We study the difference equation $x_{n+1} = \frac{x_{n-1}}{p + x_n}$, $n = 0, 1, \ldots$, where initial values $x_{-1}, x_0 \in (0, +\infty)$ and $0 < p < 1$, and obtain the set of all initial values $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$ such that the positive solution $\{x_n\}_{n=-1}^\infty$ is bounded. This answers the Open Problem 2 proposed by Kulenović and Ladas.

Kulenović and Ladas in [2] (also see [1]) studied the following difference equation:

$$x_{n+1} = \frac{x_{n-1}}{p + x_n}, \quad n = 0, 1, \ldots, \quad (1)$$

where initial values $x_{-1}, x_0 \in (0, +\infty)$ and $p \in (0, +\infty)$, and obtained the following theorem.

**Theorem 1.**

(i) If $p > 1$, then the unique equilibrium 0 of (1) is globally asymptotically stable.

(ii) If $p = 1$, then every positive solution of (1) converges to a period-two solution.

(iii) If $0 < p < 1$, then 0 and $\bar{x} = 1 - p$ are the only equilibrium points of (1), and every positive solution $\{x_n\}_{n=-1}^\infty$ of (1) with $(x_N - \bar{x})(x_{N+1} - \bar{x}) < 0$ for some $N \geq -1$ is unbounded.

They proposed the following open problem.

**Open Problem 2.** Assume that $0 < p < 1$. Determine the set of initial values $x_{-1}, x_0 \in (0, +\infty)$ for which the solution $\{x_n\}_{n=-1}^\infty$ of (1) is bounded.

In this note, we will answer the above open problem.

Write $D = (0, +\infty) \times (0, +\infty)$ and define $f : D \to D$ by, for all $(x, y) \in D$,

$$f(x, y) = \left( y, \frac{x}{p + y} \right). \quad (2)$$
2 The solutions of a difference equation

It is easy to see that if \( \{ x_n \}_{n=1}^\infty \) is a solution of (1), then \( f^n(x_{-1},x_0) = (x_{n-1},x_n) \) for any \( n \geq 0 \). From Theorem 1, we have the following corollary.

**Corollary 3.** Let \( 0 < p < 1, (x_{-1},x_0) \in D \), and \( (x_{n-1},x_n) = f^n(x_{-1},x_0) \) for any \( n \geq 0 \). If there exists \( N \geq -1 \) such that \((x_N - \bar{x})(x_{N+1} - \bar{x}) < 0\), then \( \{ x_n \}_{n=1}^\infty \) is a solution of (1).

Let

\[
A_1 = (0, \bar{x}) \times (0, \bar{x}), \quad A_2 = (\bar{x}, +\infty) \times (\bar{x}, +\infty),
\]

\[
A_3 = (0, \bar{x}) \times (\bar{x}, +\infty), \quad A_4 = (\bar{x}, +\infty) \times (0, \bar{x}),
\]

\[
R_0 = \{ \bar{x} \} \times (0, \bar{x}), \quad L_0 = \{ \bar{x} \} \times (\bar{x}, +\infty),
\]

\[
R_1 = (0, \bar{x}) \times \{ \bar{x} \}, \quad L_1 = (\bar{x}, +\infty) \times \{ \bar{x} \}.
\]

Then \( D = (\cup_{i=1}^4 A_i) \cup L_0 \cup L_1 \cup R_0 \cup R_1 \cup \{ (\bar{x}, \bar{x}) \} \).

**Lemma 4.** The following statements are true.

(i) \( f \) is a homeomorphism.

(ii) \( f(L_1) = L_0 \) and \( f(L_0) \subset A_4 \).

(iii) \( f(R_1) = R_0 \) and \( f(R_0) \subset A_3 \).

(iv) \( f(A_3) \subset A_4 \) and \( f(A_4) \subset A_3 \).

(v) \( A_2 \cup L_1 \subset f(A_2) \subset A_2 \cup L_1 \cup A_4 \) and \( A_1 \cup R_1 \subset f(A_1) \subset A_1 \cup R_1 \cup A_3 \).

