STRONG LAW OF LARGE NUMBERS FOR ρ^* -MIXING SEQUENCES WITH DIFFERENT DISTRIBUTIONS

GUANG-HUI CAI

Received 19 January 2006; Accepted 24 March 2006

Strong law of large numbers and complete convergence for ρ^* -mixing sequences with different distributions are investigated. The results obtained improve the relevant results by Utev and Peligrad (2003).

Copyright © 2006 Guang-hui Cai. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Let nonempty sets $S, T \subset \mathcal{N}$, and define $\mathcal{F}_S = \sigma(X_k, k \in S)$, and the maximal correlation coefficient $\rho_n^* = \sup \text{corr}(f, g)$ where the supremum is taken over all (S, T) with $\text{dist}(S, T) \geq n$ and all $f \in L_2(\mathcal{F}_S), g \in L_2(\mathcal{F}_T)$ and where $\text{dist}(S, T) = \inf_{x \in S, y \in T} |x - y|$.

A sequence of random variables $\{X_n, n \geq 1\}$ on a probability space $\{\Omega, \mathcal{F}, P\}$ is called ρ^* -mixing if

$$\lim_{n \rightarrow \infty} \rho_n^* < 1. \quad (1.1)$$

As for ρ^* -mixing sequences of random variables, Bryc and Smoleński [1] established the moments inequality of partial sums. Peligrad [10] obtained a CLT and established an invariance principles. Peligrad [11] established the Rosenthal-type maximal inequality. Utev and Peligrad [16] obtained invariance principles of nonstationary sequences.

As for negatively associated (NA) random variables, Joag [6] gave the following definition.

Definition 1.1 (Joag [6]). A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets T_1 and T_2 of $\{1, 2, \dots, n\}$,

$$\text{Cov}(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \leq 0, \quad (1.2)$$

2 SLLN for mixing sequences with different distributions

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

Recently, some authors focused on the problem of limiting behavior of partial sums of NA sequences. Su et al. [15] derived some moment inequalities of partial sums and a weak convergence for a strong stationary NA sequence. Lin [9] set up an invariance principal for NA sequences. Su and Qin [15] also studied some limiting results for NA sequences. More recently, Liang and Su [8], Liang [7] considered some complete convergence for weighted sums of NA sequences. Those results, especially some moment inequality by Huang and Xu [5], Shao [13], and Yang [17] undoubtedly propose important theory guide in further apply for the NA sequence.

The main purpose of this paper is to establish a strong law of large numbers and complete convergence for ρ^* -mixing sequences or NA sequences with different distributions that are investigated. The results obtained improve the relevant results by Utev and Peligrad [16].

2. Main results

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leq Cb_n$. And $a_n \ll b_n$ will mean $a_n = O(b_n)$.

In order to prove our results, we need the following lemma and the concept of complete convergence.

Definition 2.1 (Hsu and Robbins [4]). Let $\{X, X_n, n \geq 1\}$ be a sequence of random variables, if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty \quad (2.1)$$

holds, $\{X_n, n \geq 1\}$ is called completely converging to X .

As for complete convergence, let now $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables and denote $S_n = \sum_{i=1}^n X_i$. The Hsu-Robbins-Erdős law of large numbers (Hsu and Robbins [4], Erdős [3]) states that

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} P(|S_n| > \varepsilon n) < \infty \quad (2.2)$$

is equivalent to $EX = 0$ and $EX^2 < \infty$.

This is a fundamental theorem in probability theory and has been intensively investigated by many authors in the past decades as we can see by Petrov [12], Chow and Teicher [2], and Stout [14]. There have been many extensions in various directions of the Hsu-Robbins-Erdős law of large numbers.

