum given by the image of the function $q_\varphi(x)$ when $x \notin \operatorname{supp}\varphi$. We will show that every interval, where equilibrium distribution function $\varphi(x)$ vanishes, generates a branch of continuous spectrum of the corresponding operator $\mathcal{K}_\varphi$. Different cases are illustrated with several specific examples. In order to be able to consider in a unified manner several specific cases, we generalize our original problem for spaces $L_2(I)$, where $I \subseteq \mathbb{R}$ is an interval on the real line (it may coincide with $\mathbb{R}$ as above). We define operators $\mathcal{K}_\varphi$ as follows:

$$\mathcal{K}_\varphi : u(x) \longmapsto \int_I \frac{u(x)\varphi(s) - u(s)\varphi(x)}{|x-s|}\,ds. \tag{3.75}$$

Above we have shown (Theorem 3.4) that if $I = \mathbb{R}$, $\varphi(x)$ has a compact support $\operatorname{supp}\varphi$, and $\varphi(x)$ is smooth and separated from zero in $\operatorname{supp}\varphi$, then the complement to $\operatorname{supp}\varphi$ generates branch $[0,\infty)$ of absolutely continuous spectrum of double multiplicity of operator $\mathcal{K}_\varphi$. Below we demonstrate that every separated interval belonging to the complement of the support of equilibrium distribution function $\varphi(x)$ generates a branch (finite or infinite) of the spectrum of operator $\mathcal{K}_\varphi$.

Let $I = [A,B] \subseteq \mathbb{R}$ be an interval on the real line $\mathbb{R}$ (may be coinciding with $\mathbb{R}$). Let $D_N = \bigcup_j [a_j, b_j]$, $a_j < b_j < a_{j+1} < b_{j+1}$, be a union (finite or infinite) of intervals, where equilibrium distribution function $\varphi(x)$ vanishes: $\varphi(x)|_{D_N} \equiv 0$. We denote by $D_S = I \backslash D_N$ the complement to the set D_N and assume that $\varphi(x)$ is piecewise continuous on D_S. We also use standard assumptions $\varphi(x) \geq 0$ and $\int_I \varphi(x)dx = \int_{D_S} \varphi(x)dx = 1$. Following the scheme introduced above, we represent Hilbert space $L_2(I)$ as $L_2(I) = L_2(D_S) \oplus L_2(D_N)$ and denote by P_S, P_N the projection operators on subspaces $L_2(D_S)$, $L_2(D_N)$, respectively. By $\chi_S(x)$, $\chi_N(x)$, we denote the indicators of the sets D_S, D_N, respectively. Any function $u \in L_2(I)$ can be presented as follows:

$$u = \begin{pmatrix} u_S \\ u_D \end{pmatrix}, \quad u_S = P_S u \in L_2(D_S), \qquad u_N = P_N u \in L_2(D_N). \tag{3.76}$$

In this representation, operator $\mathcal{H}_\varphi$ can be written as follows:

$$\mathcal{H}_\varphi = \begin{pmatrix} K_{SS} & K_{SN} \\ K_{NS} & K_{NN} \end{pmatrix}, \tag{3.77}$$

and one can calculate the entities of this operator matrix:

$$\begin{gathered} K_{SS} = P_S\mathcal{H}_\varphi P_S : u_S(x) \longmapsto \chi_S(x)\int_{D_S} \frac{u_S(x)\varphi(s) - u_S(s)\varphi(x)}{|x-s|}ds, \\ K_{SN} = P_S\mathcal{H}_\varphi P_N : u_N(x) \longmapsto -\varphi(x)\int_{D_N} \frac{u_N(s)}{|x-s|}ds, \\ K_{NS} = P_N\mathcal{H}_\varphi P_S = 0, \\ K_{NN} = P_N\mathcal{H}_\varphi P_N : u_N(x) \longmapsto q(x)u_N(x), \end{gathered} \tag{3.78}$$

where

$$q(x) := \int_{D_S} \frac{\varphi(s)}{|x-s|}ds, \quad x \in D_N. \tag{3.79}$$

Therefore, operator $\mathcal{H}_\varphi$ in representation (3.77) can be written as follows:

$$\mathcal{H}_\varphi = \begin{pmatrix} K_{SS} & K_{SN} \\ 0 & q \end{pmatrix}, \tag{3.80}$$

where q stands for the operator of multiplication by function $q(x)$ defined in the set D_N. Hence, the spectral problem $\mathcal{H}_\varphi u = \lambda u$ is equivalent to the system of equations

$$K_{SS}u_s + K_{SN}u_N = \lambda u_S, \qquad qu_N = \lambda u_N. \tag{3.81}$$

This immediately implies the following lemma.