**Proof.** (i) Since \( f(x_1,y_1) \neq f(x_2,y_2) \) for any \( (x_1,y_1), (x_2,y_2) \in D \) with \( (x_1,y_1) \neq (x_2,y_2) \) and \( f^{-1}(u,v) = (v(p+u), u) \) is continuous, \( f \) is a homeomorphism.

(ii) Let \((x,y) \in L_1 \) and \((u,v) = f(x,y) = (y,x/(p+y)) \), then \( y = \bar{x} \) and \( x > \bar{x} \), it follows

\[
u = \frac{x}{p+y} > \frac{\bar{x}}{p+\bar{x}} = \bar{x},
\]

which implies \( f(L_1) \subset L_0 \).

On the other hand, let \((u,v) \in L_0 \) and \((x,y) = f^{-1}(u,v) = (v(p+u),u) \), then \( u = \bar{x} \) and \( v > \bar{x} \), it follows

\[
y = u = \bar{x}, \quad x = v(p+u) > \bar{x}(p+\bar{x}) = \bar{x},
\]

which implies \( f^{-1}(L_0) \subset L_1 \). Thus \( f(L_1) = L_0 \).

Now let \((x,y) \in L_0 \) and \((u,v) = f(x,y) = (y,x/(p+y)) \), then \( x = \bar{x} \) and \( y > \bar{x} \), it follows

\[
u = \frac{x}{p+y} < \bar{x},
\]

which implies \( f(L_0) \subset A_4 \).

The proof of (iii) is similar to that of (ii).
(iv) Let \((x,y) \in A_3\) and \((u,v) = f(x,y) = (y,x/(p+y))\), then \(\bar{x} < y\) and \(0 < x < \bar{x}\), from which it follows
\[
v = \frac{x}{(p+y)} < \frac{\bar{x}}{(p+\bar{x})} = \bar{x}, \quad u > \bar{x}.
\] (7)

Thus \((u,v) \in A_4\). In a similar fashion, we may show \(f(A_4) \subset A_3\).

(v) Let \((x,y) \in A_2\) and \((u,v) = f(x,y) = (y,x/(p+y))\), then \(y > \bar{x}\) and \(x > \bar{x}\), from which it follows \(u > \bar{x}\). Since \(f\) is a homeomorphism and \(L_0 \cup L_1 \cup \{(\bar{x},\bar{x})\}\) is the boundary of \(A_2\) with \(f(L_1) = L_0\) and \(f(L_0) \subset A_4\), we obtain \(A_2 \cup L_1 \subset f(A_2) \subset A_2 \cup L_1 \cup A_4\). We similarly have \(A_1 \cup R_1 \subset f(A_1) \subset A_1 \cup R_1 \cup A_3\). Lemma 4 is proven.

**Lemma 5.** If \(0 < p < 1\) and \(\{x_n\}_{n=1}^{\infty}\) is a positive solution of (1) with \(x_n \geq \bar{x} = 1 - p\) for all \(n \geq -1\) (or \(x_n \leq \bar{x} = 1 - p\) for all \(n \leq -1\)), then \(\lim_{n \to \infty} x_n = \bar{x}\).

**Proof.** We will prove the lemma for \(x_n \geq \bar{x} = 1 - p\) for all \(n \geq -1\). The case for \(x_n \leq \bar{x} = 1 - p\) for all \(n \geq -1\) is similar. From \(x_n \geq \bar{x}\) for all \(n \geq -1\) and
\[
x_{n+1} - x_n = \frac{\bar{x} - x_n}{p + x_n} x_{n-1},
\] it follows that the sequences \(\{x_{2n-1}\}\) and \(\{x_{2n}\}\) are monotone decreasing. Let \(\lim_{n \to \infty} x_{2n} = a\) and \(\lim_{n \to \infty} x_{2n+1} = b\). By (8), we have \(a = b = \bar{x}\). Lemma 5 is proven.

Set
\[
x = g_2(y) = (p+y)\bar{x} \quad (y > 0),
\] then \(y = h_2(x) = g_2^{-1}(x) = x/\bar{x} - p\) is an increasing and differentiable function which maps \((p\bar{x},+\infty)\) onto \((0, +\infty)\). Let
\[
x = g_3(y) = (p+y)h_2(y) \quad (y > p\bar{x}),
\] then \(y = h_3(x) = g_3^{-1}(x)\) is an increasing and differentiable function which maps \((0, +\infty)\) onto \((p\bar{x}, +\infty)\).

