EXTINCTION AND PERMANENCE OF TWO-NUTRIENT AND ONE-MICROORGANISM CHEMOSTAT MODEL WITH PULSED INPUT

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A chemostat model with periodically pulsed input is considered. By using the Floquet theorem, we find that the microorganism eradication periodic solution $(u_1^*(t), v_1^*(t), 0)$ is globally asymptotically stable if the impulsive period T is more than a critical value. At the same time we can find that the nutrient and microorganism are permanent if the impulsive period T is less than the critical value.

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1. Introduction

A chemostat is a piece of laboratory apparatus frequently used for culturing microorganisms. It can be used for representing all kinds of microorganism systems such as lake, waste-water treatment and reaches for commercial production of the advantage of being easily implementable in a laboratory, and hence the model has been studied by more and more people. In its simplest form, the system approximates conditions for plankton growth in lakes, where the limiting nutrients such as silica and phosphate are supplied from streams draining the watershed. Chemostat with period inputs are studied in [5, 6, 14], those with periodic washout rate in [3, 9] and those with periodic input and washout in [11]. We all know that nutrients are input into lakes and lakes are washed out when rain is falling. In fact, raining is not continuous. It occurs seasonally or in regular pulses. Thus, it is natural to describe this case in impulsive differential equations.

Impulsive differential equations are suitable for the mathematical simulation of evolutionary process in which the parameters undergo relatively long periods of smooth variation followed by a short-term rapid change (i.e., jumps) in their values. Recently, equations of this kind are found in almost every domain of applied science. Numerous examples are given in Bainov's and his collaborator's book [1, 8]. Some impulsive differential equations have been recently introduced in population dynamics in relation to impulsive birth [12, 15], impulsive vaccination [4, 13], chemotherapeutic treatment of disease [7, 10], and population ecology [2].

2 A two-nutrient and one-microorganism chemostat model

In this paper, we consider the dynamics of two-nutrient and one-microorganism chemostat with pulsed input:

$$\begin{aligned}\dot{S} &= -QS - \frac{\mu_1 x S}{\delta_1}, \\ \dot{R} &= -RQ - \frac{\mu_2 R x}{\delta_2}, \quad \bar{t} \neq nT, \\ \dot{x} &= \mu_1 x S + \mu_2 x R - Qx, \\ \Delta S &= S^0 Q, \quad \Delta R = R^0 Q, \quad \bar{t} = nT,\end{aligned}\tag{1.1}$$

where $S(t)$ and $R(t)$ denote the concentration of nutrient, and $x(t)$ denotes the concentration of the microorganism at time t . S^0 and R^0 represent that input concentration of the nutrient. Q ($0 < Q < 1$) is referred to as the dilution rate. μ_1 and μ_2 denote the predation constants of predator, δ_1 and δ_2 show yield term, and T is the period of pulse.

The aim of this work is to study the dynamical behaviors of two nutrients and one microorganism with pulsed input, and investigate how the impulsive perturbation affects the dynamical behaviors of unforced continuous system.

The variables in the above system may be rescaled by measuring $u = S/S^0$, $v = \delta_2/(\delta_1 S^0)R$, $w = x/\delta_1 S^0$, and $\bar{t} = Qt$, then we have the following system:

$$\begin{aligned}\dot{u} &= -u - Auw \\ \dot{v} &= -v - Bvw, \quad t \neq nT, \\ \dot{w} &= Auw + Bvw - w, \\ \Delta u &= Q, \quad \Delta v = p, \quad t = nT,\end{aligned}\tag{1.2}$$

where $p = \delta_2 QR^0/\delta_1 S^0$, $A = \mu_1 S^0/Q$, and $B = \mu_2 \delta_1 S^0/Q\delta_2$.

This paper is arranged as follows. In Section 2, we introduce some useful notations and definitions. In Section 3, by using Floquet theorem for the impulsive equation, small-amplitude perturbation skills and techniques of comparison, we get the local stability and global asymptotic stability of the microorganism eradication periodic solution. In Section 4, we show that the system is permanent if the impulsive period is less than some critical value. In Section 5, we give a brief discussion.