Lemma 3.5. *Image $\mathcal{R}(q)$ of the function $q(x)$, $x \in D_N$, is a subset of the spectrum of operator $\mathcal{H}_\varphi$: $\mathcal{R}(q) \subseteq \sigma(\mathcal{H}_\varphi)$.*

Proof. Suppose that $\lambda \in \mathcal{R}(q)$, that is, there exists at least one $x_\lambda \in D_N$ such that $q(x_\lambda) = \lambda$. We choose $u_N(x) = \delta(x - x_\lambda)$, so the second equation of system (3.81) is satisfied. We calculate

$$K_{SN}u_N = -\varphi(x)\int_{D_N} \frac{\delta(x-x_\lambda)}{|x-x_\lambda|}ds = -\frac{\varphi(x)}{|x-x_\lambda|}. \tag{3.82}$$

Here $x \in D_S$, $x_\lambda \in D_N$, therefore, $K_{SN}u_N \in L_2(D_S)$. First, we assume that $\lambda \notin \sigma(K_SS)$; therefore, there exists bounded operator $(K_{SS} - \lambda)^{-1}$, and from the first equation of system (3.81), we calculate $u_S = -(K_{SS} - \lambda)^{-1}(\varphi(x)/|x - x_\lambda|)$. The eigenfunction corresponding to the point λ is

$$u = \begin{pmatrix} -(K_{SS} - \lambda)^{-1}\dfrac{\varphi(x)}{|x-x_\lambda|} \\ \delta(x - x_\lambda) \end{pmatrix}. \tag{3.83}$$

If $\lambda \in \sigma(K_{SS})$, that is, there exists (generalized) eigenfunction u_S^λ such that $K_{SS}u_S^\lambda = \lambda u_S^\lambda$, we take

$$u = \begin{pmatrix} u_S^\lambda \\ 0 \end{pmatrix}, \tag{3.84}$$

and system (3.81) is again satisfied. □

Therefore, image $\mathcal{R}(q)$ of function $q(x)$, $x \in D_N = \bigcup_j [a_j, b_j]$, generates part of the spectrum of operator $\mathcal{K}_\varphi$. It is worth noticing that as $\varphi(x) \geq 0$, also $q(x) \geq 0$, that is, $\mathcal{R}(q) \subseteq \mathbb{R}_+$.

We consider $\mathcal{R}(q)$ for every interval $[a_j, b_j] \subset D_N$. For $x \in [a_j, b_j]$, we can write

$$\begin{gathered} q(x) = q_-(x) + q_+(x), \\ q_-(x) = \int_{s<a_j} \frac{\varphi(s)}{x-s} ds, \qquad q_+(x) = \int_{s>b_j} \frac{\varphi(s)}{s-x} ds. \end{gathered} \tag{3.85}$$

As $\varphi(x) \geq 0$, function $q_-(x)$ decreases monotonically in $[a_j, b_j]$ from $\nu_j^- = \int_{s<a_j} (\varphi(s)/(a_j - s))ds$ to $\tilde{\nu}_j^- = \int_{s<a_j} (\varphi(s)/(b_j - s))ds$, while function $q_+(x)$ increases monotonically from $\tilde{\nu}_j^+ = \int_{s>b_j} (\varphi(s)/(s - a_j))ds$ to $\nu_j^+ = \int_{s>b_j} (\varphi(s)/(s - b_j))ds$. If $a_j = -\infty$, then $\nu_j^- = \tilde{\nu}_j^- = \tilde{\nu}_j^+ = 0$, so $\mathcal{R}(q)|_{(-\infty, b_j]} = [0, \nu_j^+]$. Similarly, if $b_j = \infty$, then $\nu_j^+ = \tilde{\nu}_j^+ = \tilde{\nu}_j^- = 0$ and $\mathcal{R}(q)|_{[a_j, \infty)} = [0, \nu_j^-]$. Values $\nu_j^\pm$ depend on the behaviour of function $\varphi(x)$ in the points $x = a_j - 0$ and $x = b_j + 0$; they can be either finite or infinite. Now, we illustrate different cases with three specific examples.