Assume that for some positive integer \(n\) we already define increasing and differentiable functions \(h_{2n}(x)\) and \(h_{2n+1}(x)\) such that \(h_{2n}(x)\) maps \((p^n\bar{x},+\infty)\) onto \((0, +\infty)\) and \(h_{2n+1}(x)\) maps \((0, +\infty)\) onto \((p^n\bar{x}, +\infty)\). Set
\[
x = g_{2n+2}(y) = (p+y)h_{2n+1}(y) \quad (y > 0),
\] then \(y = h_{2n+2}(x) = g_{2n+2}^{-1}(x)\) is an increasing and differentiable function which maps \((p^{n+1}\bar{x}, +\infty)\) onto \((0, +\infty)\). Set
\[
x = g_{2n+3}(y) = (p+y)h_{2n+2}(y) \quad (y > p^{n+1}\bar{x}),
\] then \(y = h_{2n+3}(x) = g_{2n+3}^{-1}(x)\) is an increasing and differentiable function which maps \((0, +\infty)\) onto \((p^{n+1}\bar{x}, +\infty)\). In such a way, we construct a family of increasing and differentiable functions \(y = h_n(x)\).
4 The solutions of a difference equation

Let \( P_0 = A_2 \) and \( Q_0 = A_1 \). For any \( n \geq 1 \), write

\[
P_n = f^{-1}(P_{n-1}), \quad Q_n = f^{-1}(Q_{n-1}), \quad L_n = f^{-1}(L_{n-1}), \quad R_n = f^{-1}(R_{n-1}).
\]

(13)

From Lemma 4 we have that \( L_2 = f^{-1}(L_1) \subset P_0, \ R_2 = f^{-1}(R_1) \subset Q_0, \ P_1 = f^{-1}(P_0) \subset P_0 \) and \( Q_1 = f^{-1}(Q_0) \subset Q_0 \), which implies that for any \( n \geq 1 \),

\[
L_{n+1} \subset P_{n-1}, \quad R_{n+1} \subset Q_{n-1}, \quad P_n \subset P_{n-1}, \quad Q_n \subset Q_{n-1}.
\]

(14)

Let \( (x, y) \in L_2 \). Since \( f(L_2) = L_1 \) and \( (u, v) = f(x, y) = (y, x/(p + y)) \), it follows that

\[
\frac{x}{p+y} = v = \bar{x}, \quad y = u > \bar{x}.
\]

(15)

Thus \( x = g_2(y) = (p + y)\bar{x} > \bar{x} \) \((y > \bar{x})\) and \( L_2 = \{(x, y) : y = h_2(x), \ x > \bar{x}\} \). In a similar fashion, we may show \( R_2 = \{(x, y) : y = h_2(x), \ p\bar{x} < x < \bar{x}\} \).

Since \( f \) is a homeomorphism, \( f(P_1) = P_0 \), and \( L_0 \cup L_1 \cup \{(\bar{x}, \bar{x})\} \) is the boundary of \( P_0 \) with \( f(L_2) = L_1 \) and \( f(L_1) = L_0 \), we have

\[
P_1 = \{(x, y) : \bar{x} < y < h_2(x), \ x > \bar{x}\}.
\]

(16)

In a similar fashion, we may show

\[
Q_1 = \{(x, y) : 0 < y < \bar{x}, \ 0 < x \leq p\bar{x}\} \cup \{(x, y) : h_2(x) < y < \bar{x}, \ p\bar{x} < x < \bar{x}\}.
\]

(17)

Let \( (x, y) \in L_3 \). Since \( f(L_3) = L_2 \) and \( (u, v) = f(x, y) = (y, x/(p + y)) \in L_2 \), it follows that

\[
\frac{x}{p+y} = v = h_2(u) = h_2(y), \quad y = u > \bar{x}.
\]

(18)

Thus \( x = g_3(y) = (p + y)h_2(y) > \bar{x} \) \((y > \bar{x})\) and \( L_3 = \{(x, y) : y = h_3(x), \ x > \bar{x}\} \). In a similar fashion, we may show \( R_3 = \{(x, y) : y = h_3(x), \ 0 < x < \bar{x}\} \).