2. Preliminaries

In this section, we will give some definitions, notations, and some lemmas which will be useful for our main results.

Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_+^3 = \{x \in \mathbb{R}^3 : x > 0\}$. Denote $f = (f_1, f_2, f_3)$ the map defined by the right-hand side of the first three equations of system (1.2). Let $V : \mathbb{R}_+ \times \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$, then V is said to belong to class V_0 if

- (i) V is continuous in $(nT, (n+1)T) \times \mathbb{R}_+^3$ and for each $x \in \mathbb{R}_+^3$, $n \in \mathbb{Z}_+$, $\lim_{(t,y) \rightarrow (nT,x)} V(t,y) = V(nT^+, x)$ exists;
- (ii) V is locally Lipschitzian in x .

Definition 2.1. Let $V \in V_0$, then for $(t, x) \in (nT, (n+1)T] \times \mathbb{R}_+^3$, the upper-right derivative of $V(t, x)$ with respect to the impulsive differential system (1.2) is defined as

$$D^+V(t, x) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(t+h, x+f(t, x)) - V(t, x)]. \quad (2.1)$$

Definition 2.2. System (1.2) is said to be permanent if there are constants $m > 0$, $M > 0$ and such that each positive solution $(u(t), v(t), w(t))$ of the system (1.2) satisfies $m \leq u(t) \leq M$, $m \leq v(t) \leq M$, $m \leq w(t) \leq M$, for all $t \geq 0$. If all species of the system are permanent, then the system is called permanent.

LEMMA 2.3. Suppose $\omega(t) = (u(t), v(t), w(t))$ is a solution of (1.2) subject to $\omega(0^+) \geq 0$ for all $t \geq 0$, and further suppose $\omega(t) > 0$, $t \geq 0$ if $\omega(0^+) > 0$.

LEMMA 2.4. Let $V : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}_+$ and $V \in V_0$. Assume that

$$\begin{aligned} D^+V(t, \omega) &\leq g(t, V(t, \omega)), \quad t \neq nT, \\ V(t, \omega(t^+)) &\leq \psi_n(V(t, \omega(t))), \quad t = nT, \end{aligned} \quad (2.2)$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is continuous in $(nT, (n+1)T] \times \mathbb{R}_+$ and for $y \in \mathbb{R}_+$, $n \in \mathbb{Z}_+$, $\lim_{(t,y) \rightarrow (nT^+, y)} g(t, y)$ exists, $\psi_n : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{aligned} \frac{du}{dt} &= g(t, u), \quad t \neq nT, \\ u(t^+) &= \psi_n(u_n(t)), \quad t = nT, \\ u(0^+) &= u_0 \end{aligned} \quad (2.3)$$

existing on $[0, \infty)$. Then $V(0^+, \omega_0) \leq u_0$ implies that $V(t, \omega(t)) \leq r(t)$, $t \geq 0$, where $\omega(t)$ is any solution of system (1.2).

Consider the following system:

$$\begin{aligned} \dot{u} &= -u, & \dot{v} &= -v, & t &\neq nT, \\ \Delta u &= Q, & \Delta w &= p, & t &= nT. \end{aligned} \quad (2.4)$$

LEMMA 2.5. System (2.4) has a positive periodic solution $(u^*(t), v^*(t))$ and for every solution $(u(t), v(t))$ of (2.4), $|u(t) - u^*(t)| \rightarrow 0$, $|v(t) - v^*(t)| \rightarrow 0$ as $t \rightarrow \infty$, where $u^*(t) = Qe^{-(t-nT)}/(1 - e^{-T})$, $v^*(t) = pe^{-(t-nT)}/(1 - e^{-T})$.