Example 3.6 (absolutely continuous spectrum of infinite multiplicity). We consider here case $I = \mathbb{R}$ and infinite number of intervals $[a_j, b_j] \subset D_N$. Namely, we define equilibrium distribution function $\varphi(x)$ as follows:

$$\varphi(x) = \begin{cases} 2^{-n-2}, & x \in [2n, 2n+1],\ n \geq 1, \\ 2^{n-2}, & x \in [2n-1, 2n],\ n \leq -1, \\ 2^{-2}, & x \in [-1, 1], \\ 0, & \text{otherwise}. \end{cases} \tag{3.86}$$

One can easily calculate $\int_{-\infty}^{\infty} \varphi(x)dx = \sum_{n \geq 0} 2^{-n-1} = 1$. Note that in this case, $\varphi(x)$ does not have a compact support.

We consider interval $[2j-1, 2j]$, $j > 1$, where function $\varphi(x)$ vanishes. For $x \in [2j-1, 2j]$, function $q(x)$ can be represented as a sum of the principal part $q_j^p(x)$ (given by the integrals over neighbouring intervals from D_S) and the remaining part $\tilde{q}_j(x)$:

$$\begin{aligned} q(x) &= q_j^p(x) + \tilde{q}_j(x), \quad x \in [2j-1, 2j], \\ q_j^p(x) &= 2^{-j-1} \int_{2j-2}^{2j-1} \frac{ds}{x-s} + 2^{-j-2} \int_{2j}^{2j+1} \frac{ds}{s-x} ds \\ &= 2^{-j-1} \ln \frac{2j-2-x}{2j-1-x} + 2^{-j-2} \ln \frac{2j+1-x}{2j-x}. \end{aligned} \tag{3.87}$$

Function $q_j^p(x)$ goes to infinity at $x = 2j-1$ and $x = 2j$ and has the only minimum in the interval $x \in [2j-1, 2j]$ at the point $x = x_0 = 2j - \alpha$, $\alpha = (\sqrt{33}-5)/2$, $q_j^p(x_0) = 2^{-j-1}\ln((2-\alpha)(1+\alpha)^2/\alpha(1-\alpha))$.

The remaining part given by the integrals over the rest of the integrals can be easily estimated as

$$0 < \tilde{q}_j(x) \le \frac{C_1}{j} + C_2 2^{-j} \tag{3.88}$$

with some constants C_1 and C_2.

Therefore, function $q(x)$ maps interval $[2j-1, 2j]$ into semiinfinite interval $[M_j, \infty)$, where $0 < M_j = O(|j|^{-1})$ as $j \to \infty$. The same is true for $j < 0$. It means that open interval $(0, \infty)$ belongs to the absolute continuous spectrum of operator $\mathcal{H}_\varphi$ and has infinite spectral multiplicity.

Example 3.7 (branch of absolutely continuous spectrum with lower boundary parametrically going to infinity). Now, we consider the case $I = [-1, 1]$ with equilibrium distribution function defined as follows:

$$\varphi(x) = \begin{cases} \dfrac{1}{2(1-a)}, & x \in [-1, -a] \cup [a, 1], \\ 0, & x \in (-a, a). \end{cases} \tag{3.89}$$

One can calculate

$$q(x) = \frac{1}{2(1-a)}\left(\ln\frac{1-x}{a-x} + \ln\frac{1+x}{a+x}\right) \tag{3.90}$$

and see that in the interval $[-a, a]$, function $q(x) \to \infty$ when $x \to \pm a$ and has the only minimum at the point $x = x_0 = 0$. The minimal value of $q(x)$ is $q(x_0) = (1/(a-1))\ln a \to \infty$ as $a \to 0$. Therefore, $[(1/(a-1))\ln a, \infty)$ is a branch of absolutely continuous spectrum of operator $\mathcal{H}_\varphi$, and its lower boundary goes to infinity when the complement to the support of function $\varphi(x)$ shrinks, that is, when $a \to 0$.

Example 3.8 (finite branch of absolutely continuous spectrum). Finally, we consider a case of continuous equilibrium distribution function. Namely, let $I = \mathbb{R}$ and

$$\varphi(x) = \begin{cases} x+1, & x \in [-1, 0], \\ -x+1, & x \in [0, 1], \\ 0, & \text{otherwise}. \end{cases} \tag{3.91}$$

One can calculate

$$q(x) = |x| \ln\left(1 - 1/x^2\right) + \left| \ln \left| \frac{x+1}{x-1} \right| \right|; \tag{3.92}$$

therefore, image of $q(x)$ is $[0, 2\ln 2] \in \sigma_{ac}(\mathcal{H}_\varphi)$ with double multiplicity.