Since \( f \) is a homeomorphism, \( f(P_2) = P_1 \), and \( L_1 \cup L_2 \cup \{(\bar{x}, \bar{x})\} \) is the boundary of \( P_2 \) with \( f(L_3) = L_2 \) and \( f(L_2) = L_1 \), we have

\[
P_2 = \{(x, y) : h_3(x) < y < h_2(x), \ x > \bar{x}\}.
\]

(19)

In a similar fashion, we may show

\[
Q_2 = \{(x, y) : 0 < y < h_3(x), \ 0 < x \leq p\bar{x}\} \cup \{(x, y) : h_2(x) < y < h_3(x), \ p\bar{x} < x < \bar{x}\}.
\]

(20)
Using induction, one can easily show that for any $n \geq 2$,

$$L_n = \{(x, y) : y = h_n(x), \ x > \bar{x}\}, \tag{21}$$

and for any $n \geq 1$,

$$R_{2n} = \{(x, y) : y = h_{2n}(x), \ p^n \bar{x} < x < \bar{x}\},$$

$$R_{2n+1} = \{(x, y) : y = h_{2n+1}(x), \ 0 < x < \bar{x}\},$$

$$Q_{2n} = \{(x, y) : 0 < y < h_{2n+1}(x), \ 0 < x \leq p^n \bar{x}\}$$

$$\quad \cup \{(x, y) : h_{2n}(x) < y < h_{2n+1}(x), \ p^n \bar{x} < x < \bar{x}\},$$

$$Q_{2n+1} = \{(x, y) : 0 < y < h_{2n+1}(x), \ 0 < x \leq p^{n+1} \bar{x}\}$$

$$\quad \cup \{(x, y) : h_{2n+2}(x) < y < h_{2n+1}(x), \ p^{n+1} \bar{x} < x < \bar{x}\},$$

$$P_{2n} = \{(x, y) : h_{2n+1}(x) < y < h_{2n}(x), \ x > \bar{x}\},$$

$$P_{2n+1} = \{(x, y) : h_{2n+1}(x) < y < h_{2n+2}(x), \ x > \bar{x}\}. \tag{22}$$

By (14), it follows that for $x > 0$,

$$\bar{x} < h_3(x) \leq h_5(x) \leq \cdots \leq h_4(x) \leq h_2(x) \tag{23}$$

and for $0 < x \leq \bar{x}$,

$$\bar{x} \geq h_3(x) \geq h_5(x) \geq \cdots, \tag{24}$$

and for any $n \geq 2$ and $p^n \bar{x} < x \leq \bar{x}$

$$h_{2n-1}(x) \geq h_{2n}(x) \geq h_{2n-2}(x). \tag{25}$$

From (23), (24), and (25) we may assume that for every $x > 0$,

$$F(x) = \lim_{n \to \infty} h_{2n+1}(x), \quad G(x) = \lim_{n \to \infty} h_{2n}(x) \quad \left(n > \log_p \left( \frac{x}{\bar{x}} \right) \right). \tag{26}$$

Then $F(x) \leq G(x)$ if $x > \bar{x}$ and $F(x) \geq G(x)$ if $0 < x \leq \bar{x}$.

**Lemma 6.** $F(x)$ and $G(x)$ are continuous.

**Proof.** We first show that $F(x)$ is continuous. Let $x, x_0 \in (0, +\infty)$. Choosing $N > 0$ such that $x, x_0 \in (p^N \bar{x}, +\infty)$, then for every $n > N + 1$, there exists $c_n$ between $x$ and $x_0$ such that

$$|h_{2n+1}(x) - h_{2n+1}(x_0)| = |h'_{2n+1}(c_n)| \ |x - x_0|. \tag{27}$$
Let \( \xi_n = h_{2n+1}(c_n) \), then \( h_{2n}(\xi_n) \geq 0 \) and
\[
\begin{align*}
    h_{2n}(\xi_n) + (p + \xi_n)h'_n(\xi_n) &\geq \int_{2n} h_{2n+1}(c_n) \\
    &\geq h_{2n}(h_{2n+1}(p^Nx)) \geq h_{2N}(h_{2N+2}(p^Nx)), \\
    |h_{2n+1}(x) - h_{2n+1}(x_0)| &\leq \frac{1}{h_{2N}(h_{2N+2}(p^Nx))} |x - x_0|.
\end{align*}
\]

Thus
\[
|F(x) - F(x_0)| = \lim_{n \to -\infty} |h_{2n+1}(x) - h_{2n+1}(x_0)| \leq \frac{1}{h_{2N}(h_{2N+2}(p^Nx))} |x - x_0|,
\]

which implies \( F(x) \) is continuous. In a similar fashion, we may show that \( G(x) \) is also continuous.