3. Extinction

In this section, we study the stability of the microorganism-free periodic solution as a solution of the full system (1.2). Firstly, we present the Floquet theory for the linear T -periodic impulsive equation:

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, \quad t \neq \tau_k, \quad t \in \mathbb{R}, \\ \Delta x &= B_k x, \quad t = \Delta_k, \quad k \in \mathbb{Z}. \end{aligned} \quad (3.1)$$

Then we introduce the following conditions:

- (H₁) $A(\cdot) \in PC(\mathbb{R}, \mathbb{C}^{n \times n})$ and $A(t+T) = A(t)$ ($t \in \mathbb{R}$),
- (H₂) $B \in \mathbb{C}^{n \times n}$. $\det(E + B_k) \neq 0$, $\tau_k < \tau_{k+1}$ ($k \in \mathbb{Z}$),
- (H₃) there exists a $q \in \mathbb{N}$ such that

$$B_{k+q} = B_k, \quad \tau_{k+q} = \tau_k + T, \quad k \in \mathbb{Z}. \quad (3.2)$$

Let $\Phi(t)$ be a fundamental matrix of (3.1), then there exists a unique nonsingular matrix $M \in \mathbb{C}^{n \times n}$ such that

$$\Phi(t+T) = \Phi(t)M, \quad t \in \mathbb{R}. \quad (3.3)$$

By equality (3.3) there corresponds to the fundamental matrix $\Phi(t)$ the constant matrix M which we call the monodromy matrix of (3.1) (corresponding to the fundamental matrix of $\Phi(t)$).

All monodromy matrices of (3.1) are similar and have the same eigenvalues. The eigenvalues $\mu_1, \dots, \mu_n$ of the monodromy matrices are called the Floquet multipliers of (3.1).

LEMMA 3.1 (Floquet theory). *Let conditions (H₁)–(H₃) hold. Then the linear T -periodic impulsive (3.1) is*

- (1) *stable if and only if all multipliers μ_j ($j = 1, \dots, n$) of (3.1) satisfy the inequality $|\mu_j| \leq 1$ and, moreover, to those μ_j for which $|\mu_j| = 1$, there correspond simple elementary divisors;*
- (2) *asymptotically stable if and only if all monodromy μ_j ($j = 1, \dots, n$) of (3.1) satisfy the inequality $|\mu_j| < 1$;*
- (3) *unstable if $|\mu_j| > 1$ for some $j = 1, \dots, n$.*

THEOREM 3.2. *Let $\omega(t) = (u(t), v(t), w(t))$ be any solution of system (1.2), then $(u^*(t), v^*(t), 0)$ is globally asymptotically stable, provided that $T > AQ + Bp$.*

Proof. Firstly, we prove the locally stable. The local stability of the periodic solution $(u^*(t), v^*(t), 0)$ may be determined by considering the behavior of small-amplitude perturbations of the solution. Define $u(t) = u^*(t) + x_1(t)$, $v(t) = v^*(t) + x_2(t)$, $w(t) = w^*(t) + x_3(t)$, where x_1 , x_2 , and x_3 are small perturbations. Equation (1.2) can be expanded in a

Taylor series: after neglecting higher-order terms, the linearized equations are read as

$$\begin{aligned}\dot{x}_1 &= -x_1 - Au^*(t)x_3, \\ \dot{x}_2 &= -x_2 - Bv^*(t)x_3, \\ \dot{x}_3 &= (Au^*(t) + Bv^*(t) - 1)x_3.\end{aligned}\tag{3.4}$$

Let $\Phi(t)$ be the fundament matrix of (3.4), then $\Phi(t)$ must satisfy

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} -1 & 0 & -Au^*(t) \\ 0 & -1 & -Bv^*(t) \\ 0 & 0 & Au^*(t) + Bv^*(t) \end{pmatrix} \Phi(t).\tag{3.5}$$

Thus, linearization of (1.2) from the fourth to the sixth becomes

$$\begin{pmatrix} x_1(nT^+) \\ x_2(nT^+) \\ x_3(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(nT) \\ x_2(nT) \\ x_3(nT) \end{pmatrix}.\tag{3.6}$$

