MULTIPLE POSITIVE SOLUTIONS OF STRUM-LIOUVILLE EQUATIONS WITH SINGULARITIES

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The existence of multiple positive solutions for Strum-Liouville boundary value problems with singularities is investigated. By applying a fixed point theorem of cone map, some existence and multiplicity results of positive solutions are derived. Our results improve and generalize those in some well-known results.

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1. Introduction

In this paper, we consider the following second-order Strum-Liouville boundary value problem with singularities (BVP):

$$\begin{aligned}(p(t)u'(t))' + g(t)F(t, u) &= 0, \quad \forall t \in (0, 1), \\ \alpha u(0) - \beta \lim_{t \rightarrow 0^+} p(t)u'(t) &= 0, \\ \gamma u(1) - \delta \lim_{t \rightarrow 1^-} p(t)u'(t) &= 0,\end{aligned}\tag{1.1}$$

where $\alpha, \beta, \gamma, \delta \geq 0$, and

$$\rho = \beta\gamma + \alpha\gamma \int_0^1 \frac{dr}{p(r)} + \alpha\delta > 0.\tag{1.2}$$

We always assume that the following hypotheses hold:

- (A₁) $p \in C([0, 1], [0, +\infty])$, and $\int_0^1 dt/p(t) < +\infty$;
- (A₂) $g \in L(0, 1)$, and $g(s) \geq 0$, a.e. and there exists $[a, b] \subset (0, 1)$, such that $0 < \int_a^b g(s)ds$;
- (A₃) $F(t, u) \in C([0, 1] \times [0, +\infty], [0, +\infty])$.

The BVP(1.1) arises in many different areas of applied mathematics and physics, and only its positive solution is significant in some practice. For the special case as follows with

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$p(t) \equiv 1$ and $F(t, u) = f(u)$, BVP(1.1) was studied by many authors (see [1, 4–8]). Erbe and Wang [1] investigated this problem by using norm-type cone expansion and compression theorems; Lan and Webb [4] get the existence of positive solution for BVP(1.1) by using a well-known nonzero fixed point theorem. This problem is also studied by Ma [6] by using compact operator approximation.

Recently the authors [10] investigated the Strum-Liouville equation which has singularities at 0 and 1, the existence of one positive solution is established by applying the fixed point index theory.

However, the authors [1, 4, 6, 8, 10] only investigated the existence of positive solution for BVP(1.1). Liu and Li [5] get the existence of multiple positive solutions in the special case as follows with $p(t) \equiv 1$ and $g(t) \equiv 1$ by using the fixed pointed theorem. When we choose $g(t) \equiv 1$, Wang et al. [11] established some nonexistence, existence, and multiplicity results for the BVP(1.1) which are based on the Schauder fixed point theorem, the method of upper and lower solutions, and the Leray-Schauder degree theory. Furthermore, Ma and Thompson [7] studied the existence of multiple positive solutions by applying bifurcation techniques. The purpose of this paper is to consider BVP(1.1) in which $g(t) \in L^1(0, 1)$ and $F(t, u)$ satisfies weaker conditions than those in [1, 4–11], the existence of multiple positive solutions for BVP(1.1) is obtained by using a fixed point theorem. Our method is different from the ones in those papers and our results are often new even when $p(t) \equiv 1$, $F(t, u) = f(u)$.

A map $u \in C([0, 1], R^+) \cap C^1((0, 1), R^+)$, $p(t)u'(t) \in C^1((0, 1), R^+)$ is called a positive solution of BVP(1.1) if $x(t) > 0$, for all $t \in (0, 1)$ and $x(t)$ satisfies BVP(1.1).

2. Some lemmas

We will need the following well-known result (see, e.g., [3, Theorems 2.1 and 2.2]).

LEMMA 2.1. *Let K be a cone in a Banach space X . Assume that Ω is a bounded open subset of X with $\theta \in \Omega$ and let $A : K \cap (\overline{\Omega}) \rightarrow K$ be a completely continuous operator. If $Au \neq \lambda x$ for $x \in K \cap \partial\Omega$, $\lambda \geq 1$, then $i(A, K \cap \Omega, K) = 1$.*

LEMMA 2.2. *Assume that $A : K \cap (\overline{\Omega}) \rightarrow K$ is completely continuous, and there exists $B : K \cap \partial\Omega \rightarrow K$ which is completely continuous, such that*

- (i) $\inf_{x \in K \cap \partial\Omega} \|Bu\| > 0$;
 - (ii) $x - Ax \neq \lambda Bx$ for $x \in K \cap \partial\Omega$, and $\lambda \geq 0$,
- then $i(A, K \cap \Omega, K) = 0$.