Let \( S \) be the set of initial values \((x_{-1}, x_0) \in D\) such that the positive solution \( \{x_n\}_{n=1}^\infty \) of \((1)\) is bounded. Then we have the following theorem.

**Theorem 7.** Let \( 0 < p < 1 \), then \( S = W_1 \cup \{ (\overline{x}, \overline{x}) \} \cup W_2 \), where \( W_1 = \{(x, y) : F(x) \leq y \leq G(x), \overline{x} < x \} \) and \( W_2 = \{(x, y) : G(x) \leq y \leq F(x), 0 < x < \overline{x} \} \). Moreover, every positive solution \( \{x_n\}_{n=1}^\infty \) of \((1)\) with initial value \((x_{-1}, x_0) \in S\) converges to \( \overline{x} \).

**Proof.** Let \((x_{-1}, x_0) \in W_1 \cup \{ (\overline{x}, \overline{x}) \} \cup W_2\) and \( \{x_n\}_{n=1}^\infty \) is a positive solution of \((1)\) with initial value \((x_{-1}, x_0)\).

If \((x_{-1}, x_0) = (\overline{x}, \overline{x})\), then \( \{x_n\}_{n=1}^\infty \) is a trivial solution of \((1)\), which implies \( \lim_{n \to -\infty} x_n = \overline{x} \) and \((x_{-1}, x_0) \in S\).

If \((x_{-1}, x_0) \in W_1\), then \((x_{-1}, x_0) \in P_n\) for any \( n \geq 0 \), which implies \( f^n(x_{-1}, x_0) = (x_{n-1}, x_n) \in A_2\) for any \( n \geq 0 \). Thus it follows from Lemma 5 that \( \lim_{n \to -\infty} x_n = \overline{x} \) and \((x_{-1}, x_0) \in S\). In a similar fashion, we may show that if \((x_{-1}, x_0) \in W_2\), then \( \lim_{n \to -\infty} x_n = \overline{x} \) and \((x_{-1}, x_0) \in S\).

Now let \((x_{-1}, x_0) \in D - W_1 \cup \{ (\overline{x}, \overline{x}) \} \cup W_2\) and \( \{x_n\}_{n=1}^\infty \) is a positive solution of \((1)\) with initial value \((x_{-1}, x_0)\).

If \((x_{-1}, x_0) \in A_3 \cup A_4 \cup R_0 \cup R_1 \cup L_0 \cup L_1\), then by Lemma 4 we have \( f^2(x_{-1}, x_0) = (x_1, x_2) \in \{(x, y) : (x - \overline{x})(y - \overline{x}) < 0\}\), it follows from Corollary 3 that \((x_{-1}, x_0) \notin S\).

If \((x_{-1}, x_0) \in A_2 - W_1\), then there exists \( n \geq 0 \) such that
\[
(x_{-1}, x_0) \in P_n - P_{n+1} = f^{-n}(A_2) - f^{-n-1}(A_2),
\]
from which it follows
\[
f^n(x_{-1}, x_0) = (x_{n-1}, x_n) \in A_2 - f^{-1}(A_2).
\]

By Lemma 4, we have \( f^{n+1}(x_{-1}, x_0) \in A_4 \cup L_1\), which implies \( f^{n+3}(x_{-1}, x_0) = (x_{n+2}, x_{n+3}) \in A_4\), it follows from Corollary 3 that \((x_{-1}, x_0) \notin S\). In a similar fashion, we may show that if \((x_{-1}, x_0) \in A_1 - W_2\), then it follows that \((x_{-1}, x_0) \notin S\). Theorem 7 is proven.

\(\Box\)
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