Thus, the monodromy matrix of (3.4) is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \Phi(T).\tag{3.7}$$

From (3.5) we have $\Phi(T) = \Phi(0) \exp(\int_0^T A dt) \stackrel{\Delta}{=} \Phi(0) \exp(\bar{A})$, where $\Phi(0)$ is the identity matrix. Let λ_1, λ_2 , and λ_3 be eigenvalues of matrix $\bar{A}$, then $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 = AQ + Bp - T < 0$. Therefore, all eigenvalues of M , namely, $e^{\lambda_1}, e^{\lambda_2}$, and e^{λ_3} have absolute values less than one if and only if $T > AQ + Bp$. According to Floquet theorem (Lemma 3.1), the $(u^*(t), v^*(t), 0)$ is locally stable.

In the following, we prove the global attractivity. Choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\delta = AQ + \varepsilon_1 T + Bp + \varepsilon_2 T - T < 0.\tag{3.8}$$

Noting that $\dot{u} \leq -u, \dot{v} \leq -v$ and considering the following impulsive differential equations:

$$\begin{aligned}\dot{y}_1(t) &= -y_1(t), \quad t \neq nT, \\ \Delta y_1 &= Q, \quad t = nT, \\ \dot{y}_2(t) &= -y_2(t), \quad t \neq nT, \\ \Delta y_2 &= p, \quad t = nT,\end{aligned}\tag{3.9}$$

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we have $u(t) \leq y_1(t)$, $v(t) \leq y_2(t)$, and $y_1(t) \rightarrow y_1^*(t)$, $y_2(t) \rightarrow y_2^*(t)$ as $t \rightarrow \infty$, where $y_1^*(t)$, $y_2^*(t)$ is the periodic solution. Then

$$\begin{aligned} u(t) &\leq y_1(t) < y_1^*(t) + \varepsilon_1, \\ v(t) &\leq y_2(t) < y_2^*(t) + \varepsilon_2, \end{aligned} \quad (3.10)$$

for t large enough. For simplification, we may assume (3.10) holds for all $t \geq 0$. From (1.2), we can get

$$\dot{w} \leq w(A(u^*(t) + \varepsilon_1) + B(v^*(t) + \varepsilon_2) - 1), \quad (3.11)$$

integrate (3.11) on $(nT, (n+1)T]$ which yields

$$w(n+1)T \leq w(nT^+) \exp \int_{nT}^{(n+1)T} [A(u^*(t) + \varepsilon_1) + B(v^*(t) + \varepsilon_2) - 1] dt, \quad (3.12)$$

therefore, we have $w(n+1)T \leq w(nT^+) \exp(\delta)$. Thus $w(nT) \leq w(0^+) \exp(n\delta)$ and $w(nT) \rightarrow 0$ as $n \rightarrow \infty$, since $0 < w(t) < w(nT)$. Therefore $w(t) \rightarrow 0$ as $t \rightarrow \infty$.

Next we prove that $u(t) \rightarrow u^*(t)$, $v(t) \rightarrow v^*(t)$ as $t \rightarrow \infty$, if $\lim_{t \rightarrow \infty} w(t) = 0$. For $0 < \varepsilon_3 < \min\{1/A, 1/B\}$, there exists a $\hat{T} > 0$ such that $0 < w(t) < \varepsilon_3$, for all $t \geq \hat{T}$. Then we have

$$-u(1 + A\varepsilon_3) \leq \dot{u} \leq -u. \quad (3.13)$$

By Lemma 2.4 we obtain $y_3(t) \leq u(t) \leq y_1(t)$, and $y_1(t) \rightarrow u^*(t)$ and $y_3(t) \rightarrow y_3^*(t)$ as $t \rightarrow \infty$, where $y_3(t)$ is the solution of