4. Gaussian equilibrium distributions

Contrary to most of the above-considered cases, physically important Gaussian equilibrium distributions [8, 9, 10] have infinite support and the corresponding operators cannot be analyzed using the above described technique. We consider a family of operators K_{φ_a} with truncated Gaussian equilibrium distribution function

$$\varphi(x) = C_{a\beta}\chi_{[-a,a]}(x)e^{-\beta^2x^2}, \qquad C_{a\beta}^{-1} = \int_{-a}^{a} e^{-\beta^2x^2}dx = \frac{\sqrt{\pi}}{\beta}\operatorname{erf}(a\beta), \tag{4.1}$$

on the interval $[-a, a]$. Without loss of generality, one can take $\beta = 1$. Infinite integration limits in the original operator (1.1) are always understood as the limit of the integral over the interval $[-a, a]$ when $a \to \infty$. One can have an intuitive feeling [10] that, as truncated Gaussian functions are "very similar" for different but large values of the truncation parameter a, the spectral properties of the corresponding operators $\mathcal{H}_{\varphi_a}$ will also be similar for different large values of a. However, that is not true. Namely, there is no regular limit of the operator $\mathcal{H}_{\varphi_a}$ at $a \to \infty$. Therefore, there is no way to develop a successful perturbation theory for the spectrum of the operator $\mathcal{H}_{\varphi_a}$ with respect to the parameter $1/a$. Previously [9], we have proved analytically that the first two eigenvalues λ_1, λ_2 of operator $\mathcal{H}_{\varphi_a}$ go to zero $\sim a^{-1}$ when $a \to \infty$ and have confirmed it numerically for several other lower eigenvalues. A stronger result [8] is an analytic proof of the fact that zero becomes a point of spectral concentration when $a \to \infty$, that is, the number of the eigenvalues in arbitrary small vicinity of zero increases unlimitedly as $a \to \infty$.

As the discrete spectrum of the original operator $\mathcal{H}_\varphi$ is determined by the spectrum of the restricted operator K_φ, in this section, we study a family of operators

$$K_{\varphi_a} : u(x) \longmapsto \int_{-a}^{a} \frac{u(x)\varphi(s) - u(s)\varphi(x)}{|x-s|}ds \tag{4.2}$$

on the interval $[-a, a]$ with the equilibrium distribution function $\varphi(x)$ given by (4.1) with $\beta = 1$. One can see that a simple change of variables makes the spectral problem for operator K_{φ_a} equivalent to the spectral problem for operator

$$K_a : u(x) \longmapsto \int_{-1}^{1} \frac{u(x)\varphi_a(s) - u(s)\varphi_a(x)}{|x-s|}ds \tag{4.3}$$

on the interval $[-1, 1]$, where

$$\varphi_a(x) := C_a e^{-a^2x^2}. \tag{4.4}$$

Indeed, introducing the notations $s' = s/a$, $x' = x/a$, and $\tilde{u}(x) = u(ax)$, one can calculate

$$\begin{aligned} K_{\varphi_a} u &= \int_{-a}^{a} \frac{u(x)e^{-s^2} - u(s)e^{-x^2}}{|x-s|} ds \\ &= a \int_{-1}^{1} \frac{u(x)e^{-a^2 s'^2} - u(as')e^{-x^2}}{|x-as'|} ds' \\ &= \int_{-1}^{1} \frac{u(ax')e^{-a^2 s^2} - u(as')e^{-a^2 x'^2}}{|x'-s'|} ds' \\ &= \int_{-1}^{1} \frac{\tilde{u}(x')e^{-a^2 s'^2} - \tilde{u}(s')e^{-a^2 x'^2}}{|x'-s'|} ds' = K_a \tilde{u}. \end{aligned} \tag{4.5}$$

It is more convenient to study the spectral properties of our operator in the form (4.3). In this form, it is obvious that functions $\varphi_a(x)$ given by (4.4) are not "similar" for different but large values of the parameter a.

We use notations

$$\langle u, v \rangle := \int_{-1}^{1} u(x)\bar{v}(x) \frac{dx}{\varphi_a(x)} \tag{4.6}$$

for the inner product in space $L_2([-1,1], dx/\varphi_a(x))$ and

$$(u, v) := \int_{-1}^{1} u(x)\bar{v}(x) dx \tag{4.7}$$

for the inner product in the space $L_2([-1,1], dx)$.

We denote by $E_a[-M,M]$ the spectral measure of the operator K_a on the interval $[-M,M] \subset \mathbb{R}$. By $\mathcal{H}_a = L_2([-1,1], dx/\varphi_a(x))$, we denote Hilbert space where operator K_a acts as a selfadjoint operator. The main result [8] presented in this section is the following theorem.