LEMMA 2.2 (Utev and Peligrad [16]). *Let $\{X_i, i \geq 1\}$ be a ρ^* -mixing sequence of random variables, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then there exists $C = C(p)$,*

such that

$$E \max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i \right|^p \leq C \left\{ \sum_{i=1}^n E |X_i|^p + \left(\sum_{i=1}^n EX_i^2 \right)^{p/2} \right\}. \quad (2.3)$$

LEMMA 2.3 (Shao [13]). *Let $\{X_i, i \geq 1\}$ be a sequence of NA random variables, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then there exists $C = C(p)$, such that*

$$E \max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i \right|^p \leq C \left\{ \sum_{i=1}^n E |X_i|^p + \left(\sum_{i=1}^n EX_i^2 \right)^{p/2} \right\}. \quad (2.4)$$

Now we state the main result of this paper.

THEOREM 2.4. *Let $\{X, X_i, i \geq 1\}$ be ρ^* -mixing sequence with $E|X|^p < \infty$, $0 < p < 2$. Let $S_n = \sum_{i=1}^n X_i$, $P(|X_i| > x) \ll P(|X| > x)$, for all $x > 0$, $i \geq 1$. When $1 \leq p < 2$, let $EX = 0$. Then,*

$$\forall \varepsilon > 0, \quad \sum_{n=1}^{\infty} n^{-1} P \left(\max_{1 \leq j \leq n} |S_j| > \varepsilon n^{1/p} \right) < \infty. \quad (2.5)$$

Proof of Theorem 2.4. For any $i \geq 1$, let $X_i^{(n)} = X_i I(|X_i| \leq n^{1/p})$, $S_j^{(n)} = \sum_{i=1}^j (X_i^{(n)} - EX_i^{(n)})$. for all $\varepsilon > 0$, then

$$\begin{aligned} & P \left(\max_{1 \leq j \leq n} |S_j| > \varepsilon n^{1/p} \right) \\ & \leq P \left(\max_{1 \leq j \leq n} |X_j| > n^{1/p} \right) + P \left(\max_{1 \leq j \leq n} \left| S_j^{(n)} + \sum_{i=1}^j EX_i^{(n)} \right| > \varepsilon n^{1/p} \right) \\ & \leq P \left(\max_{1 \leq j \leq n} |X_j| > n^{1/p} \right) + P \left(\max_{1 \leq j \leq n} |S_j^{(n)}| > \varepsilon n^{1/p} - \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EX_i^{(n)} \right| \right). \end{aligned} \quad (2.6)$$

When n large enough, first we show that

$$n^{-1/p} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EX_i^{(n)} \right| \rightarrow 0. \quad (2.7)$$

In fact

(i) if $p < 1$, then

$$\begin{aligned} n^{-1/p} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EX_i^{(n)} \right| & \leq n^{-1/p} \sum_{i=1}^n E |X_i| I(|X_i| \leq n^{1/p}) \\ & \leq n^{1-1/p} E |X| I(|X| \leq n^{1/p}) \\ & = n^{1-1/p} \sum_{k=1}^n E |X| I(k-1 < |X|^p \leq k), \end{aligned} \quad (2.8)$$

4 SLLN for mixing sequences with different distributions

because

$$\begin{aligned} \sum_{k=1}^{\infty} k^{1-1/p} E|X|I(k-1 < |X|^p \leq k) &\leq \sum_{k=1}^{\infty} E|X|^p I(k-1 < |X|^p \leq k) \\ &\leq \sum_{k=1}^{\infty} E|X|^p I(k-1 < |X|^p \leq k) = E|X|^p < \infty. \end{aligned} \quad (2.9)$$

By Kronecker lemma, we get $n^{1-1/p} \sum_{k=1}^n E|X|I(k-1 < |X|^p \leq k) \rightarrow 0, n \rightarrow \infty$, so

$$n^{-1/p} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EX_i^{(n)} \right| \rightarrow 0, \quad n \rightarrow \infty; \quad (2.10)$$

(ii) if $1 \leq p < 2$, by $EX = 0$, then

$$\begin{aligned} n^{-1/p} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EX_i^{(n)} \right| &\leq n^{-1/p} \sum_{i=1}^n E|X_i|I(|X_i| > n^{1/p}) \\ &\leq E|X|^p I(|X| > n^{1/p}) \rightarrow 0. \end{aligned} \quad (2.11)$$

Equations (2.10) and (2.11) imply (2.7).