$$\begin{aligned} \dot{y}_3 &= -y_3(1 + B\varepsilon_3), & t \neq nT, \\ \Delta y_3 &= Q, & t = nT, \end{aligned} \quad (3.14)$$

and $y_3^*(t) = Q \exp(-1 - B\varepsilon_3)(t - nT) / (1 - \exp(-1 - \varepsilon_3 B)T)$, $nT < t \leq (n+1)T$, therefore $y_3^*(t) - \varepsilon_4 < u(t) < u^*(t) + \varepsilon_4$, $\varepsilon_4 > 0$, for t large enough. Let $\varepsilon_3 \rightarrow 0$, we get $y_3^*(t) \rightarrow u^*(t)$. Hence $u(t) \rightarrow u^*(t)$ as $t \rightarrow \infty$. By the same method, we can prove $v(t) \rightarrow v^*(t)$, as $t \rightarrow \infty$. This completes the proof. $\square$

4. Permanence

First, we show that all solutions of (1.2) are uniformly ultimately bounded.

THEOREM 4.1. *There exists a constant $M > 0$ such that $u(t) \leq M$, $v(t) \leq M$, $w(t) \leq M$ for each positive solution $\omega(t) = (u(t), v(t), w(t))$ of (1.2) with t large enough.*

Proof. Define a function $V(t, \omega(t)) = u + v + w$, then $V(t, \omega(t)) \in V_0$ and the upper-right derivative of $V(t, \omega(t))$ along a solution of (1.2) is described as

$$\begin{aligned} D^+ V(t, \omega(t)) &\leq -V, & t \neq nT, \\ \Delta V &= p + Q, & t = nT, \end{aligned} \quad (4.1)$$

we obtain

$$V(t) \leq V(0^+)e^{-t} + (p+Q)\frac{e^{-(t-T)}}{1-e^{-T}} + \frac{(p+Q)e^T}{e^T-1} \rightarrow \frac{(p+Q)e^T}{e^T-1}, \quad t \rightarrow \infty. \quad (4.2)$$

By the definition of $V(t, \omega(t))$, we obtain that each positive solution of (1.2) is uniformly ultimately bounded. $\square$

Next we give the conditions of permanence.

THEOREM 4.2. *System (1.2) is permanent if $T < AQ + Bp$.*

Proof. Suppose $\omega(t) = (u(t), v(t), w(t))$ is a solution of (1.2) with positive initial value. From Theorem 4.1 we may assume $u(t) \leq M$, $v(t) \leq M$, $w(t) \leq M$, $t \geq 0$, and $M > 0$. From system (1.2), let $m_1 = Qe^{-T}/(1-e^{-T}) - \varepsilon_5$, $\varepsilon_5 > 0$, and $m_2 = pe^{-T}/(1-e^{-T}) - \varepsilon_6$, $\varepsilon_6 > 0$. According to Lemma 2.4, we have $u(t) \geq m_1$ and $v(t) \geq m_2$ for t large enough.

In the following, we want to find m_3 such that $w(t) \geq m_3$ for t large enough. We will do it in the following two steps for convenience.

Step 1. Let $m_3 > 0$ and $\varepsilon_7, \varepsilon_8$ be small enough, such that

$$\rho = AQ + Bp - \varepsilon_7T - \varepsilon_8T - T > 0. \quad (4.3)$$

We will prove $w(t) < m_3$ cannot hold for all $t \geq 0$. Otherwise,

$$\frac{du}{dt} \geq -(1 + Am_3)u, \quad (4.4)$$

from Lemma 2.4, we have $u(t) \geq z(t)$ and $z(t) \rightarrow z^*(t)$, $t \rightarrow \infty$, where $z(t)$ is the solution of

$$\begin{aligned} \frac{dz}{dt} &= -(1 + Am_3)z, & t \neq nT, \\ \Delta z &= Q, & t = nT, \end{aligned} \quad (4.5)$$

$$z^*(t) = \frac{Q \exp(-1 - Am_3)(t - nT)}{1 - \exp(-1 - Am_3)T}, \quad t \in (nT, (n+1)T].$$