We denote by $G(t, s)$ Green's function for the homogeneous boundary value problem

$$\begin{aligned} (p(t)u'(t))' &= 0, \quad \forall t \in (0, 1), \\ \alpha u(0) - \beta \lim_{t \rightarrow 0^+} p(t)u'(t) &= 0, \\ \gamma u(1) - \delta \lim_{t \rightarrow 1^-} p(t)u'(t) &= 0. \end{aligned} \tag{2.1}$$

We know that $G(t, s)$ is nonnegative on $[0, 1] \times [0, 1]$ and is expressed by

$$G(t, s) = \begin{cases} \frac{1}{\rho} \left(\beta + \alpha \int_0^s \frac{dr}{p(r)} \right) \left(\delta + \gamma \int_t^1 \frac{dr}{p(r)} \right), & 0 \leq s \leq t \leq 1, \\ \frac{1}{\rho} \left(\beta + \alpha \int_0^t \frac{dr}{p(r)} \right) \left(\delta + \gamma \int_s^1 \frac{dr}{p(r)} \right), & 0 \leq t \leq s \leq 1, \end{cases} \quad (2.2)$$

where $\rho = \alpha\delta + \alpha\gamma \int_0^1 (dr/p(r)) + \beta\gamma$.

LEMMA 2.3 [2]. *Green's function $G(t, s)$ has the following properties:*

- (i) $G(t, s) \leq G(s, s) \leq (1/\rho)(\beta + \alpha \int_0^1 (dr/p(r)))(\delta + \gamma \int_0^1 (dr/p(r))) := \theta < \infty$;
- (ii) *for all $t \in [a, b] \subset (0, 1)$, $s \in [0, 1]$, there is $G(t, s) \geq \sigma G(s, s)$,*

where

$$\sigma = \min \left\{ \frac{\delta + \gamma \int_b^1 (dr/p(r))}{\delta + \gamma \int_0^1 (dr/p(r))}, \frac{\beta + \alpha \int_0^a (dr/p(r))}{\beta + \alpha \int_0^1 (dr/p(r))} \right\}. \quad (2.3)$$

It is obvious that $0 < \sigma < 1$. Let $X = C[0, 1]$, $\Omega_h = \{u \in X : \|u\| \leq h\}$ for any $h > 0$, and $K = \{u \in C[0, 1] : u \geq 0, \text{ and } x(t) \geq \sigma x(\tau) \text{ for } t \in [a, b], \tau \in [0, 1]\}$. Then K is a positive cone in X .

Define an operator A by

$$(Au)(t) = \int_0^1 G(t, s)g(s)F(s, u(s))ds. \quad (2.4)$$

It is well known that $u \in C[0, 1] \cap C^1(0, 1)$ is a positive solution of BVP(1.1) if and only if u is a fixed point of the operator A in K .

LEMMA 2.4. $A : K \rightarrow K$ is completely continuous.

Proof. Similar to the proof of [10, Lemma 2.1], we can prove that A is a completely continuous operator. $\square$

3. Main result

THEOREM 3.1. *Assume that (A_1) – (A_3) hold, if*

(H₁) there exists $p \in (0, 1)$ such that $0 < \liminf_{u \rightarrow 0^+} \min_{t \in [0, 1]} (F(t, u)/u^p) \leq +\infty$,

(H₂) there exists $q \in (0, 1]$ such that $0 \leq \limsup_{u \rightarrow +\infty} \max_{t \in [0, 1]} (F(t, u)/u^q) < +\infty$,
then BVP(1.1) has at least one positive solution.

Proof. Without loss of generality, we assume that there exists $\epsilon > 0$ such that $x \neq Ax$ for $x \in K$ with $0 < \|x\| < \epsilon$, otherwise there is a fixed point in K and this would complete the proof.