Theorem 4.1. *For any $M > 0$,*

$$\dim\left(E_a[-M,M]\mathcal{H}_a\right) \longrightarrow \infty \quad \text{as } a \longrightarrow \infty. \tag{4.8}$$

This theorem means that the number of eigenvalues (counted with multiplicity) of the operator K_a (and, consequently, of the operator K_{φ_a}) in arbitrary small vicinity of zero increases to infinity when the truncation parameter a goes to infinity. Indeed, due to Theorem 3.4, the spectrum of the operators K_{φ_a} is purely discrete for all $a < \infty$. Hence the increase of the spectral measure on the interval $[-M,M]$ can be caused only by the increase of the number of the eigenvalues (counted with multiplicity) on this interval. Therefore, zero is a point of spectral concentration for the limit operator $\mathcal{K}_\infty = \lim_{a\to\infty} K_{\varphi_a}$.

Proof of Theorem 4.1. We will prove this theorem using the bilinear form approach. In order to prove our theorem, it is enough [1, 2] to construct for all $N > 0$ a linear set $F_N^a \subset D(K_a)$, $\dim F_N^a = N$, such that for any $M > 0$, there exists $a_0(N,M)$ such that for all

$a > a_0(N,M)$, inequality

$$|\langle K_a u, u\rangle| \le M\langle u,u\rangle \tag{4.9}$$

is true for all $u \in F_N^a$.

We construct F_N^a as a linear span:

$$F_N^a := \bigvee_{k=0}^{N-1} u_k, \qquad u_k(x) := p_k(x)\varphi_a^{1/2}(x), \tag{4.10}$$

where $p_k(x)$ are Legendre polynomials normalized in space $L_2([-1,1],dx)$ (the eigenfunctions of operator K_0). Functions $u_k(x)$ are orthogonal in the space $L_2([-1,1], dx/\varphi_a(x))$; therefore, $\dim F_N^a = N$ for all a.

Any function $u \in F_N^a$ can be represented as $u(x) = \sum_{k=0}^{N-1} \alpha_k u_k(x)$. Obviously,

$$\langle u,u\rangle = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \alpha_k\bar{\alpha}_l(p_k,p_l) = \sum_{k=0}^{N-1} |\alpha_k|^2. \tag{4.11}$$

On the order hand, using representation (3.51), we have

$$\langle K_a u,u\rangle = (K_0 u,u) - \langle (K_0\varphi_a)u,u\rangle. \tag{4.12}$$

We first estimate the term

$$\begin{aligned}
|\langle (K_0\varphi_a)u,u\rangle| &= \left|\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \alpha_k\bar{\alpha}_l(K_0(\varphi_a)p_k,p_l)\right| \\
&= \left|\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \alpha_k\bar{\alpha}_l\int_{-1}^{1}\int_{-1}^{1} dx\,ds\,p_k(x)p_l(x)\frac{\varphi_a(x)-\varphi_a(s)}{|x-s|}\right| \\
&= \left|\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \alpha_k\bar{\alpha}_l\int_{-1}^{1}\int_{-1}^{1} dx\,ds\left[\frac{\varphi_a(x)p_k(x)p_l(x)}{|x-s|} - \frac{\varphi_a(x)p_k(s)p_l(s)}{|x-s|}\right]\right| \\
&= \left|\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \alpha_k\bar{\alpha}_l(\varphi_a,K_0(p_k p_l))\right|
\end{aligned} \tag{4.13}$$

with a simple change of variables between x and s in the second term. We estimate the terms $(\varphi_a,K_0(p_k p_l))$. As p_k, $p_l \in F_N^a$, then $k,l \le N-1$; therefore, the product of these Legendre polynomials is a polynomial of the power not higher than $2N-2$. Therefore, one can represent

$$p_k(x)p_l(x) = \sum_{m=0}^{2N-2} \gamma_{kl}^m p_m(x), \tag{4.14}$$

where

$$\gamma_{kl}^{m} := \int_{-1}^{1} p_k(x)p_l(x)p_m(x)dx < \infty \tag{4.15}$$

for all k, l, m. Therefore,

$$|(\varphi_a, K_0(p_k p_l))| = \left| \sum_{m=0}^{2N-2} \gamma_{kl}^{m} (\varphi_a, K_0 p_m) \right| = \left| \sum_{m=0}^{2N-2} \gamma_{kl}^{m} \mu_m (\varphi_a, p_m) \right|. \tag{4.16}$$