Therefore, there exists a $n_1 > 0$, $t > n_1T$ such that $w(t) \geq z(t) > z^*(t) - \varepsilon_7$ at the same time, we have

$$v(t) \geq -(1 + Bm_3)v(t); \quad (4.6)$$

similarly, we have $v(t) \geq z_1(t)$, $z_1(t) \rightarrow z_1^*(t)$ ($t \rightarrow \infty$), where $z_1^*(t)$ is the solution of

$$\begin{aligned} \dot{z}_1(t) &= -(1 + Bm_3)z_1, & t \neq nT, \\ \Delta z_1 &= p, & t = nT, \end{aligned} \quad (4.7)$$

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and $z_1^*(t) = p \exp(-1 - Bm_3)(t - nT)/(1 - \exp(-1 - Bm_3)T)$, $t \in (nT, (n+1)T]$. For $n > n_2 > n_1$, we have

$$\frac{dw}{dt} \geq [A(z^*(t) - \varepsilon_7) + B(z_1^*(t) - \varepsilon_8) - 1]w. \quad (4.8)$$

Integrating (4.8) on $t \in (nT, (n+1)T]$, $n > n_2$, we obtain

$$w(n+1)T \geq w(nT) \exp \int_{nT}^{(n+1)T} [A(z^*(t) - \varepsilon_7) + B(z_1^*(t) - \varepsilon_8) - 1] dt; \quad (4.9)$$

therefore,

$$w((n+1)T) \geq w(nT) \exp(\rho). \quad (4.10)$$

Then $w(N_1 + k)T \geq w(N_1 T) \exp(k\rho) \rightarrow \infty$ as $t \rightarrow \infty$, which is a contradiction to the boundedness of $w(t)$.

Step 2. If $w(t) \geq m_3$ for all $t > t_1$, then our aim is obtained. Otherwise, $w(t) \leq m_3$ for some $t \geq t_1$. Setting $t^* = \inf_{t > t_1} \{w(t) < m_3\}$, then we have $w(t) \geq m_3$ for $t \in [t, t^*)$, and $w(t^*) = m_3$. Since $w(t)$ is continuous, suppose $t^* \in (n_1 T, (n_1 + 1)T]$, $n_1 \in \mathbb{N}$, select $n_2 \in \mathbb{N}$, $n_3 \in \mathbb{N}$, such that

$$\begin{aligned} n_2 T &> \frac{1}{-1 - Am_3} \ln \frac{\varepsilon_7}{M + Q}, \\ n_2 T &> \frac{1}{-1 - Bm_3} \ln \frac{\varepsilon_8}{M + p}, \end{aligned} \quad (4.11)$$

$$\exp(\eta(n_2 + 1)T) \exp(n_3 \rho) > 1,$$

where $\eta = -1 + Am_1 + Bm_2 < 0$. Let $T' = n_2 T + n_3 T$, we claim that there must exist a $t' \in ((n_1 + 1)T, (n_1 + 1)T + T']$ such that $w(t) \geq m_3$; otherwise $w(t) < m_3$, $t \in ((n_1 + 1)T, (n_1 + 1)T + T']$. Consider (4.5) and (4.7) with $z((n_1 + 1)T^+) = u((n_1 + 1)T^+)$, $z_1((n_1 + 1)T^+) = v((n_1 + 1)T^+)$, we have

$$\begin{aligned} z(t) &= \left(z((n_1 + 1)T^+) - \frac{Q}{1 - e^{-(1+Am_3)T}} \right) e^{-(1+Am_3)(t-(n_1+1)T)} + z^*(t), \\ z_1(t) &= \left(z_1((n_1 + 1)T^+) - \frac{p}{1 - e^{-(1+Bm_3)T}} \right) e^{-(1+Bm_3)(t-(n_1+1)T)} + z_1^*(t), \end{aligned} \quad (4.12)$$