By virtue of (H_1) , there exists $\tau > 0$ and $\epsilon_1 > 0$ such that

$$F(t, u) \geq \tau u^p, \quad \text{for } 0 \leq u \leq \epsilon_1. \quad (3.1)$$

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Define $B : C[0, 1] \rightarrow C[0, 1]$ by

$$Bu = \phi, \quad \forall u \in C[0, 1], \quad (3.2)$$

where $\phi(t) \equiv 1$, $\phi \in C[0, 1]$. Then it is easy to verify that $B : K \cap \partial\Omega \rightarrow K$ is completely continuous and $\inf_{K \cap \partial\Omega_r} \|Bu\| > 0$, $\phi \in K \setminus \theta$ with $\|\phi\| = 1$.

Choose

$$\epsilon_2 = \min \left\{ \epsilon, \epsilon_1, \left(\sigma \delta \int_a^b G(s, s) g(s) ds \right)^{1/(1-p)} \right\} \quad (3.3)$$

and $r \in (0, \epsilon_2]$; we now prove that

$$u - Au \neq \lambda Bu, \quad \text{for } u \in K \cap \partial\Omega_r, \lambda \geq 0. \quad (3.4)$$

In fact, if not, there are $\lambda_0 \geq 0$ and $u_0 \in K \cap \partial\Omega_r$ such that $u_0 - Au_0 = \lambda_0 Bu_0$. So $\lambda_0 > 0$, then we have $u_0 = Au_0 + \lambda_0 Bu_0 \geq \lambda_0 \phi$. Let $\lambda^* = \sup \{ \lambda : u_0(s) \geq \lambda \phi(s), s \in [a, b] \}$, then $\lambda^* \in [\lambda_0, +\infty)$ and $u_0(s) \geq \lambda^*$ for $s \in [a, b]$. So the inequality

$$\lambda^* \leq u_0(s) \leq \|u_0\| = r \leq \left(\sigma \delta \int_a^b G(s, s) g(s) ds \right)^{1/(1-p)}, \quad \forall s \in [a, b]. \quad (3.5)$$

Then if $t \in [a, b]$, we have

$$\begin{aligned} u_0(t) &= \int_0^1 G(t, s) g(s) F(s, u_0(s)) ds + \lambda_0 \phi(t) \\ &\geq \int_a^b G(t, s) g(s) F(s, u_0(s)) ds + \lambda_0 \\ &\geq \sigma \int_a^b G(s, s) g(s) \delta (u_0(s))^p ds + \lambda_0 \\ &\geq \sigma \delta (\lambda^*)^p \int_a^b G(s, s) g(s) ds + \lambda_0 \geq \lambda^* + \lambda_0 \end{aligned} \quad (3.6)$$

which contradicts the definition of λ^* , hence (3.4) holds, by Lemma 2.2

$$i(A, K \cap \partial\Omega_r, K) = 0. \quad (3.7)$$

In view of (H_2) , there exists $\eta > 0$ and $N > 0$ such that $F(t, u) \leq \eta u^q$, for $u \geq C_0$, then

$$0 \leq F(t, u) \leq M + \eta u^q, \quad \text{for } u \in [0, +\infty), \quad (3.8)$$

where $M = \max \{ F(t, u) : 0 \leq u \leq C_0 \}$.

Choose sufficiently large $R > 0$ such that

$$MR^{-1} + \eta R^{q-1} < \left(\theta \int_0^1 g(s) ds \right)^{-1}. \quad (3.9)$$

We will prove that

$$Au \neq \lambda u, \quad \text{for } u \in K \cap \partial\Omega_R, \lambda \geq 1. \quad (3.10)$$

In fact, if not, there exist $u_1 \in K \cap \partial\Omega_R$ and $\lambda_1 \geq 1$ such that $Au_1 = \lambda_1 u_1$, then if $t \in [0, 1]$, we have

$$\begin{aligned} u_1(t) &\leq \lambda_1 u_1(t) = \int_0^1 G(t, s)g(s)F(s, u_1(s))ds \\ &\leq \int_0^1 G(t, s)g(s)(M + \eta u_1(s)^q)ds \\ &\leq (M + \eta R^q) \int_0^1 G(t, s)g(s)ds \\ &\leq (M + \eta R^q)\theta \int_0^1 g(s)ds. \end{aligned} \quad (3.11)$$