Using the Laplace method [3], we find the asymptotics

$$\begin{aligned} (\varphi_a, p_m) &= C_a \int_{-1}^{1} p_m(x) e^{-a^2x^2} dx \\ &= C_a \sqrt{\pi} p_m(0) a^{-1} (1 + O(a^{-1})) \quad \text{as } a \longrightarrow \infty. \end{aligned} \tag{4.17}$$

Obviously, $C_a < C_1$ for $a > 1$. Thus we got the estimate at $a \to \infty$:

$$\begin{aligned} &\left| \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \alpha_k \bar{\alpha}_l (\varphi_a, K_0(p_k p_l)) \right| \\ &\quad \le 2C_1 \sum_{m=0}^{2N-2} \mu_m |p_m(0)| \max_{0\le k\le N-1} \max_{0\le l\le N-1} |\gamma_{kl}^{m}| \left| \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \alpha_k \bar{\alpha}_l \right| a^{-1}(1 + O(a^{-1})). \end{aligned} \tag{4.18}$$

Obviously,

$$|\alpha_k \bar{\alpha}_l| \le \frac{(|\alpha_k| + |\alpha_l|)^2}{2}, \tag{4.19}$$

therefore,

$$\left| \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \alpha_k \bar{\alpha}_l \right| \le 2N^2 \max_{0\le k\le N-1} |\alpha_k|^2 \le 2N^2 \sum_{k=0}^{N-1} |\alpha_k|^2. \tag{4.20}$$

Finally, we have obtained the estimate

$$\begin{aligned} |((K_0\varphi_a)u, u)| &= \left| \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \alpha_k \bar{\alpha}_l (\varphi_a, K_0(p_k p_l)) \right| \\ &\le A(N) a^{-1} \sum_{k=0}^{N-1} |\alpha_k|^2 (1 + O(a^{-1})), \end{aligned} \tag{4.21}$$

where the coefficient

$$A(N) := 4C_1N^2 \sum_{m=0}^{2N-2} \mu_m \left| p_m(0) \right| \max_{0\le k\le N-1} \max_{0\le l\le N-1} \left| \gamma_{kl}^m \right| \tag{4.22}$$

does not depend on a and finite for any $N < \infty$.

Now, we estimate the term

$$(K_0u,u) = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \alpha_k\bar{\alpha}_l (K_0(p_k\varphi_a^{1/2}), p_l\varphi_a^{1/2}). \tag{4.23}$$

We have

$$(K_0(p_k\varphi_a^{1/2}), p_l\varphi_a^{1/2}) = \int_{-1}^{1} dx\, p_l(x)\varphi_a^{1/2}(x) \int_{-1}^{1} ds \frac{p_k(x)\varphi_a(x) - p_k(s)\varphi_a(s)}{|x-s|}. \tag{4.24}$$

We split the domain of integration into two parts, one where $|x-s| < a^{-1/2}$, and the other where $|x-s| \ge a^{-1/2}$. Then we can write

$$(K_0(p_k\varphi_a^{1/2}), p_l\varphi_a^{1/2}) = \mathcal{I}_1 + \mathcal{I}_2, \tag{4.25}$$

$$\mathcal{I}_1 := C_a \iint_{|x-s|<a^{-1/2}} p_l(x)e^{-a^2x^2/2} \frac{p_k(x)e^{-a^2x^2/2} - p_k(s)e^{-a^2s^2/2}}{|x-s|} ds\,dx; \tag{4.26}$$

$$\mathcal{I}_2 := C_a \iint_{|x-s|\ge a^{-1/2}} p_l(x)e^{-a^2x^2/2} \frac{p_k(x)\exp^{-a^2x^2/2} - p_k(s)e^{-a^2s^2/2}}{|x-s|} ds\,dx; \tag{4.27}$$

and estimate integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ separately.

For $\mathcal{I}_1$, we have

$$\begin{aligned} \left|\mathcal{I}_1\right| &\le C_a \max_{x\in[-1,1]} \left| p_l(x) \right| \iint_{|x-s|\le a^{-1/2}} e^{-a^2x^2/2} \left| \frac{1}{x-s} \int_s^x \left(p_k(t)e^{-a^2t^2/2}\right)' dt \right| ds\,dx \\ &\le C_a \max_{x\in[-1,1]} \left| p_l(x) \right| \max_{x\in[-1,1]} \left| \left(p_k(x)e^{-a^2x^2/2}\right)' \right| \iint_{|x-s|\le a^{-1/2}} e^{-a^2x^2/2} ds\,dx. \end{aligned} \tag{4.28}$$