for $t \in (nT, (n+1)T]$, $n_1 + 1 < n \leq n_1 + n_2 + n_3 + 1$. Then

$$|z(t) - z^*(t)| \leq (M + Q)e^{-(1+Am_3)(t-(n_1+1)T)} < \varepsilon_7, \quad (4.13)$$

$$|z_1(t) - z_1^*(t)| \leq (M + p)e^{-(1+Bm_3)(t-(n_1+1)T)} < \varepsilon_8, \quad (4.14)$$

and $u(t) \geq z(t) > z^*(t) - \varepsilon_7$, $v(t) \geq z_1(t) > z_1^*(t) - \varepsilon_8$, for $(n_1 + 1 + n_2)T \leq t \leq (n_1 + 1)T + T'$, which implies (4.8) holds. For $(n_1 + 1 + n_2)T \leq t \leq (n_1 + 1)T + T'$, as in Step 1, we

have $w((n_1 + n_2 + n_3 + 1)T) \geq w((n_1 + n_2 + 1)T) \exp(n_3 \rho)$; there are two possible cases for $t \in (t^*, (n_1 + 1)T]$.

Case 1. If $w(t) < m_3$ for $t \in (t^*, (n_1 + 1)T]$, then $w(t) < m_3$ for all $t \in (t^*, (n_1 + 1 + n_2)T]$; system (1.2) gives

$$\dot{w}(t) \geq w(t)(-1 + Am_1 + Bm_2) = \eta w(t), \quad (4.15)$$

integrating (4.15) on $(t^*, (n_1 + 1 + n_2)T]$, which yields $w((n_1 + n_2 + 1)T) \geq m_3 \exp(\eta(n_2 + 1))$, then $w(n_1 + 1 + n_2 + n_3)T \geq m_3 \exp(\eta(n_2 + 1)T) \exp(n_3 \rho) > m_3$, which is a contraction. Let $\bar{t} = \inf_{t > t^*} \{w(t) \geq m_3\}$, then $w(\bar{t}) = m_3$ and (4.15) holds for $t \in [t^*, \bar{t})$. Then integrating (4.15) on $[t^*, \bar{t})$ yields $w(t) \geq w(t^*) \exp(\eta(t - t^*)) \geq m_3 \exp(\eta(1 + n_2 + n_3)T) \triangleq \bar{m}_3$. For $t > \bar{t}$, the same argument can be continued. Since $w(\bar{t}) \geq m_3$, hence $w(t) \geq \bar{m}_3$ for all $t > t_1$.

Case 2. There exists a $t' \in (t^*, (n_1 + 1)T]$ such that $w(t') \geq m_3$. Let $\hat{t} = \inf_{t > t^*} \{w(t) \geq m_3\}$, then $w(t) < m_3$ for $t \in [t^*, \hat{t})$ and $w(\hat{t}) = m_3$ for $t \in [t^*, \hat{t})$, (4.15) holds, and integrating (4.15) on $[t^*, \hat{t})$, we have $w(t) \geq w(t^*) \exp(\eta(t - t^*)) \geq m_3 \exp(\eta T) > \bar{m}_3$. This process can be continued since $w(\hat{t}) \geq m_3$, and we have $w(t) \geq \bar{m}_1$ for $t > t_1$. Thus in both cases, we conclude $w(t) \geq \bar{m}_3$ for all $t > t_1$.

Incorporating into Theorem 4.1, the proof is complete. $\square$

5. Discussion

In this paper, we have investigated the model for a chemostat with two nutrients and one microorganism and periodically pulsed substrate. We have proved that microorganism eradication periodic solution $(u^*(t), v^*(t), 0)$ is globally asymptotically stable if $T > AQ + Bp$, which is showed in Figure 5.1. We can see that the variables $u(t)$, $v(t)$ oscillate in a stable periodical cycle, in contrast with $w(t)$ rapidly decrease to zero. At the same time we also have proved the system (1.2) is permanent if $T < AQ + Bp$, which is simulated in Figure 5.2. The variables $u(t)$, $v(t)$, $w(t)$ oscillate in a stable periodical cycle, respectively. So we can find that $T = AQ + Bp$ is a threshold. In fact, when the period of pulses is less than the threshold, the nutrients and microorganism coexist. If the period is more than the threshold, the microorganism will become extinct.