Hence $R \leq (M + \eta R^q)\theta \int_0^1 g(s)ds$. That is,

$$R \leq (M + \eta R^q)\theta \int_0^1 g(s)ds \quad (3.12)$$

which yields

$$MR^{-1} + \eta R^{q-1} \geq \left(\theta \int_0^1 g(s)ds \right)^{-1}, \quad (3.13)$$

which is a contradiction to (3.9), so (3.10) holds. By Lemma 2.1, we have

$$i(A, K \cap \partial\Omega_R, K) = 1, \quad (3.14)$$

(3.7) and (3.14) together imply

$$i(A, K \cap (\Omega_R \setminus \overline{\Omega_r}), K) = i(K \cap \partial\Omega_R, K) - i(A, K \cap \partial\Omega_r, K) = 0 - 1 = -1. \quad (3.15)$$

Consequently, according to [2, Theorem 2.3.2], A has a fixed point $u^* \in K \cap (\Omega_R \setminus \Omega_r)$, so BVP(1.1) has at least one positive solution u^* . This completes the proof. $\square$

THEOREM 3.2. Assume that (A_1) – (A_3) hold, if

(H₃) there exists $k \in (1, +\infty)$ such that $0 < \limsup_{u \rightarrow +\infty} \min_{t \in [0, 1]} (F(t, u)/u^k) \leq +\infty$,

(H₄) there exists $l \in (1, +\infty)$ such that $0 \leq \liminf_{u \rightarrow 0^+} \max_{t \in [0, 1]} (F(t, u)/u^l) < +\infty$,

then BVP(1.1) has at least one positive solution.

Proof. By virtue of (H₃), there exist $\xi > 0$ and $C_0 > 0$ such that $F(t, u) \geq \xi u^k$, for $u \geq C_0$. Choose

$$R > \max \left\{ N\sigma^{-1}, \left[\min_{t \in [a, b]} \int_a^b G(t, s)g(s)ds \xi \right]^{(-1/k-1)} \right\}. \quad (3.16)$$

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Without loss of generality, we assume that $u \neq Au$ for $u \in K \cap \partial\Omega_R$, otherwise the conclusion holds. Define $B : C[0, 1] \rightarrow C[0, 1]$ by

$$Bu = \psi, \quad \forall u \in C[0, 1], \quad (3.17)$$

where $\psi(t) = \min_{t \in [a, b]} \int_a^b G(t, s)g(s)ds$, $\psi \in C[0, 1]$. Then it is easy to verify $B : K \cap \partial\Omega \rightarrow K$ is completely continuous and $\inf_{K \cap \partial\Omega_R} \|Bu\| > 0$, $\phi \in K \setminus \theta$ with $\|\psi\| = C$, C is a const.

We now prove that

$$u - Au \neq \lambda Bu, \quad \text{for } u \in K \cap \partial\Omega_R, \lambda \geq 0. \quad (3.18)$$

In fact, if not, there are $\lambda_1 \geq 0$ and $u_1 \in K \cap \partial\Omega_r$ such that $u_1 - Au_1 = \lambda_1 Bu_1$. So $\lambda_1 > 0$, then we have $u_1 = Au_1 + \lambda_1 Bu_1 \geq \lambda_1 \psi$. Let $\lambda^* = \sup\{\lambda : u_1(s) \geq \lambda \psi(s), s \in [a, b]\}$, then $\lambda^* \in [\lambda_1, +\infty]$ and $u_1(s) \geq \lambda^* \psi$ for $s \in [a, b]$. Then if $t \in [a, b]$, we have

$$\begin{aligned} u_1(t) &= \int_0^1 G(t, s)g(s)F(s, u_1(s))ds + \lambda_1 \psi(t) \\ &\geq \int_a^b G(t, s)g(s)F(s, u_1(s))ds + \lambda_1 \psi \\ &\geq \int_a^b G(t, s)g(s)\xi(u_1(s))^p ds + \lambda_1 \psi \\ &\geq \xi(\lambda^* \psi)^k \min_{t \in [a, b]} \int_a^b G(t, s)g(s)ds + \lambda_1 \psi \geq (\lambda^* + \lambda_1) \psi, \end{aligned} \quad (3.19)$$

which contradicts the definition of λ^* , hence (3.18) holds, by Lemma 2.2

$$i(A, K \cap \partial\Omega_R, K) = 0. \quad (3.20)$$

By virtue of (H_4) , there are $\mu > 0$ and $\epsilon > 0$ such that $0 \leq F(t, u) \leq \mu u^l$ for $0 \leq u \leq \epsilon$. Take

$$0 \leq r \leq \min \left\{ \epsilon, R, \left(\mu \max_{t \in [0, 1]} \int_0^1 G(t, s)g(s)ds \right)^{-1/(l-1)} \right\}, \quad (3.21)$$

then we now prove that

$$Au \neq \lambda u, \quad \text{for } u \in K \cap \partial\Omega_r, \mu \geq 1. \quad (3.22)$$