One can estimate

$$\begin{aligned} \max_{x\in[-1,1]} \left| \left(p_k(x)e^{-a^2x^2/2}\right)' \right| &\le \max_{x\in[-1,1]} \left| p_k'(x) \right| + a^2 \max_{x\in[-1,1]} \left| p_k(x) \right| \max_{x\in[-1,1]} \left| xe^{-a^2x^2/2} \right| \\ &= \max_{x\in[-1,1]} \left| p_k'(x) \right| + ae^{-1/2} \max_{x\in[-1,1]} \left| p_k(x) \right|. \end{aligned} \tag{4.29}$$

Now, we estimate

$$\iint_{|x-s|\le a^{-1/2}} e^{-a^2x^2/2}\,ds\,dx \le \int_{-1}^{1} ds \int_{s-a^{-1/2}}^{s+a^{1/2}} dx\, e^{-a^2x^2/2} = \mathcal{I}_3 + \mathcal{I}_4, \tag{4.30}$$

where

$$\begin{aligned} \mathcal{I}_3 &= \int_{2a^{-1/2}\le|s|\le 1} ds \int_{s-a^{-1/2}}^{s+a^{1/2}} dx\, e^{-a^2x^2/2}; \\ \mathcal{I}_4 &= \int_{-2a^{-1/2}}^{2a^{-1/2}} ds \int_{s-a^{-1/2}}^{s+a^{1/2}} dx\, e^{-a^2x^2/2}. \end{aligned} \tag{4.31}$$

Everywhere in the domain of integration of $\mathcal{I}_3$, we have $|x| \ge a^{-1/2}$; therefore,

$$\mathcal{I}_3 \le 2a^{-1/2} \max_{|x|\ge a^{-1/2}} e^{-a^2x^2/2} = 2a^{-1/2}e^{-a/2}. \tag{4.32}$$

On the other hand,

$$\mathcal{I}_4 \le \int_{-2a^{-1/2}}^{2a^{-1/2}} ds \int_{-a^{-1/2}}^{a^{1/2}} dx\, e^{-a^2x^2/2} = 4a^{-1/2} \int_{-a^{-1/2}}^{a^{1/2}} dx. \tag{4.33}$$

Using the Laplace method [3], we get the asymptotics

$$\int_{-a^{-1/2}}^{a^{1/2}} dx\, e^{-a^2x^2/2} = a^{-1}\sqrt{\frac{\pi}{2}}\left(1+O\left(a^{-1/2}\right)\right) \quad \text{as } a\to\infty; \tag{4.34}$$

therefore,

$$\mathcal{I}_4 \le a^{-3/2}4\sqrt{\frac{\pi}{2}}\left(1+O\left(a^{-1/2}\right)\right) \quad \text{as } a \longrightarrow \infty. \tag{4.35}$$

Combining (4.28), (4.29), (4.30), (4.32), and (4.35), we get at $a\to\infty$,

$$|\mathcal{I}_1| \le a^{-1/2}C_1 4\sqrt{\frac{\pi}{2}}e^{-1/2} \max_{x\in[-1,1]} |p_l(x)| \max_{x\in[-1,1]} |p_k(x)| \left(1+O\left(a^{-1/2}\right)\right). \tag{4.36}$$

Now, we estimate $\mathcal{I}_2$:

$$\begin{aligned} |\mathcal{I}_2| &\le a^{1/2}C_a \int_{-1}^{1}\int_{-1}^{1} dx\,ds\, \left| p_l(x)e^{-a^2x^2/2}\left(p_k(x)e^{-a^2x^2/2} - p_k(s)e^{-a^2s^2/2}\right)\right| \\ &\le a^{1/2}2C_a \max_{x\in[-1,1]} |p_l(x)| \max_{x\in[-1,1]} |p_k(x)| \int_{-1}^{1} e^{-a^2x^2/2}dx. \end{aligned} \tag{4.37}$$

Again using the Laplace method [3], we have the asymptotics

$$\int_{-1}^{1} e^{-a^2x^2/2}dx = \sqrt{\pi}a^{-1}(1+O(a^{-1})) \quad \text{as } a \longrightarrow \infty, \tag{4.38}$$

and therefore,

$$|\mathcal{I}_2| \le a^{-1/2}2C_1\sqrt{\pi}\max_{x\in[-1,1]}|p_l(x)|\max_{x\in[-1,1]}|p_k(x)|(1+O(a^{-1})). \tag{4.39}$$