From (2.6) and (2.7) it follows that for n large enough, we have $P(\max_{1 \leq j \leq n} |S_j| > \varepsilon n^{1/p}) \leq \sum_{j=1}^n P(|X_j| > n^{1/p}) + P(\max_{1 \leq j \leq n} |S_j^{(n)}| > \varepsilon/2 n^{1/p})$.

Hence we need only to prove that

$$\begin{aligned} I &=: \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^n P(|X_j| > n^{1/p}) < \infty, \\ II &=: \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} |S_j^{(n)}| > \frac{\varepsilon}{2} n^{1/p}\right) < \infty. \end{aligned} \quad (2.12)$$

By $E|X|^p < \infty$, then

$$I \leq C \sum_{n=1}^{\infty} P(|X| > n^{1/p}) \ll E|X|^p < \infty. \quad (2.13)$$

By Lemma 2.2, it follows that

$$\begin{aligned} II &\ll \sum_{n=1}^{\infty} n^{-1-\alpha q} E \max_{1 \leq j \leq n} |S_j^{(n)}|^q \\ &\ll \sum_{n=1}^{\infty} n^{-1-\alpha q} \left\{ \sum_{j=1}^n E|X_j^{(n)}|^q + \left(\sum_{j=1}^n E|X_j^{(n)}|^2 \right)^{q/2} \right\} \\ &=: II_1 + II_2. \end{aligned} \quad (2.14)$$

Let $q = 2$, we have

$$\begin{aligned}
 II_1 &\leq C \sum_{n=1}^{\infty} n^{-\alpha q} E|X|^q I(|X| \leq n^{1/p}) \\
 &= \sum_{n=1}^{\infty} n^{-\alpha q} \sum_{k=1}^n E|X|^q I(k-1 < |X|^p \leq k) \\
 &= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} n^{-\alpha q} E|X|^q I(k-1 < |X|^p \leq k) \\
 &\ll \sum_{k=1}^{\infty} k P(k-1 < |X|^p \leq k) \ll E|X|^p < \infty.
 \end{aligned} \tag{2.15}$$

Let $q = 2$, then $II_2 = II_1 < \infty$. So $II < \infty$. Now we complete the proof of Theorem 2.4. $\square$

COROLLARY 2.5. *Under the conditions of Theorem 2.4,*

$$\lim_{n \rightarrow \infty} \frac{S_n}{n^{1/p}} = 0 \quad \text{a.s.} \tag{2.16}$$

Proof of Corollary 2.5. For all $\varepsilon > 0$, by (2.5), we have

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} |S_j| > \varepsilon n^{1/p}\right) < \infty. \tag{2.17}$$

Then we have

$$\sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} (2^{k+1} - 1)^{-1} P\left(\max_{1 \leq j \leq 2^k} |S_j| > \varepsilon n^{1/p}\right) < \infty. \tag{2.18}$$

So

$$\sum_{k=0}^{\infty} P\left(\max_{1 \leq j \leq 2^k} |S_j| > \varepsilon 2^{(k+1)/p}\right) < \infty. \tag{2.19}$$

By the Borel-Cantelli lemma, we have

$$\max_{1 \leq j \leq 2^k} \frac{|S_j|}{2^{k/p}} \longrightarrow 0 \quad \text{a.s.} \tag{2.20}$$

For all positive integers n , there exists a nonnegative integer k_0 , such that $2^{k_0} \leq n < 2^{k_0+1}$.

Thus

$$\frac{|S_n|}{n^{1/p}} \leq \max_{1 \leq j \leq 2^{k_0+1}} \frac{|S_j|}{2^{(k_0+1)/p}} \longrightarrow 0 \quad \text{a.s.} \tag{2.21}$$

Thus we have

$$\lim_{n \rightarrow \infty} \frac{S_n}{n^{1/p}} = 0 \quad \text{a.s.} \quad (2.22)$$