If we replace the pulse input in system (1.2) with continuous input, the system (1.2) becomes

$$\begin{aligned} \dot{u} &= \frac{Q}{T} - u - Auw, \\ \dot{v} &= \frac{p}{T} - v - Bvw, \\ \dot{w} &= Auw + Bvw - w, \end{aligned} \quad (5.1)$$

there also exists a microorganism eradication equilibrium for system (5.1), that is, $(Q/T, p/T, 0)$, which is globally asymptotically stable if $T > AQ + Bp$, see the appendix. The result is the same as our system (1.2), the result is simulated in Figure 5.3. We can obtain impulsive input effect that is the same as the continuous input.

(a)

(b)

(c)

Figure 5.1. Time series of the system (3.1) with pulse, $A = 5/9$, $p = 0.54$, $B = 20/9$, $Q = 0.9$, and $T = 5$.

Figure 5.2. Time series of the system (3.1) with pulse, $A = 5$, $B = 20/3$, $p = 0.6$, $Q = 0.6$, and $T = 3$.

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(a)

(b)

(c)

Figure 5.3. Time series of the system (3.1) without pulse, $A = 5/9$, $B = 20/9$, $p = 0.54$, $Q = 0.9$, and $T = 10$.

Appendix

In this appendix, we will prove that the microorganism eradication equilibrium $(\tilde{u}, \tilde{v}, 0)$ is globally stable with respect to $\text{int } \mathbb{R}_+^3$ if $T > AQ + Bp$, where $\tilde{u} = Q/T$, $\tilde{v} = p/T$, $\mathbb{R}_+^3 = \{(u, v, w) : u > 0, v > 0, w > 0\}$.

Let us denote the Jacobian matrix system (5.1) evaluated at the equilibrium point $E(Q/T, p/T, 0)$ by $J(E)$. Then

$$\begin{pmatrix} -1 & 0 & -\frac{AQ}{T} \\ 0 & -1 & -\frac{Bp}{T} \\ 0 & 0 & \frac{AQ}{T} + \frac{Bp}{T} - 1 \end{pmatrix}. \quad (\text{A.1})$$

It is easy to see that all eigenvalues of $J(E)$ are negative if and only if $T > AQ + Bp$. Therefore, $E(Q/T, p/T, 0)$ is locally stable if $T > AQ + Bp$. Let $V(u, v, w)$ be positive definite function about (u, v, w) given by

$$V(u, v, w) = c_1 \left(u - \tilde{u} - \tilde{u} \ln \frac{u}{\tilde{u}} \right) + c_2 \left(v - \tilde{v} - \tilde{v} \ln \frac{v}{\tilde{v}} \right) + c_3 w, \quad (\text{A.2})$$

where $c_i > 0$, $i = 1, 2, 3$. Then the derivative of V along with the solution of system (5.1) is $\dot{V} = -c_1(u - \tilde{u})^2/u - c_2(v - \tilde{v})^2/v + A(c_3 - c_1)w(u - \tilde{u}) + B(c_3 - c_2)w(v - \tilde{v}) + c_3(AQ/T + Bp/T - 1)w$, we choose c_i ($i = 1, 2, 3$), such that $c_1 = c_2 = c_3$. Then V is negative definite in $\text{int } \mathbb{R}_+^3$ if and only if $T > AQ + Bp$. So the microorganism eradication equilibrium $(\tilde{u}, \tilde{v}, 0)$ is globally stable in $\text{int } \mathbb{R}_+^3$.

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