From (4.25), (4.36), and (4.39), we see now that at $a \to \infty$,

$$\begin{aligned} &|(K_0(p_k\varphi_a^{1/2}), p_l\varphi_a^{1/2})| \\ &\quad \le a^{-1/2}2C_1\sqrt{\pi}(1+\sqrt{2}e^{-1/2})\max_{x\in[-1,1]}|p_l(x)|\max_{x\in[-1,1]}|p_k(x)|(1+O(a^{-1})); \end{aligned} \tag{4.40}$$

therefore at $a \to \infty$,

$$\begin{aligned} |(K_0u,u)| &= \left|\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\alpha_k\bar{\alpha}_l(K_0(p_k\varphi_a^{1/2}), p_l\varphi_a^{1/2}a)\right| \\ &\le B(N)a^{-1/2}\sum_{k=0}^{N-1}|\alpha_k|^2(1+O(a^{-1})), \end{aligned} \tag{4.41}$$

where the coefficient

$$B(N) := 4N^2C_1\sqrt{\pi}(1+\sqrt{2}e^{-1/2})\max_{0\le k\le N-1}\max_{x\in[-1,1]}|p_k(x)|^2 \tag{4.42}$$

does not depend on a and finite for any $N < \infty$. Formulae (4.11), (4.12), (4.21), and (4.41) mean that for any $N > 0$, any $M > 0$, and any function $u(x) = \sum_{k=0}^{N-1}\alpha_k u_k(x) \in F_N^a$, inequality (4.9) is satisfied for sufficiently large a, $a > a_0(N,M)$. Taking into account the normalization of the Legendre polynomials $p_k(x)$ in $L_2([-1,1],dx)$, we get very rough estimate $a_0(N,M) \le N^6M^{-2}16\pi C_1^2(1+\sqrt{2}e^{-1/2})^2$. □

5. Discussion and open problems

In the present paper, we have summarized the known results on the spectral analysis of a class of integral-difference operators $\mathcal{H}_\varphi$ for different classes of equilibrium distribution functions $\varphi(x)$. The most advanced results can be obtained in the case when equilibrium distribution function has compact support and uniformly separated from zero in the whole support. However, even in this case, there is no example of exactly solvable spectral problem except for the reference operator $\mathcal{H}_0$ given by (3.2). In the case of compact support, the spectral analysis of the original operator $\mathcal{H}_\varphi$ is essentially reduced to the analysis of the restricted operator $K_\varphi = P_{\operatorname{supp}\varphi}\mathcal{H}_\varphi P_{\operatorname{supp}\varphi}$. For the reference operator $\mathcal{H}_0$, it is shown that the corresponding restricted operator K_0 commutes with second-order differential operator L generating Legendre polynomials. It would be rather interesting to find such differential or pseudodifferential operator that commutes with K_φ for some other φ.

Spectral properties of operators K_φ (and, consequently, $\mathcal{H}_\varphi$) seem to be in crucial dependence on the property of uniform separation of $\varphi|_{\operatorname{supp}\varphi}$ from zero: $\varphi|_{\operatorname{supp}\varphi} \geq \varepsilon > 0$ (condition (iii) of Theorem 3.4). Technically, the reason is that in the opposite case, function $1/\varphi$ is not bounded and one cannot use resolvent comparison technique based on (3.53). This may be also the reason of very different spectral properties of the operators corresponding to continuous equilibrium distribution functions $\varphi(x)$ with infinite support (like Gaussian distribution). Indeed, as $\varphi(x)$ is nonnegative and integrable, it cannot simultaneously have an infinite support and be uniformly separated from zero on the whole axis $\mathbb{R}$.

We have observed these different spectral properties for Gaussian equilibrium distribution function. However, the known result (Theorem 4.1) is an asymptotic result on the concentration of the discrete spectrum in a vicinity of zero when the truncation parameter a goes to infinity. Other spectral properties of the limit operator are still unknown. In particular, it would be very interesting to clarify if the discrete spectrum of the truncated operators condenses into continuous one when $a \to \infty$.

Another difficulty for functions $\varphi(x)$ with infinite support is due to the fact that in this case, we have no relation similar to (3.51). Indeed, reference operator with constant equilibrium distribution function like K_0 can be defined only for finite intervals, otherwise, the corresponding integrals diverge. This brings an idea that for $\varphi(x)$ with infinite support, another reference operator should be chosen and investigated. However, no results in this direction are yet known.

Concluding this discussion, we can say that the considered operators have a rich spectral structure and demand different mathematical tools for their investigation. Many open problems related to different fields of mathematics make this kind of problems rather challenging.

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