Now we complete the proof of Corollary 2.5. $\square$

THEOREM 2.6. *Let $\{X, X_i, i \geq 1\}$ be NA sequence with $E|X|^p < \infty$, $0 < p < 2$. Let $S_n = \sum_{i=1}^n X_i$, $P(|X_i| > x) \ll P(|X| > x)$, for all $x > 0, i \geq 1$. When $1 \leq p < 2$, let $EX = 0$. Then,*

$$\forall \varepsilon > 0, \quad \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} |S_j| > \varepsilon n^{1/p}\right) < \infty. \quad (2.23)$$

Proof of Theorem 2.6. Using Lemma 2.3 instead of Lemma 2.2, the proof of Theorem 2.6 is similar to the proof of Theorem 2.4. $\square$

COROLLARY 2.7. *Under the conditions of Theorem 2.6,*

$$\lim_{n \rightarrow \infty} \frac{S_n}{n^{1/p}} = 0 \quad \text{a.s.} \quad (2.24)$$

Proof of Corollary 2.7. The proof of Corollary 2.7 is similar to the proof of Corollary 2.5. $\square$

Acknowledgments

This paper is supported by Key Discipline of Zhejiang Province (Key Discipline of Statistics of Zhejiang Gongshang University) and National Natural Science Foundation of China.

References

- [1] W. Bryc and W. Smoleński, *Moment conditions for almost sure convergence of weakly correlated random variables*, Proceedings of the American Mathematical Society **119** (1993), no. 2, 629–635.
- [2] Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, 3rd ed., Springer Texts in Statistics, Springer, New York, 1997.
- [3] P. Erdős, *On a theorem of Hsu and Robbins*, Annals of Mathematical Statistics **20** (1949), 286–291.
- [4] P. L. Hsu and H. Robbins, *Complete convergence and the law of large numbers*, Proceedings of the National Academy of Sciences of the United States of America **33** (1947), no. 2, 25–31.
- [5] W.-T. Huang and B. Xu, *Some maximal inequalities and complete convergences of negatively associated random sequences*, Statistics & Probability Letters **57** (2002), no. 2, 183–191.
- [6] K. Joag-Dev and F. Proschan, *Negative association of random variables, with applications*, The Annals of Statistics **11** (1983), no. 1, 286–295.
- [7] H.-Y. Liang, *Complete convergence for weighted sums of negatively associated random variables*, Statistics & Probability Letters **48** (2000), no. 4, 317–325.
- [8] H.-Y. Liang and C. Su, *Complete convergence for weighted sums of NA sequences*, Statistics & Probability Letters **45** (1999), no. 1, 85–95.
- [9] Z. Y. Lin, *An invariance principle for negatively dependent random variables*, Chinese Science Bulletin **42** (1997), no. 3, 238–242 (Chinese).

- [10] M. Peligrad, *On the asymptotic normality of sequences of weak dependent random variables*, Journal of Theoretical Probability **9** (1996), no. 3, 703–715.
- [11] M. Peligrad and A. Gut, *Almost-sure results for a class of dependent random variables*, Journal of Theoretical Probability **12** (1999), no. 1, 87–104.
- [12] V. V. Petrov, *Limit Theorems of Probability Theory. Sequences of Independent Random Variables*, Oxford Studies in Probability, vol. 4, The Clarendon Press, Oxford University Press, New York, 1995.
- [13] Q.-M. Shao, *A comparison theorem on moment inequalities between negatively associated and independent random variables*, Journal of Theoretical Probability **13** (2000), no. 2, 343–356.
- [14] W. F. Stout, *Almost Sure Convergence*, Academic Press, New York, 1974.
- [15] C. Su and Y. S. Qin, *Two limit theorems for negatively associated random variables*, Chinese Science Bulletin **42** (1997), no. 3, 243–246.
- [16] S. Utev and M. Peligrad, *Maximal inequalities and an invariance principle for a class of weakly dependent random variables*, Journal of Theoretical Probability **16** (2003), no. 1, 101–115.
- [17] S. C. Yang, *Moment inequality of random variables partial sums*, Science in China. Series A **30** (2000), 218–223.

Guang-hui Cai: Department of Mathematics and Statistics, Zhejiang Gongshang University,
 Hangzhou 310035, China
E-mail address: cghzju@163.com