In fact, if it is not true, there exist $u_0 \in K \cap \partial\Omega_r$ and $\lambda_0 \geq 1$ such that $Au_0 = \lambda_0 u_0$. Then if $t \in [0, 1]$,

$$\begin{aligned} u_0(t) &\leq \lambda_0 u_0(t) = \int_0^1 G(t, s)g(s)F(s, u_0(s))ds \\ &\leq \int_0^1 G(t, s)g(s)\mu(u_0(s))^l ds \\ &\leq r^l \left(\mu \max_{t \in [0, 1]} \int_0^1 G(t, s)g(s)ds \right), \end{aligned} \quad (3.23)$$

that is,

$$r \leq \mu r^l \max_{t \in [0,1]} G(t,s)g(s)ds \quad (3.24)$$

which contradicts the definition of r , so (3.22) holds. According to Lemma 2.1,

$$i(A, K \cap \partial\Omega_r, K) = 1, \quad (3.25)$$

(3.20) and (3.25) together imply

$$i(A, K \cap (\Omega_R \setminus \overline{\Omega_r}), K) = i(K \cap \partial\Omega_R, K) - i(A, K \cap \partial\Omega_r, K) = 0 - 1 = -1. \quad (3.26)$$

Consequently, according to [2, Theorem 2.3.2], A has a fixed point $u^* \in K \cap (\Omega_R \setminus \Omega_r)$, so BVP(1.1) has at least one positive solution u^* . This completes the proof. $\square$

THEOREM 3.3. *Assume that (A_1) – (A_3) hold, (H_1) and (H_3) are satisfied. In addition if (H_5) there exists T_1 , such that*

$$T_1 > \max_{(t,u) \in [0,1] \times [\sigma T_1, T_1]} F(t,u) \max_{t \in [0,1]} \int_0^1 G(t,s)g(s)ds, \quad (3.27)$$

then BVP(1.1) has at least two positive solutions.

Proof. By the proof of Theorems 3.1 and 3.2, there exist $0 < r < T_1 < R$ such that (3.7) and (3.20) hold, respectively.

We now prove that

$$Au \neq \lambda u, \quad \text{for } u \in K \cap \partial\Omega_{T_1}, \lambda \geq 1. \quad (3.28)$$

Otherwise, there are $u_2 \in K \cap \partial\Omega_{T_1}$ and $\lambda_2 \geq 1$ such that $Au_2 = \lambda u_2$. Then if $t \in [0,1]$,

$$\begin{aligned} u_2(t) &\leq \lambda_2 u_2(t) = \int_0^1 G(t,s)g(s)F(s, u_2(s))ds \\ &\leq \max_{(t,u) \in [0,1] \times [\sigma T_1, T_1]} F(t,u) \int_0^1 G(t,s)g(s)ds \\ &\leq \max_{(t,u) \in [0,1] \times [\sigma T_1, T_1]} F(t,u) \max_{t \in [0,1]} \int_0^1 G(t,s)g(s)ds, \end{aligned} \quad (3.29)$$

that is,

$$T_1 \leq \max_{(t,u) \in [0,1] \times [\sigma T_1, T_1]} F(t,u) \max_{t \in [0,1]} \int_0^1 G(t,s)g(s)ds, \quad (3.30)$$

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which contradicts with (H_5) , hence (3.28) holds. According to Lemma 2.1,

$$i(A, K \cap \partial\Omega_{T_1}, K) = 1, \quad (3.31)$$

(3.7), (3.20), and (3.31) together imply

$$\begin{aligned} i(A, K \cap (\Omega_R \setminus \overline{\Omega}_{T_1}), K) &= i(K \cap \partial\Omega_R, K) - i(A, K \cap \partial\Omega_{T_1}, K) = 0 - 1 = -1, \\ i(A, K \cap (\Omega_{T_1} \setminus \overline{\Omega}_r), K) &= i(K \cap \partial\Omega_{T_1}, K) - i(A, K \cap \partial\Omega_r, K) = 1 - 0 = 1. \end{aligned} \quad (3.32)$$

Then according to [2, Theorem 2.3.2], A has two fixed points $u_1^* \in K \cap (\Omega_R \setminus \overline{\Omega}_{T_1})$ and $u_2^* \in K \cap (\Omega_{T_1} \setminus \overline{\Omega}_r)$, so BVP(1.1) has at least two positive solutions u_1^*, u_2^* . This completes the proof. $\square$

THEOREM 3.4. *Assume that (A_1) – (A_3) hold, (H_2) and (H_4) are satisfied. In addition if (H_6) there exists T_2 , such that*

$$0 < T_2 < \min_{(t,u) \in [a,b] \times [\sigma T_2, T_2]} F(t, u) \min_{t \in [a,b]} \int_a^b G(t, s) g(s) ds, \quad (3.33)$$

then BVP(1.1) has at least two positive solutions.

Proof. By the proof of Theorems 3.1 and 3.2, there exist $0 < r < T_2 < R$ such that (3.14) and (3.25) hold, respectively.

Define $B : C[0, 1] \rightarrow C[0, 1]$ by

$$Bu = u, \quad \forall u \in C[0, 1]. \quad (3.34)$$

Then it is easy to verify $B : K \cap \partial\Omega \rightarrow K$ is completely continuous and $\inf_{K \cap \partial\Omega_r} \|Bu\| > 0$.

We now prove that

$$u - Au \neq \lambda Bu, \quad \text{for } u \in K \cap \partial\Omega_{T_2}, \lambda \geq 0. \quad (3.35)$$

Otherwise, there are $u_3 \in K \cap \partial\Omega_{T_2}$ and $\lambda_3 \geq 1$ such that $u_3 - Au_3 = \lambda_3 Bu_3$. Then if $t \in [a, b]$,

$$\begin{aligned} u_3(t) &= Au_3(t) + \lambda_3 Bu_3 \\ &\geq \int_a^b G(t, s) g(s) F(s, u_3(s)) ds \\ &\geq \min_{(t,u) \in [a,b] \times [\sigma T_2, T_2]} F(t, u) \int_a^b G(t, s) g(s) ds \\ &\geq \min_{(t,u) \in [a,b] \times [\sigma T_2, T_2]} F(t, u) \min_{t \in [a,b]} \int_a^b G(t, s) g(s) ds, \end{aligned} \quad (3.36)$$

that is,

$$T_2 > \min_{(t,u) \in [a,b] \times [\sigma T_2, T_2]} F(t, u) \min_{t \in [a,b]} \int_a^b G(t, s) g(s) ds, \quad (3.37)$$

which contradicts with (H_6) , hence (3.35) holds. According to Lemma 2.2,

$$i(A, K \cap \partial\Omega_{T_2}, K) = 0, \quad (3.38)$$

(3.14), (3.25), and (3.38) together imply

$$\begin{aligned} i(A, K \cap (\Omega_R \setminus \overline{\Omega_{T_2}}), K) &= i(K \cap \partial\Omega_R, K) - i(A, K \cap \partial\Omega_{T_1}, K) = 1 - 0 = 1, \\ i(A, K \cap (\Omega_{T_2} \setminus \overline{\Omega_r}), K) &= i(K \cap \partial\Omega_2, K) - i(A, K \cap \partial\Omega_r, K) = 0 - 1 = -1. \end{aligned} \quad (3.39)$$

Then according to [2, Theorem 2.3.2], A has two fixed points $u_1^* \in K \cap (\Omega_R \setminus \Omega_{T_2})$ and $u_2^* \in K \cap (\Omega_{T_2} \setminus \Omega_r)$, so BVP(1.1) has at least two positive solutions u_1^*, u_2^* . This completes the proof. $\square$

THEOREM 3.5. *Assume that (A_1) – (A_3) hold, $T_1 < T_2$, (H_1) , (H_2) , (H_5) , and (H_6) are satisfied. Then BVP(1.1) admits at least three positive solutions.*

Proof. By the proof of Theorem 3.1, there exist $0 < r < T_1$ and $R > T_2$ such that (3.7) and (3.14) hold, respectively. By Theorems 3.3 and 3.4, (3.31) and (3.38) are valid. Therefore BVP(1.1) has at least three positive solutions $u_1^* \in K \cap (\Omega_R \setminus \overline{\Omega_{T_2}})$, $u_2^* \in K \cap (\Omega_{T_2} \setminus \overline{\Omega_{T_1}})$, $u_3^* \in K \cap (\Omega_{T_1} \setminus \overline{\Omega_r})$. This completes the proof. $\square$

Similar to the proof of Theorem 3.5, we can get the following theorem.

THEOREM 3.6. *Assume that (A_1) – (A_3) hold, $T_1 > T_2$, (H_3) , (H_4) , (H_5) , and (H_6) are satisfied. Then BVP(1.1) admits at least three positive solutions.*

Remark 3.7. In fact, if $T_1 = T_2 = T$, the conditions of Theorems 3.5 and 3.6 both cannot ensure that BVP(1.1) has at least three positive solutions or even one positive solution. The reason is that the conditions (H_5) and (H_6) imply that the inequalities

$$\begin{aligned} Au &\not\leq u, \quad \text{for } u \in K \cap \Omega_T, \\ Au &\not\geq u, \quad \text{for } u \in K \cap \Omega_T, \end{aligned} \quad (3.40)$$

hold, respectively, however, the latter two contradict each other.

Remark 3.8. In this paper, if $p(t) = 1$, $F(t, u) = f(u)$, all the theorems above still hold and the results are new. Here the function $f(u)$ and the boundary conditions are more general than in [1, 6, 7] where $f(u)$ only satisfies $\lim_{u \rightarrow 0^+} f(u)/u = 0$ (or ∞), $\lim_{u \rightarrow \infty} f(u)/u = \infty$ (or 0) and only the cases $\beta = 0$, $\delta = 0$ are considered. In addition, our method is different from those methods in [1, 6, 7].

Remark 3.9. In the proof of theorems, one of the key steps is to find the operator B . We note that it is more general than the ones in [6, 8–11]. We think not only about the superlinear, sublinear cases but also the general cases. Hence, our results improve and generalize those in some well-known papers.

4. Examples

In this section, we provide some examples to illustrate the validity of the results established in Section 2.

Example 4.1. Consider the following boundary value problem:

$$u'' + \frac{1}{\sqrt{t(1-t)}} \left[\frac{1}{4}(1+t)u^{3/2}(t) + u^{1/2}(t) \right], \quad t \in (0, 1), \quad (4.1)$$

$$u(0) = u(1) = 0.$$

Conclusion. BVP (4.1) has at least two positive solutions.

Proof. Let

$$p(t) = 1, \quad g(t) = \frac{1}{\sqrt{t(1-t)}}, \quad F(t, u) = \frac{1}{4}(1+t)u^{3/2}(t) + u^{1/2}(t), \quad (4.2)$$

and we choose $[a, b] = [1/4, 3/4] \subset (0, 1)$.

It is easy to verify that the conditions (A_1) – (A_3) of Theorem 3.3 are satisfied. In term of (2.1) and (2.2), the corresponding Green function is

$$G(t, s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases} \quad (4.3)$$

$$\sigma = \frac{1}{4}, \quad G(s, s) = s(1-s), \quad \int_0^1 \frac{dt}{\sqrt{t(1-t)}} = \pi.$$

We first verify the conditions (H_1) and (H_3) . In fact let $p = 1/2$, $k = 3/2$, we have

$$\lim_{u \rightarrow 0^+} \min_{t \in [0, 1]} \frac{F(t, u)}{u^p} = \lim_{u \rightarrow 0^+} \min_{t \in [0, 1]} \frac{(1/4)(1+t)u^{3/2}(t) + u(t)}{u^{1/2}} = \frac{1}{4}, \quad (4.4)$$

$$\lim_{u \rightarrow +\infty} \min_{t \in [0, 1]} \frac{F(t, u)}{u^k} = \lim_{u \rightarrow +\infty} \min_{t \in [0, 1]} \frac{(1/4)(1+t)u^{3/2}(t) + u^{1/2}(t)}{u^{3/2}} = 1.$$

Choosing $T_1 = 4$, we have

$$\max_{(t, u) \in [0, 1] \times [1, 4]} F(t, u) \max_{t \in [0, 1]} \int_0^1 G(s, s) g(s) ds \quad (4.5)$$

$$= \max_{(t, u) \in [0, 1] \times [1, 4]} \left[\frac{1}{4}(1+t)u^{3/2}(t) + u^{1/2}(t) \right] \frac{1}{8}\pi = 6 \cdot \frac{1}{8}\pi = \frac{3}{4}\pi < 4.$$

So condition (H_5) holds. Consequently by Theorem 3.3, BVP (4.1) has at least two positive solutions. $\square$

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